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Scale Nomogram for Stereo Plotters

. . . allows the operator to select the optimum combination of instrument base, projection distance and height counter.

Assume that the following conditions apply:

- The camera characteristics, i.e., format and focal length are known, viz., 23×23-cm format, 152-mm camera constant.
- Forward overlap is the usual 60 percent.
- Approximate flying height above the terrain is known and hence nominal picture scale is known.

The usual sequence in plotting in analogue instruments is to connect the photo scale to the model scale and subsequently the model scale to that of the manuscript. Thus the following mathematical relationships exist:

$$M_m = \mu_{\text{machine}} \cdot M_B \tag{1}$$

$$M_p = \mu_{\text{gear}} \cdot M_m \tag{2}$$

Determination of Machine Scale and Limits

It is noted, referring to Figure 1, that the following relationships hold from similar triangles:

$$M_m = \frac{b}{B} = \frac{z}{Z} = \frac{d}{D} = \frac{x}{X} = \frac{\text{model}}{\text{terrain}} \tag{4}$$

where lower-case symbols refer to model parameters and upper-case symbols refer to the equivalent terrain parameters, and b is the model base, z is the projection distance, d is the half of effective width of coverage, and x is the half of effective length of coverage

Abstract: *A general method of determining the correct settings of the instrument base, height counter, and gear ratios used in analogue instruments includes examples. The method presupposes vertical photography with normal overlap conditions indicating how simple nomograms may be constructed for determining the correct machine base, model scale, height counter and plot scale under varying conditions as normally encountered by operators. These determinations are useful in both the planning and production stages.*

and thus

$$M_p = \mu_{\text{gear}} \cdot \mu_{\text{machine}} \cdot M_B \tag{3}$$

where M_m is the scale, M_p is the manuscript scale, M_B is the photo scale, μ_{machine} or z/f is the machine magnification, u_{gear} is the machine gear ratio, z is the machine projection distance, and f is the camera constant.

In general, u_{gear} is one of a finite number of combinations supplied with the instrument, e.g., 2:1, and there is usually a maximum and minimum value with a reasonable number of intermediate ratios available. The main problem is that M_m has to be chosen so that it allows the use of one of the standard height counters supplied and the attainment of the current plotting scale commensurate with the instrument's design limitations.

In order that Equation 4 is satisfied, the following relationships must be satisfied by the above four parameters in the model space:

$$\begin{aligned} z_{\min} &< z < z_{\max} \\ d_{\min} &= 0 < d < d_{\max} \\ b_{\min} &< b < b_{\max} \\ x_{\min} &= 0 < x < x_{\max}. \end{aligned} \tag{5}$$

By virtue of Equation 4, Equations 5 may be subsequently expressed as

$$\begin{aligned} z_{\min} &< M_m Z < z_{\max} \\ d_{\min} &= 0 < M_m D < d_{\max} \\ b_{\min} &< M_m B < b_{\max} \\ x_{\min} &= 0 < M_m X < x_{\max} \end{aligned}$$

where

$b_{\min}$ is the minimum machine base in mm, e.g., 65 mm in Wild A-8;

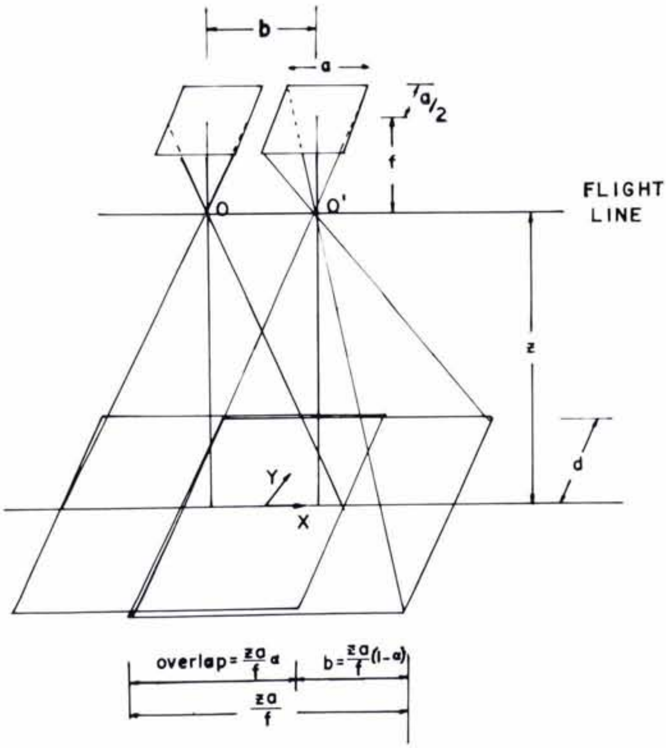

FIG. 1. Diagram representing the model space.

