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Image Transformations

A suitable one represents an important means for improving image quality and increasing the accuracy of analytical photogrammetric processing.

ABSTRACT: *The systematic deformation of a photogrammetric image is treated from the point of view of defining a suitable corrective transformation. As no analytical formulation can entirely cope with all the physical phenomena causing image deformation, a theoretical analysis shows a general distribution of residual systematic errors after applying transformation corrections. The effect of this error compensation and redistribution depends on the internal structures of the transformations and on the number or configuration of fiducial marks used. Different distribution schemes for random changes and errors are analyzed independently of the systematic errors.*

INTRODUCTION

THE GEOMETRY of a photogrammetric image suffers due to various sources of distortion which occur at different phases in the process of forming and preserving the image prior to its final photogrammetric evaluation. The effect of the resulting deformation of the image can be partially eliminated by applying a suitable transformation to the set of measured photo coordinates. The chance to improve the geometry of the image is limited to the introduction of a suitable scale change if the photographs are processed with analog plotters. For the analytical treatment of photographs, however, a wide range of mathematical transformations in two dimensions is applicable. Because of the physical nature of distortion causing phenomena, which can be neither predicted nor fully controlled, one can never remove the effect of the existing deformation from the image completely. Nevertheless, the refinement of the geometry can at least supress the main part of original positional image errors.

Different choices of mathematical or statistical means are available to define a suitable transformation. Theoretically, the transformation task consists of a two-dimensional interpolation of corrections determined from a stable framework of fiducial or réseau marks. Generally, polynomial formulations using linear parameters are given preference in photogrammetric production to more complex systems with non-linear parameters. The number and selection of terms incorporated into the solution depend on the adopted degree of the transformation used. The linear changes of the image usually prevail over the non-linear ones. Consequently, the linear terms determine the most important transformation parameters, such as scaling and skew factors, which introduce linear changes in dimensions and angles. The non-linear changes are smaller and can be interpreted as causing a variable concentration or extension of the image information along straight lines and a bend of hypothetical straight lines.

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FORMULATION OF A SUITABLE TRANSFORMATION

There is a great variety of transformations to be utilized for the purpose of correcting the image geometry. In general, the polynomial transformation necessary to convert the vector $\boldsymbol{x}$ of primary coordinates into the vector $\boldsymbol{X}$ of corrected coordinates can be expressed by

$$X = x + Ag \tag{1}$$

where A is the matrix of the transformation and g, the vector of the individual transformation parameters. Considering non-réseau cameras, the main restriction in selecting a suitable type of transformation is imposed by the limited number of fiducial marks available in the camera frame. As each fiducial mark contributes to the formation of two equations, the number of transformation parameters is then twice as large. Most cameras are nowadays equipped with four fiducial marks so the following simple transformations represent an appropriate practical choice: an affine transformation with six parameters and the relevant matrix,

$$A = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 \\ 0 & 1 & 0 & x & 0 & y \end{bmatrix} \tag{2}$$

a similarity (linear conformal) transformation with four parameters,

$$A = \begin{bmatrix} 1 & 0 & x & -y \\ 0 & 1 & y & x \end{bmatrix} \tag{3}$$

a bilinear transformation with eight parameters

$$A = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 & xy & 0 \\ 0 & 1 & 0 & x & 0 & y & 0 & xy \end{bmatrix} \tag{4}$$

or a projective transformation with eight parameters (shown in linearized form),

$$A = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 & xy & x^2 \\ 0 & 1 & 0 & x & 0 & y & y^2 & xy \end{bmatrix}. \tag{5}$$

As the transformations are used for the elimination of systematic distortions, there is no special need to require redundancy in setting the equation system. Therefore, Equations 4 and 5 are usually preferred to the simpler Equations 2 and 3. The photographs with eight fiducial marks could be preferably treated by using a more general transformation containing all second order, and also some third order, terms. In general, the number of fiducials predetermines the usable number of unknown parameters in the selected transformation.

The use of réseau photographs requires different reasoning. It would not be reasonable to increase the number of parameters in proportion with the number of réseau points; such an approach would be very inefficient. Instead, one takes full advantage of mosaic-type transformation of smaller sections of the image. This is more efficient, as well as more dependable, than a simultaneous complex transformation. The non-linear changes of the image can be eliminated better in sections of smaller size. Any complex transformation can be replaced by a set of simpler transformations which are applied to fields derived by suitable partitioning of the original image area. In an extreme example, the smallest réseau squares are used for the simplest possible transformation consisting only of a suitable shift of the coordinate origin. More accurately, four corner crosses of a local réseau square will accommodate any of Transformations 2 to 5. Only a bilinear Transformation 4 could, however, provide an exact linkage of individual local transformations in ties.

Another important aspect is worth mentioning. Not only the degree of complexity or the number of parameters for any particular transformation plays a role, but also the internal structure of the transformation is important. One can wonder whether or not local x, y deformations are mutually independent. Undoubtedly, x and y physical changes in real photographs have to be naturally correlated to a certain extent. Accordingly, some of the transformation parameters should be constrained. The actual analytical form of these variable constraints is difficult to be determined or predicted. A typical example of a physically correlated two-dimensional deformation is the radial effect of an improper flattening of the film.

