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Camera Orientation with Peripheral Circles

Images of circles such as storage tanks, sewer treatment plants, etc., offer solutions, but most oblique exposures do not generate right cones.

Abstract: *Camera orientation is treated with geometric rigor using image coordinates of an object circle generating an oblique cone with the lens nodal point as the vertex. In contrast, the solution for camera orientation is that of a single conic where the lens nodal point and the object circle generate a right cone. The equations for a right conic however are not valid for an oblique cone. Yet most oblique exposures of circles do not generate right cones because the lens nodal point seldom lies on a perpendicular to the center of the object circle.*

Application

Chance photography of unscheduled events frequently provide the only record that may be examined and analyzed after the event or events have transpired. In such a situation metrical analysis of a photograph with less than optimum parameters is not a matter of choice but necessity. Generally such records contain no control data. The expedient procedure then is to determine the orientation of the record with condition equations that assume certain geometric properties inherent in the object or objects photographed. Vanishing points associated with systems of parallel lines in object space and orthogonality associated with man-made structures have been employed since the beginning of photogrammetry. To some extent, images of circles such as storage tanks, sewer treatment plants, and numerous other man-made mechanisms have been employed. Historically, the application of circular geometry to camera orientation has been the application of approximate right-cone equations to full circles.

The Right Cone

A right cone is generated wherever the lens nodal point and the center of the object circle define a perpendicular to the plane of the circle. A photogrammetric right cone is illustrated in Figure 1. This case is easily solved inasmuch as any peripheral image and the axis of the cone define a constant angle k regardless of the inclination of the focal plane to the conic generated. The cosine of the constant angle k is expressed with the angle between two lines equation:

$$\frac{x_1}{l_1}\left(\frac{x_z}{l_z}\right)+\frac{y_1}{l_1}\left(\frac{y_z}{l_z}\right)+\frac{f}{l_1}\left(\frac{f_z}{l_z}\right)=\cos k.$$

Specifically a 3×3 matrix of the form is solved for a, b and c:

$$\frac{x_1}{l_1}a+\frac{y_1}{l_1}b+\frac{f}{l_1}c=1$$

$$\frac{x_2}{l_2}a+\frac{y_2}{l_2}b+\frac{f}{l_2}c=1$$

$$\frac{x_3}{l_3}a+\frac{y_3}{l_3}b+\frac{f}{l_3}c=1$$

where subscripts 1, 2 and 3 denote peripheral image points and

$$x_z=\frac{a}{c}f$$

$$y_z=\frac{b}{c}f.$$

These equations are not applicable to an oblique conic which is the usual case. An oblique cone is generated wherever the line connecting and lens nodal point and the center of the circle does not define a perpen-

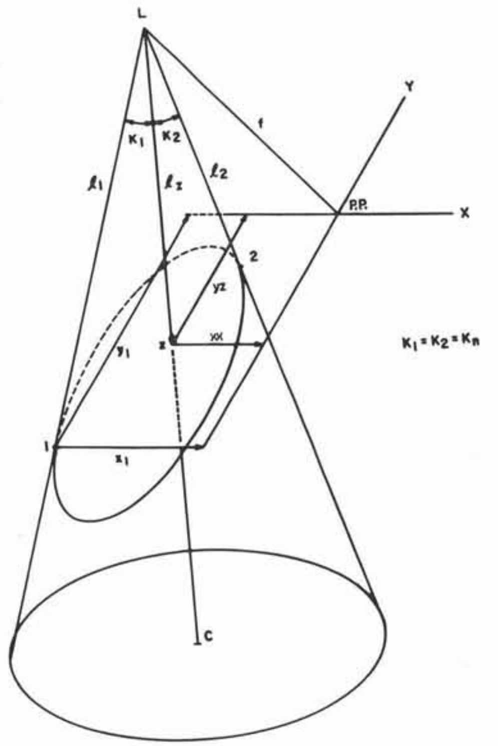

FIG. 1. Right cone intercepted by a focal plane.

dicular to the plane of the circle. The various approximate treatments of camera orientation with images of circles have not distinguished between right and oblique cones. These treatments have varied from the ratios of the minimum and maximum image diameters to anharmonic ratios if the center of the circle is displayed and permits equal segments to be assumed. It has been shown that inherent in the geometry of the photograph is an exact solution to camera orientation if a right cone is generated. In the special case, therefore, there is no justification for loose geometry.

