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Analytical Topographic Surfaces by Spatial Intersection

The problem of representing detailed map surfaces by mathematical functions deserves more consideration.

INTRODUCTION

THE GENERAL PROBLEM of photogrammetry has been divided into two parts: the simultaneous restitution of the orientation of any number of photographic records, and the reconstruction of three-dimensional space by the intersection of corresponding rays.[1] Many theoretical and practical solutions of the first part of the general problem have been developed as a result of analytical photogrammetry. In particular, these solutions have been concerned with orientation, resection, aerotriangulation, camera calibration, image coordinate correction, and similar problems. The second part of the general problem, concerned ultimately with reconstruction of surfaces in space, has been less responsive to analytical solutions. A major aspect of the problem, namely that part concerned with surfaces rather than point intersections, is ordinarily solved by either manual or automated analogic photogrammetry. Advanced cartographic methods, involving numerical and polynomial approximations are also under development.[2,3] These methods are associated with stereo-model coordinates and with computer-controlled plotters. Consequently, they provide a combined computational graphical substitute for analogic photogrammetry. This does not quite approach the potentiality of analytical photogrammetry from an idealistic point of view. A need seems to proceed directly from photocoordinates to an analytical definition of surfaces in three-dimensions.

ABSTRACT: *The projective transformation equations and their derivatives, as used in the spatial intersection problem of analytical photogrammetry, provide solutions for discrete sets of object-space coordinates. These coordinates are ordinarily used in analytical aero-triangulation. However, the recent development of analytical surface equations, in which the geometry of topographic surfaces is expressed as a function of on-surface data-point coordinates, leads to consideration of spatial intersection as a direct means of deriving analytical equations of topography. It is shown that a linear system of surface equations with unknown coefficients can be combined with a linearized system of projective transformation equations involving unknown corrections to coordinates. The combined system is solved simultaneously for the analytical surface coefficients as well as the corrections to the coordinate data. Methods of evaluating the surface to produce contour maps and providing other geometric information deserve consideration.*

Thus, a major purpose of this paper is to demonstrate the intrinsic relationship between the point solutions of the projective transformation equations and the solutions for coefficients in certain analytical surface equations. First we postulate that a

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topographic surface is the locus of an indefinitely dense array of spatially intersected points. Analytical point intersections are implicitly contained in any number of line pairs emanating from each point of a topographic surface provided the spatial orientation of each line is known, and it passes through a known position. In photogrammetry we are explicitly concerned with ray pairs reflected to controlled vertices of two overlapping perspective bundles of rays. A spatially oriented photographic record of each perspective bundle permits a continuous model to be projected for stereoscopic viewing and analogical measurement. Alternatively, we may consider a point-by-point analytical reconstruction of the scene by using projective transformation. The latter procedure, without modification, is very inefficient; thus, it has been customary to determine the coordinates of a few control points by analytical methods, leaving the topographic compilation to less analytical (analog stereoplotting instruments), but currently more productive methods of surface analysis.

A potentially competitive analytical solution is possible if the indefinitely large array of intersected points can be reduced to a manageable finite array. To do this satisfactorily, an analytical function, applicable to topography, is needed to define the surface continuously between relatively few data points. Multiquadric surface equations seem to satisfy this requirement.[4,5] They not only fit data points exactly, but provide a logical interpolation of the surface at intermediate points as applied to topography. Furthermore, these equations are convenient for a simultaneous solution of data-point coordinates and of coefficients of the analytical surface by spatial intersection. This can be accomplished using a revised version of the linearized projective transformation equations of analytical photogrammetry.

This approach is somewhat different than usual numerical surface methods; numerical methods do not determine a true analytical surface over any extended area. In addition numerical methods are based on model coordinates. The more general analytical method to be described here is flexible enough to be used with image-space coordinates of central perspective bundles. Moreover, the derived surface is truly analytical over an extended area.

Multiquadric Surfaces

To aid readers who may not have immediate access to limited publications in this area,[4,5] a brief review of essential elements of multiquadric surface theory is presented here.

A general symbolic expression for multiquadric surfaces is:

$$\sum_{j=1}^{n} C_j[Q(X_j, Y_j, X, Y)] = Z \qquad (1)$$

in which Z is an analytical function of X and Y, resulting from the summation of a single class of individual quadric surfaces Q each with its vertical axis located at the data point coordinates X_j and Y_j. The associated coefficients C_j determine the algebraic sign and flatness of each quadric surface.