b_{max} is the maximum machine base in mm, e.g., 220 mm in Wild A-8;
z_{min} is the minimum machine base in mm, e.g., 175 mm in Wild A-8;
z_{max} is the maximum machine base in mm, e.g., 350 mm in Wild A-8;
d_{max} is half the maximum y range in mm, e.g., 220 mm in Wild A-8;
x_{max} is half the maximum x range in mm, e.g., 168 mm in Wild A-8.

Thus it is seen from Equations 6 that M_m has to be chosen such that the following important condition is fulfilled: greatest lower limit $< M_m <$ smallest upper limit.

Unfortunately, B, Z, D and X are not constant but vary considerably both within a flight and from flight to flight.

Considering Figure 1, it is readily seen from similar triangles that the following relationships exist:

$$d = (z/f)(a/2) \tag{7}$$

$$b = (za/f)(1 - \alpha) \tag{8}$$

$$x = (z/f)(a\alpha/2) \tag{9}$$

where α is the forward overlap expressed as a decimal and a is the dimension of the photograph, assumed square.

Assume now that Z varies by 10 percent, either due to nature of the terrain, or due to the pilot's inability to maintain a constant flying altitude, and that the forward overlap varies between the limits of $0.55 \leq \alpha \leq 0.65$. These variations are considered acceptable and normal by most photogrammetrists and serve as a suitable starting point for the following discussion. It is immediately noted that if Z varies by 10 percent, then z, the instrument projection distance, must also vary by 10 percent. To ensure that a 10 percent variation in the instrument's projection distance is available, the nominal projection distance is defined as:

$$(10/9)z_{min} < M_m Z < (10/11)z_{max}. \tag{10}$$

Variations in d, b and x are computable as follows. The y-range of the instrument is affected only by variations in z, whereas the base and x ranges are affected by both z and α variations. To first-order accuracies these variations are

$$\delta d = \frac{d}{z}\,\delta z \tag{11}$$

$$\delta b = \frac{\partial b}{\delta z}\,\delta z + \frac{\partial b}{\partial \alpha}\,\delta\alpha = \frac{b}{z}\,\delta z - \frac{b}{(1-\alpha)}\,\delta\alpha \tag{12}$$

$$\delta x = \frac{\partial x}{\partial z}\delta z + \frac{\partial x}{\partial \alpha}\delta\alpha = \frac{x}{z}\delta z + \frac{x}{\alpha}\delta\alpha. \tag{13}$$

It is necessary and sufficient for this study to be concerned with the magnitude of the possible variations without regard to sign as both cumulative and noncumulative cases occur. Thus Equations 11, 12 and 13 are without respect to sign of the deviations

$$\delta d = \frac{d}{z}\delta z \tag{11a}$$

$$\delta b = \frac{b}{z}\delta z + \frac{b}{1-\alpha}\delta\alpha \tag{12a}$$

$$\delta x = \frac{x}{z}\delta z + \frac{x}{\alpha}\delta\alpha. \tag{13a}$$

Rearrangement and substitution of the appropriate values yield

$$\frac{\delta d}{d} = 0.1 \tag{14}$$

$$\frac{\delta b}{b} = 0.1 + \frac{0.05}{0.40} = 0.225 \tag{15}$$

$$\frac{\delta x}{x} = 0.1 + \frac{0.05}{0.60} = 0.183. \tag{16}$$

Thus to ensure that these variations are attainable, the following conditions apply:

$$\frac{10}{9}d_{\min} = 0 < M_m D < \frac{10}{11}d_{\max} \tag{17}$$

$$\frac{11}{9}b_{\min} < M_m B < \frac{11}{14}b_{\max} \tag{18}$$

$$x_{\min} = 0 < M_m X < \frac{9}{11}x_{\max} \tag{19}$$

These conditions, together with the condition specified in Equation 10 are the necessary and sufficient conditions which govern the range of values that M_m can safely take without overrunning the design limitations of the plotter under the specified working conditions. This *safe region* and other particular operating regions are most easily determined from a nomogram (Figure 2) which is constructed by using Equations 7 to 10 and 17 to 19 and the assumed initial conditions.