The Oblique Conic

The geometric properties of an oblique conic are shown in Figure 2. The solution for a right cone is developed from the property that any peripheral image subtends a constant angle with the center of the object circle at the lens. This property does not hold for an oblique cone. The oblique cone solution is based on the theorem that a chord of a circle subtends a constant angle k for all points on its circumference. These angles projected to any plane not parallel to the plane of the object circle are systematically unequal. The angles subtended by a chord are unequal if projected to any plane not coincident with the plane of the object circle.

If the perpendicular to the plane is the z axis, the corresponding horizon plane passing through the lens is parallel to the plane of the circle. In this plane the inscribed angles are equal. The image points on the horizon line are defined by the image lines passing from the inscribed points through the terminals of the chord. Plane a_1lb_1 is defined by the direction of z. Planes al_1 and bl_1 are defined by images a, b and 1 with the lens nodal point. The solution is the determination of what values of x_z and y_z that will generate a plane producing equal values of k for all pairs of planes defined by the inscribed image points and the imaged terminal of a selected chord. The sine and cosine k may be expressed in the following analytic form:

$$\sin k = \left(\frac{xa_1yb_1 - xb_1ya_1}{la_1lb_1}\right)\frac{l_z}{f}\,.$$

$$\cos k = \frac{xa_1xb_1 + ya_1yb_1 + f^2}{la_1lb_1}\,.$$

Whence

$$\frac{\tan kf}{l_z} = \frac{xa_1yb_1 - xb_1ya_1}{xa_1xb_1 + ya_1yb_1 + f^2}\,.$$

The normals to planes al_1 and bl_1 are also normal to la_1 and lb_1. These normals are expressed as follows:

$$xna_1xa + yna_1ya + f^2 = 0$$
$$xna_1x_1 + yna_1y_1 + f^2 = 0$$
$$xnb_1xb + ynb_1yb + f^2 = 0$$
$$xnb_1x_1 + ynb_1y_1 + f^2 = 0,$$

or

$$xna_1 = \frac{f^2(ya - y_1)}{xay_1 - x_1ya}$$
$$yna_1 = \frac{f^2(x_1 - xa)}{xay_1 - x_1ya}$$
$$xnb_1 = \frac{f^2(yb - y_1)}{xby_1 - x_1xb}$$
$$ynb_1 = \frac{f^2(x_1 - xb)}{xby_1 - x_1yb}\,.$$

The normal l_z is also normal to lines la_1 and lb_1, whence coordinates of the horizon points a_1 and b_1 may similarly be expressed:

$$xa_1xna_1 + ya_1yna_1 + f^2 = 0$$
$$xa_1x_z \quad + ya_1y_z \quad + f^2 = 0$$
$$xb_1xnb_1 + yb_1ynb_1 + f^2 = 0$$
$$xb_1x_z \quad + yb_1y_z \quad + f^2 = 0$$

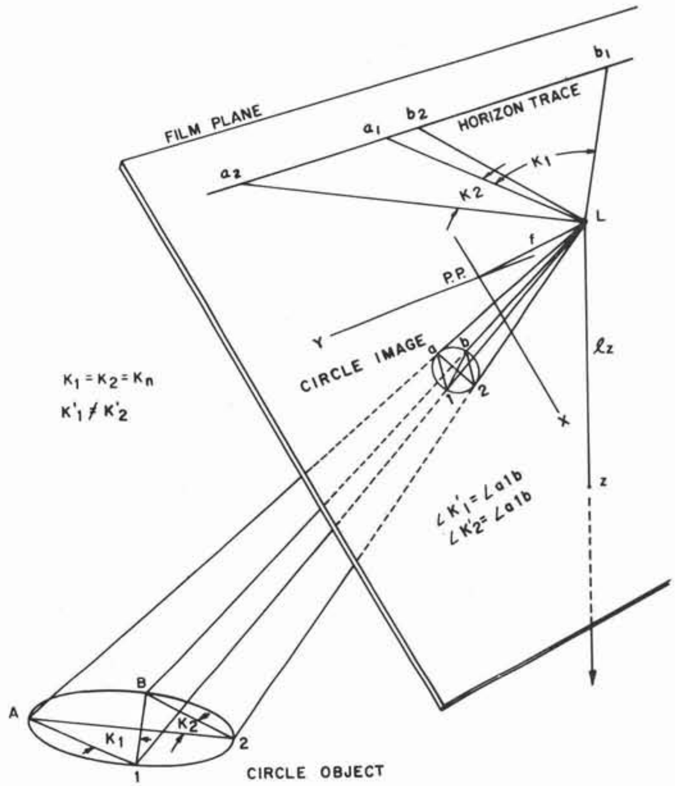

FIG. 2. Geometry of the general equation of camera orientation with peripheral coordinates referred to an arbitrary chord *ab*.