A summation of cones and sharp-nosed hyperboloids have been shown to be quite effective in representing various types of topography. Thus, Equation 1 may be expressed for this particular application as:

$$\sum_{j=1}^{n} C_j[(X_j - X)^2 + (Y_j - Y)^2 + C]^{\frac{1}{2}} = Z \qquad (2)$$

in which each term of the multiquadric surface is a single cone, or hyperboloid, dependent respectively upon whether C equals 0, or is a positive constant.

An early experiment with multiquadric equations was based on a topographic model from Krumbein.[6] His model was contoured from a 9 by 10 grid sample of part

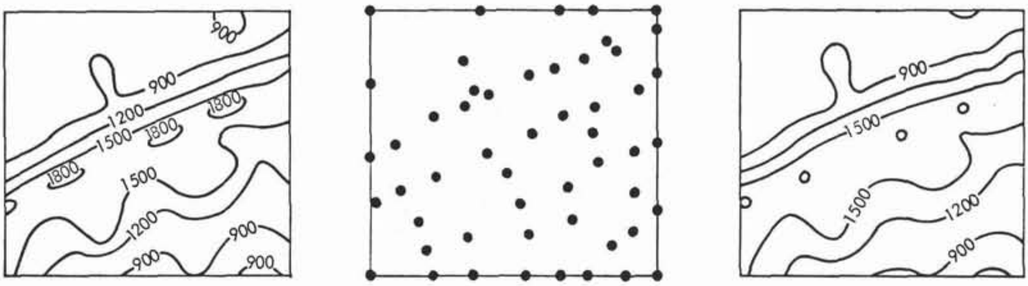

FIG. 1. (*Left*) Model of topography based on McClure, Pennsylvania, quadrangle. (*Center*) Location of 52 significant data points. These were taken from the lefthand figure and used to derive the corresponding multiquadric equation. (*Right*) Contour map from the multiquadric equation of topography. The equation was evaluated at 2,601 points (51 by 51 grid) to define the location of contours as shown.

of the 15-minute USGS quadrangle map, McClure, Pennsylvania. This model is reproduced at reduced scale in Figure 1 (*left*). The ground area represented by the model is about 23.2 square miles. The original map had a contour interval of 20 ft and the 90 surface elevations at the grid intersections were assigned values to the nearest 10 ft. However, the original map and grid data were not used in the multiquadric analysis. The horizontal coordinates and elevations of 52 significant points were measured and interpolated from the contours on Krumbein's generalized model. The horizontal locations of these significant points are shown in Figure 1 (*center*). The corresponding multiquadric surface, as determined by the 52 coefficients, is shown in Figure 1 (*right*). This solution was determined by using Equation 2 with $C=0$, i.e., a conic summation, which is adequate but not necessarily the optimum. The general procedure is the same regardless of the basic quadric or higher degree mathematical surface chosen.

The initial data in the example just shown consisted of Cartesian coordinates on the model surface ranging from X_1, Y_1, Z_1 to X_n, Y_n, Z_n; the quadric term coefficients C_1 to C_n were unknown. Equation 2 was expanded into a system of n linear equations with n unknowns. The left side was arranged in i rows and j columns whereas the right side consisted of a single column in i rows. This then gave:

$$\sum_{j=1}^{n} C_j[(X_j - X_i)^2 + (Y_j - Y_i)^2]^{1/2} = Z_i, \qquad i = 1, \cdots n. \tag{3}$$

Using matrix notation, the n-vector of unknowns was:

$$X = [c_j] = \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{bmatrix}.$$

Each unknown element was:

$$a_{ij} = [(X_j - X_i)^2 + (Y_j - Y_i)^2]^{1/2}$$

from which an $n \times n$ coefficient matrix was determined:

$$A = [a_{ij}].$$

Also, the n-vector of absolute terms was:

$$B = [Z_i] = \begin{bmatrix} Z_1 \\ Z_2 \\ \cdot \\ \cdot \\ \cdot \\ Z_n \end{bmatrix}.$$

Thus, Equation 2 was reduced to:

$$AX = B$$

for which, as usual, the solution was:

$$X = A^{-1}B.$$

After the known coefficients C_j were used in Equation 2, we obtained the required equation of topography that fit the data points exactly and provided the logical interpolation at intermediate points as shown in Figure 1.