METHOD OF CONSTRUCTING THE NOMOGRAM

The nomogram has four axes, corresponding to each of the four variables being considered. They are set down, on a plane, in the following manner:

(1) The projection range of the instrument, z, is along the usual $-y$ direction.
(2) Half of the y-range of the instrument, d, is along the usual $-x$ direction.

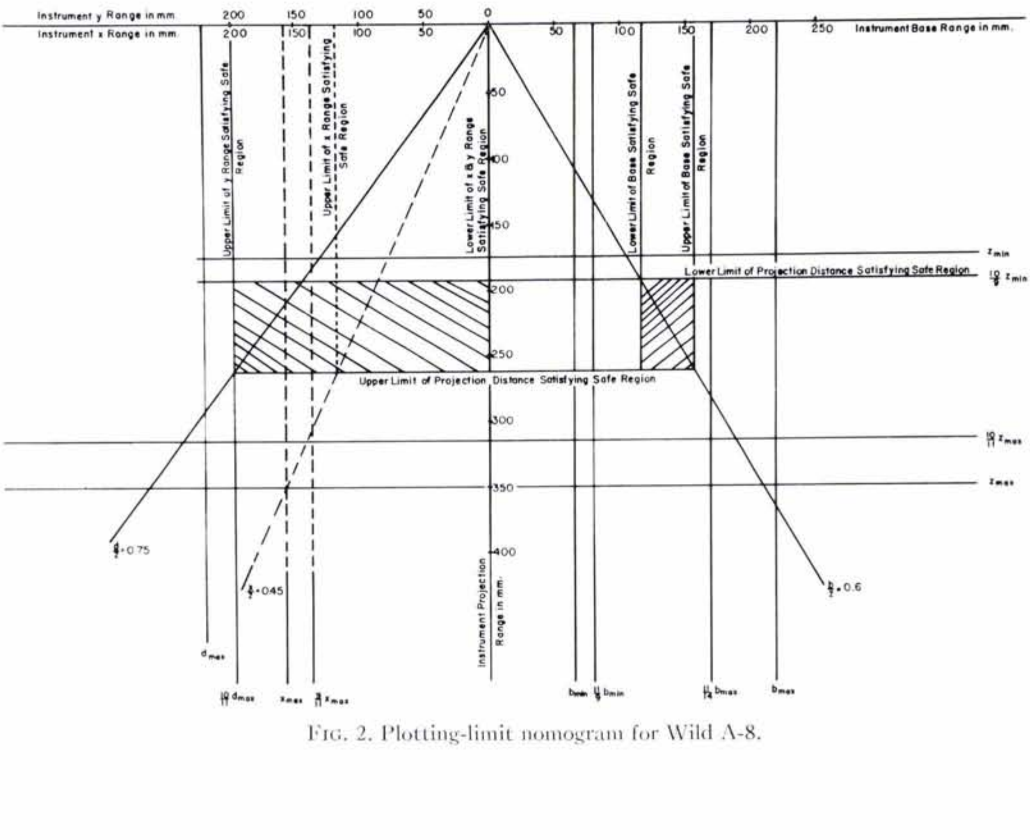

FIG. 2. Plotting-limit nomogram for Wild A-8.

(3) The base range of the instrument, b, is along the usual x direction.
(4) Half of the x-range of the instrument, x, is along the usual $-x$ direction.

Such an axial system forms a T-junction at the intersection of the axes. Each axis is labeled so that it shows both the design maxima and minima, as well as the operating maxima and minima, computable from Equations 10, 17, 18 and 19, under the chosen working conditions. For the Wild A-8 and the chosen conditions, the operational limits are:

$$\begin{aligned} 193 &\leq M_m Z = z \leq 315 \\ 0 &\leq M_m D = d \leq 198 \\ 80 &\leq M_m B = b \leq 171 \\ 0 &\leq M_m X = x \leq 137. \end{aligned} \tag{20}$$

Equations 7, 8 and 9 are next evaluated to obtain the connecting relationships between them and the projection distance which is common to all three. Under the initial assumptions they are:

$$\begin{aligned} d/z &= 0.75 \\ b/z &= 0.60 \\ x/z &= 0.45. \end{aligned} \tag{21}$$

These relationships are used to connect the base, y-range, and x-range to the projection distance, thus defining the *safe region* by their intersection with the respective b, x, and y limits.

This completes the construction of the nomogram.