$$xa_1 = \frac{f^2(yna_1 - y_z)}{xna_1y_z - x_a yna_1}$$

$$ya_1 = \frac{f^2(x_z - xna_1)}{xna_1y_z - x_z yna_1}$$

$$xb_1 = \frac{f^2(ynb_1 - y_z)}{xnb_1y_z - x_z ynb_1}$$

$$yb_1 = \frac{f^2(x_z - xnb_1)}{xnb_1y_z - x_z ynb_1}.$$

Substituting in the equation for $\tan kf/l_z$ an equation of the form is obtained:

$$\left(\frac{\tan k}{l_z}\right) A + \left(\frac{\tan k}{l_z} x_z^2\right) B + \left(\frac{\tan k}{l_z} y_z^2\right) C + \left(\frac{\tan k}{l_z} x_z y_z\right) D + \left(\frac{\tan k}{l_z} x_z\right) E + \left(\frac{\tan k}{l_z} y_z\right) F + (x_z)G + (y_z)H = I$$

where

$$A = [xna_1xnb_1 + yna_1ynb_1]$$
$$B = [1 + (yna_1ynb_1)/f^2]$$
$$C = [1 + (xna_1xnb_1)/f^2]$$
$$D = -[xna_1ynb_1 + xnb_1yna_1]/f^2$$
$$E = -[xna_1 + xnb_1]$$
$$F = -[yna_1 + ynb_1]$$
$$G = -[yna_1 - ynb_1]/f$$
$$H = [xna_1 - xnb_1]/f$$
$$I = [xnaynb - xnb - yna]/f.$$

With approximate values of x_z' and y_z' an approximate average value of k' can be computed. Letting

$$A + x_z'^2B + y_z'^2C + x_z'y_z'D + x_z'E + y_z'F = M,$$

an approximate value of I' can be computed:

$$I' = \frac{\tan k'}{l'} M + x_z'G + y_z'H,$$

whence

$$dI = I - I' = d\,\frac{\tan k'}{l_z'} M + x_z'G + y_z'H$$

which reduces to

$$\Delta x_zP + \Delta y_zQ + \Delta k''R = dI$$

where

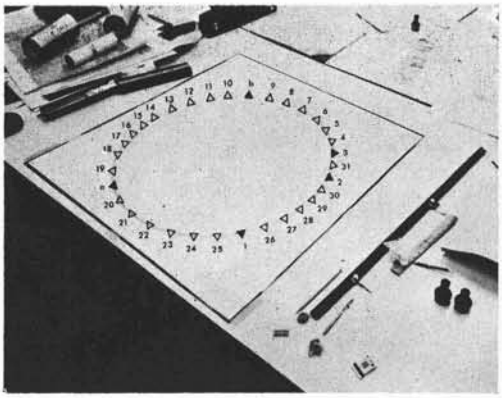

FIG. 3. A test circle.

$$P = \left[\frac{\tan k'}{l_z'}\left(2x_z'B + y_z'D + E - \frac{Mx_z'}{l_z^{2'}}\right) + G\right]$$

$$Q = \left[\frac{\tan k'}{l_z'}\left(2y_z'C + x_z'D + F - \frac{My_z'}{l_z^{2'}}\right) + H\right]$$

$$R = \frac{\sec^2 k'M \sin 1''}{l_z'}$$

The above equation has three unknowns and therefore requires three or more image coordinates of inscribed points to form a 3×3 matrix. In practice n points are used. The equations formed with estimates of x_z' and y_z' are normalized and solved iteratively until

$$\Sigma dI^2 = \text{minimum}.$$

The number of iterations seldom exceed three.

TEST CASE

The test of the equations was accomplished with image coordinates measured on an oblique exposure of a circle accurately scribed on a sheet of mylar. The exposure camera was a Hasselbald 10 feet above the circle. Thirty-four points were selected. The test (Figure 3) demonstrated the following:

(1) Points may be selected from either or both sides of the chord as long as the direction of the angles are preserved. Thus, if in the use of Point 1 the coefficients are formed from a to b, then for Point 15 on the opposite side of the chord the coefficient must be formed from b to a.

(2) No significant accuracy is gained in using more than 15 points.

(3) An accurate solution is obtainable with as little as one-fourth of a circle.

(4) The solution is unreliable if the diameter of the circular image is less than one-tenth the focal length.

(5) Finally, the chord selected cannot fail to be parallel to the image of the horizon by more than 30°. For example, if a chord is selected 90° to the horizon, one of the points of each pair will not cut the horizon.