It has been discovered recently that the coefficients C_j in a profile version of Equation 2, such as,

$$\sum_{j=1}^{n} C_j \left| X_j - X \right| = Z \tag{4}$$

can be determined with simple algebraic formulas based on the coordinates of data points, eliminating the need for matrix inversion. A related solution probably exists for the three-dimensional application. If so, these will be the subject of a future paper, because the computational advantages will become tremendously significant.

Spatial Intersection

Assume that the given parameters for a pair of photos are: $-f$, X_{C1}, Y_{C1}, Z_{C1}, ω_{C1}, ϕ_{C1}, κ_{C1}, X_{C2}, Y_{C2}, Z_{C2}, ω_{C2}, ϕ_{C2}, and κ_{C2}, referring to the camera focal length, camera coordinates, and camera orientation angles of two camera stations. The unknowns for n intersected surface points are: dX_{01}, dY_{01}, dZ_{01}, dX_{02}, dY_{02}, dZ_{02}, $\cdots dX_n$, dY_n, dZ_n which represent corrections to the assumed coordinates X_{01}, Y_{01}, Z_{01}, X_{02}, Y_{02}, Z_{02}, $\cdots X_n$, Y_n, Z_n. The observed parameters (plate coordinates) for the first point are: x_{0101}, y_{0101}, x_{0201}, and y_{0201}, where the subscript pairs, *02* and *01* in x_{0201}, for example, refer respectively to the x-coordinate on the second photo of the image of the first point.

Then, from the image-space coordinates, four linear equations for the first surface point are formulated as follows:

$$\begin{aligned} x_{0101} - F(x)_{0101} &= a_{0101}dX_{01} + b_{0101}dY_{01} + c_{0101}dZ_{01} \\ y_{0101} - F(y)_{0101} &= \bar{a}_{0101}dX_{01} + \bar{b}_{0101}dY_{01} + \bar{c}_{0101}dZ_{01} \\ x_{0201} - F(x)_{0201} &= a_{0201}dX_{01} + b_{0201}dY_{01} + c_{0201}dZ_{01} \\ y_{0201} - F(y)_{0201} &= \bar{a}_{0201}dY_{01} + \bar{b}_{0201}dY_{01} + \bar{c}_{0201}dZ_{01}. \end{aligned} \tag{5}$$

In this system, which is repeated in sets of four equations for each additional surface point in the region covered by the two photos, the theoretical values $F(x)_{0101}$, $F(y)_{0101}$, corresponding to observations x_{0101}, y_{0101} are:

$$\begin{aligned} F(x)_{0101} &= \frac{-f[(X_{01} - X_{C1})m_{11} + (Y_{01} - Y_{C1})m_{12} + (Z_{01} - Z_{C1})m_{13}]}{(X_{01} - X_{C1})m_{31} + (Y_{01} - Y_{C1})m_{32} + (Z_{01} - Z_{C1})m_{33}} \\ F(y)_{0101} &= \frac{-f[(X_{01} - X_{C1})m_{21} + (Y_{01} - Y_{C1})m_{22} + (Z_{01} - Z_{C1})m_{23}]}{(X_{01} - X_{C1})m_{31} + (Y_{01} - Y_{C1})m_{32} + (Z_{01} - Z_{C1})m_{33}} \end{aligned} \tag{6}$$

which are well-known projective transformation equations subscripted for the first two equations of System 5. The m's in Equation 6 refer to direction cosines of image coordinates relative to the XYZ object-space coordinate system, using a well-known convention.

The absolute terms of Equation 5, namely $x-F(x)$ and $y-F(y)$, are determined by subtracting the theoretical values $F(x)$ and $F(y)$ from the observed quantities x and y respectively. The theoretical values in Equation 6 are evaluated utilizing estimated values of X_{01}, Y_{01}, and Z_{01}. Here all other parameters, X_{C1}, Y_{C1}, and Z_{C1}, and the m's, are known from the resected camera coordinates and orientation matrix.