Use of Nomogram

TO SET APPROXIMATE INSTRUMENT BASE AND TO SELECT CORRECT HEIGHT COUNTER

It is immediately evident, from Figure 2, that there is a *safe region* which simultaneously satisfies Equations 5 and 6. This region is

$$\begin{aligned} 195 &\leq M_m Z = z \leq 265 \\ 0 &\leq M_m D = d \leq 198 \\ 116 &\leq M_m B = b \leq 158 \\ 0 &\leq M_m X = x \leq 118. \end{aligned} \tag{22}$$

Operation outside this region is possible but not assured, the principle restriction being the instruments limited y-range which can be disregarded when the whole plate is not being plotted. It is also noted, by virtue of the limits placed on z, that the limits on μ_{machine} are definable. For the example under consideration these limits are:

$$1.29 \leq \mu_{\text{machine}} = \frac{M_m}{M_B} \leq 1.74 \tag{23}$$

The application of Equations 1, 2 and 3 subjected to the restrictions imposed by Equations 22 and 23 will yield a workable combination of μ_{machine}, μ_{gear}, and height counter for any given M_B and desired M_p. Consider the following example. Given photography whose scale is approximately 1:6,800 from which a map at a scale of 1:2.000, with 5-ft contour intervals is required. Then by application of Equation 3,

$$\begin{aligned} \mu_{\text{gear}} \cdot \mu_{\text{machine}} &= M_p/M_B \\ &= (1/2{,}000) \div (1/6{,}800) = 3.4. \end{aligned} \tag{24}$$

Substitution of this into Equation 23 yields, after rearranging,

$$1.87 \leq \mu_{\text{gear}} \leq 2.66. \tag{25}$$

Using this equation, u_{gear} is now chosen. The best choices for μ_{gear} is the ratio that is the smallest, as this guarantees that the maximum possible model scale will be chosen and consequently error propagation will be minimized. In the context of this paper, using the standard set of gears available on the Wild A-8, $u_{\text{gear}} = 2$, i.e., the transmission ratio between the model space and the manuscript space is 1:2. From Equation 24 the consistent model scale is determined as 1:4,000. If perchance a height counter is not available for the consistent model scale, then other choices of μ_{gear} satisfying Equation 17 are tried.

The approximate base setting required to achieve the above can be computed by applying $\mu_{\text{machine}} = z/f$, and hence Equation 8, or instead of 8 the nomogram can be used because z and b are shown as functions of each other.

For the example under consideration, which is a wide-angle camera producing 1:6,800 scale photographs for use in mapping at a scale of 1:2,000 with 5-ft contour intervals on a Wild A-8, the consistent instrument settings are as follows:

The projection distance distance z:

(1) $z = \mu_{machine} \cdot f = 1.72 \times 152 = 258$ mm.
(2) The instrument base from nomogram or Equation 8 is 155 mm.
(3) The model scale is 1:4,000, hence, choose the 1:4,000 height counter, a supplied standard scale.
(4) The transmission ratio is 1:2, model space to manuscript space.

TO PLAN FOR SUITABLE PHOTOGRAPHS.

From the basic geometry of stereophotogrammetry it is known that

$$dZ = (Z/b)d_{px} \tag{26}$$

holds to the first approximation where d_{p_x} is a small change in x parallax and hence

$$\sigma_Z = (Z/b)\sigma_{p_x}. \quad (27)$$

Unfortunately, σ_{p_x} is not a simple function of the operator's acuity but a complex function of instrument, emulsion, terrain, weather, and other conditions, such that the following form is often taken under as repeatable conditions as possible.

$$(\sigma_Z)_{\text{total}} = k(\sigma_Z)_{\text{observer}} \quad (28)$$

where k is an empirical constant determined by the operator from observations over a long period of time. The usual requirements for vertical points is that at publication scale, ". . . not more than 10 percent of elevations tested shall be in error more than one-half of the contour interval. . . " Assuming that errors are normally distributed with mean zero, the above requirement can be formulated as reading

$$1.645\,(\sigma_Z)_{\text{total}} \not> 0.5\,\Delta Z. \quad (29)$$

Substituting Equation 28 into 29 yields

$$(\sigma_Z)_{\text{total}} = k(\sigma_Z)_{\text{observer}} \not> (1/3.28)\,\Delta Z \approx 0.3 \quad (30)$$

which reduces under the assumption that $(\sigma_Z)_{\text{observer}}$ has the representative value of $Z/18{,}000$ to

$$\frac{\Delta Z}{Z} > \frac{k}{5400} \cdot \quad (31)$$

The ratio $\Delta Z/Z$ is the well known C-factor, or Contour-Height Ratio, which takes on a multitude of values depending on k. A representative value for k, for second-order instruments such as the Wild A-8 under normal contour production conditions, is $k=5$. Thus, the numerical value of the Contour-Height Ratio is 1/1,080 under the assumed conditions.