The a, $\bar{a}$, b, $\bar{b}$, c, and $\bar{c}$ terms in System 5 are linearized coefficients needed to solve the transcendental projective transformation equations. In particular, these are contained in an expression for the total differentials of $dF(x)$ and $dF(y)$. Thus, from Equations 5 and 6:

$$dF(x) = x - F(x) = \frac{\partial x}{\partial X} dX + \frac{\partial x}{\partial Y} dY + \frac{\partial x}{\partial Z} dZ$$

$$dF(y) = y - F(y) = \frac{\partial y}{\partial X} dX + \frac{\partial y}{\partial Y} dX + \frac{\partial y}{\partial Z} dZ.$$

Then the explicit correlation the of a, $\bar{a}$, b, $\bar{b}$, c, and $\bar{c}$ terms with the partial differential coefficients are:

$$\begin{aligned} a_{nm} &= \left(\frac{\partial x}{\partial X}\right)_{nm}, \quad \bar{a}_{nm} = \left(\frac{\partial y}{\partial X}\right)_{nm} \\ b_{nm} &= \left(\frac{\partial x}{\partial Y}\right)_{nm}, \quad \bar{b}_{nm} = \left(\frac{\partial y}{\partial Y}\right)_{nm} \\ c_{nm} &= \left(\frac{\partial x}{\partial Z}\right)_{nm}, \quad \bar{c}_{nm} = \left(\frac{\partial y}{\partial Z}\right)_{nm}. \end{aligned} \qquad (7)$$

These coefficients and the absolute terms of Equation 5 are evaluated initially with assumed values of X_{01}, Y_{01} and Z_{01}, and of other surface coordinates. As is normal for a transcendental function, the coefficients of the so-called linear system in System 5 are re-linearized, and the absolute terms recomputed, in each cycle of an iterative process until the final increments, dX_{01}, dY_{01}, and dZ_{01}, approach zero or are at least consistent with the precision of the observations.

Simultaneous Solution of Surface Coordinates and Coefficients of the Equations of Topography

The shape of a multiquadric surface is invariant in a normal three-dimensional Cartesian system. Therefore, the Z-coordinate of the XY-reference plane may be arbitrarily equated with zero at the level of average terrain. As to utilization of the projective transformation equations, it is convenient for the basic estimate of the Z-values, $Z_{01}, Z_{02} \cdots Z_n$, to be the same value for all data points, namely the average terrain height minus the average camera height above terrain. The average camera height above terrain may be directly and accurately estimated from radar altimetry, for example. Then the XY-datum plane of the camera coordinate system is made to coincide with the XY-datum plane of the multiquadric surface. If the camera coordinates are based on a datum at sea level, for example, the transformation only requires the subtraction of all camera coordinates from the average terrain height above sea level. An adjustment of the X and Y coordinates is not necessary.

Assume that the estimate of camera height above terrain is made carefully so that the variations of the actual terrain heights from the datum plane are small compared with the camera height above terrain. Then the terrain variations may be treated as approximations ΔZ. Further, assuming that nearly vertical photographs are used, the height error caused by treating a terrain variation as a differential approximation is negligible. If these assumptions are not applicable in certain instances, then the required rigor may certainly be developed; however, the following presentation will not be encumbered with these details, instead we place more emphasis on solving the basic problem.

Based on the preceding assumptions, the Z's of the multiquadric surface summation can take on the same meaning as $dZ_{01}, dZ_{02} \cdots dZ_n$ through the approximation $dZ \approx \Delta Z$. Thus, for the general case where $C \neq 0$, Equation 3 becomes:

$$\sum_{j=1}^{n} C_j[(X_j - X_i)^2 + (Y_j - Y_i)^2 + C]^{\frac{1}{2}} = dZ_{0i}, \qquad i = 1, \cdots n. \tag{8}$$

Then the first pair of equations in System 5 become:

$$\begin{aligned} x_{0101} - F(x)_{0101} = {} & a_{0101}dX_{01} + b_{0101}dY_{01} + c_{0101}C_{01}[(X_{01} - X_{01})^2 + (Y_{01} - Y_{01})^2 + C]^{\frac{1}{2}} \\ & + c_{0101}C_{02}[(X_{02} - X_{01})^2 + (Y_{02} - Y_{01})^2 + C]^{\frac{1}{2}} + \cdots \\ & + c_{0101}C_n[(X_n - X_{01})^2 + (Y_n - Y_{01})^2 + C]^{\frac{1}{2}} \\ y_{0101} - F(y)_{0101} = {} & \bar{a}_{0101}dX_{01} + \bar{b}_{0101}dY_{01} + \bar{c}_{0101}C_{01}[(X_{01} - X_{01})^2 + (Y_{01} - Y_{01})^2 + C]^{\frac{1}{2}} \\ & + \bar{c}_{0101}C_{02}[(X_{02} - X_{01})^2 + (Y_{02} - Y_{01})^2 + C]^{\frac{1}{2}} + \cdots \\ & + \bar{c}_{0101}C_n[(X_n - X_{01})^2 + (Y_n - Y_{01})^2 + C]^{\frac{1}{2}}. \end{aligned} \tag{9}$$