It is now required to map at a scale of 1:600 (1 in = 50 ft) with 2-ft contour lines a proposed interstate highway route. The specifications for such mappings normally only call for a narrow strip (800 to 1000 feet in width) to be mapped whence much of the photograph is not normally used. However this can be partially overcome as follows.

From the Contour-Height Ratio and the requirements of the project, it is determined that a nominal flying altitude, above ground, of 2,160 feet and the resulting photoscale of 1:4,320 will supply contours according to the above assumptions and specifications. The total magnification required to compile the map from this photography is 7.2, that is, $\mu_{\text{machine}}\cdot\mu_{\text{gear}}=7.2$. Additionally, it is determined from Equations 4, 7 and 8 that the effective nominal coverage of each model is 1,300×3,200 feet.

Referring now to the nomogram, or Equation 23, it is noted that the *safe region* limits μ_{machine} to the range $1.29 \leq \mu_{\text{machine}} \leq 1.74$ and, consequently, unless doubling gears are available, the limits of the manuscript with respect to the photo scale are $0.32 \leq \mu_{\text{machine}}\cdot\mu_{\text{gear}} \leq 7.0$. However, as it is not required to plot all of the photograph, a greater μ_{machine} can be chosen without regard for the y-range of the instrument the *safe region* limits become the following operational limits

$$\begin{aligned} 195 \leq M_m Z = z \leq 283 \\ 0 \leq M_m D = d \leq 198 \\ 116 \leq M_m B = b \leq 170 \\ 0 \leq M_m X = x \leq 128, \end{aligned} \quad (32)$$

which in turn places the following limits on μ_{machine}

$$1.29 \leq \mu_{\text{machine}} \leq 1.84. \quad (33)$$

These limits of course can be further modified if further assumptions warrant it. Thus the limits on magnification become,

$$0.32 \leq \mu_{\text{machine}}\cdot\mu_{\text{gear}} \leq 7.45,$$

and hence the required magnification, 7.2, is within the instrument's capability. It is to be noted that the *plottable* model now has the dimensions of 1300×2800 feet, which is still very much greater than the area to be plotted. Furthermore, should it be necessary to triangulate the strip and all control lies within 1,600 feet of the center line, then this photography will also suffice for the triangulation

Adaption to the Block Mode

The previous development was limited to a discussion of either single models, or strips of models. Fortunately the procedure is readily adaptable to the problem of blocks. In this instance Equation 7 becomes

$$d = (z/f)(a/2)(1-\beta) \quad (34)$$

where β is the side overlap expressed as a decimal. This equation shows that d, like b, is now effected by variations in z and in β. Thus it may be treated in exactly an analogous manner to that of the base b. Hence the variations in d are expressible as

$$\frac{\delta d}{d} = \frac{\delta z}{z} + \frac{\delta\beta}{(1-\beta)} \cdot \quad (35)$$

Equation 35 is analogous to Equation 12 and is treated in this manner in constructing new limits for d on a nomogram. The effect of Equation 35 is to increase the d safety region while increasing the slope on the (d/z)-curve, thus overall increasing the usable z and consequently the usable b range. It is interesting to note that for $\beta=0.2$ and $\delta\beta=0.10$ the y-range and the instrument base both limit the same projection distance, $z=283$ mms, making a much better balanced system.

Conclusions

(1) The use of nomograms, such as that of Figure 2, allows the operator to select quickly the combination of instrument base, projection distance, and height counter commensurate with the job in hand and that will limit the propagation of errors. The combination so chosen will, under the assumed conditions, safeguard the operator from overreaching any one of the instrument's limitations.

(2) Once a carefully determined C factor is available, it is possible to choose the flying height that will result in the desired accuracy from the smallest photo scale. It is under such conditions that the most favorable economic conditions to both the producer and the user are to be found. However, the requirements for each task must be separately incorporated into the mathematical model, and checked out, to ensure that a consistent system is attained.

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Reference

1. Wild Heerbrugg, *Wild A-8 Autograph Directions for Use*. Valid for Autographs No. 1192 onwards, Revised, 1966.