The expanded multiquadric equivalent of dZ_{01}, as shown in the two equations above, would occur in the first four equations of System 5. The differential dZ_{02} would be replaced in the second four equations, dZ_{03} in the third four equations, etc., as System 5 is expanded. Thus, four equations are added to System 5 for each surface point, and there is an increase of only three unknowns; in effect, one unknown multiquadric coefficient, C_j, is substituted for an unknown dZ as each new surface point is added to the system. On a point-for-point basis, a multiquadric system of equations can be combined with the spatial intersection system of equations without introducing additional unknowns. However, the application of the combined system to the detailed definition of topographic surfaces will result in a need to greatly increase the density of control points over that used in aerotriangulation.

The combining of the linear multiquadric system of equations with the linearized spatial intersection system of equations does not change the basic requirement for relinearization of the system in each cycle of an iterative least-squares solution. The corrections, $dX_{01} \cdots dX_n$ and $dY_{01} \cdots dY_n$, as determined by the first solution are applied to the original estimates of $X_{01} \cdots X_n$ and $Y_{01} \cdots Y_n$. Then the corrected values of $X_{01} \cdots X_n$, $Y_{01} \cdots Y_n$ are used in the multiquadric terms as well as in the expressions for the partial differential coefficients. It should be noted that dZ no longer appears as such in the expanded system of equations, as illustrated in System 9, and therefore need not be computed; in fact, it would be incorrect in this problem to do so, even if possible, because the XY plane at $Z=0$ is a fixed-reference plane for determining the final values of all dZ's. In effect, each dZ is corrected in each iteration by the response of the linear multiquadric coefficients $C_{01} \cdots C_n$ to the revised X and Y coordinates of that cycle. In the final iteration, as the incremental changes in the dX's and dY's approach zero, the values of the coefficients $C_{01} \cdots C_n$ define a surface which, at data points in particular, is the height variation between the actual terrain and the datum plane, i.e., $dZ_{01} \cdots dZ_n$. As with multiquadric surfaces in general, subject to the density and distribution of data points, it can be expected that the solution will provide a logical interpolation of the terrain between data points. Upon completion of the iterative photogrammetric solution, the known coefficients C_j may be used in Equation 2 to separately express the equation of topography for future use.

Concluding Remarks

The theory developed above has been checked with a few data points, but it is not known whether it is practical for a large number of points, without modification. The possibility of solving for a large number of coefficients by formula rather than matrix inversion is an attractive consideration, but this has not been completely confirmed. In any event, it has been shown, theoretically, that analytical photogrammetry can solve both parts of the general problem of photogrammetry, which includes the representation of surfaces in three-dimensional space. If this theory is reduced to practice, it is probable that it will be combined with automatic image-matching of photo-coordinates, thus resulting in a procedure that could be called automated, analytical sterocompilation.

Now we conclude with a few remarks about the possible advantages of analytical solutions of topography as opposed to graphical and numerical solutions. An equation of topography can be evaluated digitally or analytically, depending on the application. For the automatic production of contoured maps, an analytical mode is probably preferred. Automatic contouring can become a computer-plotter problem in analytic geometry, i.e., to determine and plot the intercept equations of level surfaces passing through a three-dimensional equation of topography. This approach could lead to a reconsideration of the nature of the need for digitized cartographic data. Topographic maps may be stored in analytical coefficient form, much like a mathematical subroutine. A recalled topographic map may be digitized, if needed, by computer, without having been produced in graphical form. Problems involving map use—determining unobstructed lines of sight, areas of defilade, volumes of earth, minimum length of surface curves, and others—could involve a direct application of analytical geometry and calculus to the interrelationship of these parameters with a mathematical surface of topography. For these reasons, the problem of representing a topographic surface in detail by analytic functions, using analytical photogrammetric methods, deserves increased consideration.

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