A good insight into the matter of analytical internal correlations could be provided by the comparison of two above mentioned transformations. It is obvious that the similarity Transformation 3 represents a special case of more general affine Transformation 2. Both matrices in question use the same individual coefficients in a different arrangement. Using notation p and g for the parameters applied to Matrices 3 and 2, respectively, two basic constraints can be found

$$p_3 = g_3 = g_6, \quad p_4 = g_4 = -g_5 \tag{6}$$

to specify the similarity Transformation 3 where p_1, p_2 are identical with g_1, g_2.

The x, y correlations are even more obvious for higher-order conformal transformation as represented by the following matrix

$$A = \begin{bmatrix} 1 & 0 & x & -y & (x^2-y^2) & -2xy & (x^3-3xy^2) & (y^3-3x^2y) \\ 0 & 1 & y & x & 2xy & (x^2-y^2) & -(y^3-3x^2y) & (x^3-3xy^2) \end{bmatrix}. \tag{7}$$

This matrix can now be compared with the general third-degree transformation expressed by independent partial matrices A_x, A_y:

$$A_x = A_y = [1\ x\ y\ xy\ x^2\ y^2\ x^2y\ xy^2\ x^3\ y^3]. \quad (8)$$

All the elements of the general Matrix 8 are used in the Matrix 7 but in specific combinations which comply with the conditions of conformality.

The conformal transformation is used in this analysis without any further practical implications, only to illustrate the relationship between conditioned x, y changes and relevant transformation parameters. In general, any other physically or geometrically correlated deformation gives rise to a constraint among the transformation parameters, and any known constraint lowers the number of the necessary parameters, saving one unknown in the solution. On the other hand, if the aforementioned correlation is not properly known, one can use more general transformation with a higher degree of freedom instead of that specifically constrained one. In this instance, the desirable internal transformation fit would be established only at the price of using additional fiducial marks.

There is no general prescription how to formulate the optimum transformation because of the variety of factors affecting the dimensional stability of the photogrammetric image. Statistical analysis of the significance of different terms which should participate in the formation is very useful in this regard. The experience indicates that under normal conditions the systematic deformations of the image could be classified as first- to third-order changes. There is no need to use any higher order terms for describing these changes and for introducing corrections. The general third-degree polynomial as expressed by Equation 8 could probably compensate any significant distortions leaving local residuals of negligible magnitude in the image.

Unfortunately the number of parameters applied to this transformation is too high for any practical use. It amounts to 20 and would require 10 fiducial marks to give the solution. For the purpose of further analysis, let us consider this type of image deformation as a standard model which could be approximated by using a lower number of reference points. Simpler transformation can eliminate the main part of deformations, partially compensating some others and redistributing the rest of them.

Remaining Systematic Errors

The effect of the error compensation and redistribution is different depending on the selection of the terms for the approximating transformation, on the type of inherent constraints and on the number of configuration of used fiducial marks.

GENERAL THEORY

The general transformation Formula 1 can be applied to define an equation system formed from the necessary number of points

$$X = x + Ag$$

where A now represents the square matrix of desirable transformation. No redundancy is taken into account in this definition and no secondary random errors are considered. The matrix A can not be formulated properly and is approximated by the matrix A_0 which will further be referred to as *incomplete*. The internal structures of these two matrices differ as shown in Figure 1. The ideal matrix A is divided into two parts. The first one consists of common terms that are recognized and used also in the matrix A_0 whereas the second part of additional terms is missing in A_0. On the other hand, some other terms in A_0 can be considered as redundant because they do not appear in the ideal matrix A. The *incomplete* matrix A_0 should be completed by the supplementary matrix A_1 to form the *ideal* matrix A in accordance with

$$A = A_0 + A_1. \quad (9)$$

The vector g is to be split in a similar way to define the transformation

$$X - x = Ag = A_0g_0 + A_1g_1. \quad (10)$$

Using only the first part A_0g_0 of the ideal transformation Ag, one compensates for the remaining terms described by A_1g_1. Thus, an incorrect vector of parameters $\hat{g}_0$ is determined which, if multiplied by A_0, tries to substitute the ideal transformation Ag. It holds true

$$A_0\hat{g}_0 = Ag \quad (11)$$

so that

$$\hat{g}_0 = A_0^{-1}Ag. \quad (12)$$

Obviously, the square matrix A_0 must be nonsingular, which depends on the number and

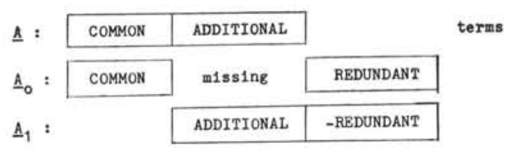

FIG. 1. Internal structure of transformation matrices.

configuration of fiducial marks. Substituting Equation 10 into 11 one gets

$$\hat{g}_0 = g_0 + dg \qquad (13)$$

where

$$dg = A_0^{-1}A_1g_1 \qquad (14)$$

can be considered as an actual error of deriving g_0, which compensates for the neglected vector g_1.

The residual errors in individual points, remaining after this imperfect transformation are expressed by the difference

$$dx = \bar{A}_0\hat{g}_0 - \bar{A}g,$$

where symbols $\bar{A}$, $\bar{A}_0$ refer to the two-row-matrices formed from coordinates of individual points. Applying previous Equations 10 and 13, one gets

$$dx = \bar{A}_0dg - \bar{A}_1g_1.$$

The first term on the right side of the previous equation represents the compensation achieved by the imperfect transformation where the latter term expresses the originally neglected part of the distortion. The final substitution of Equation 14 brings about

$$dx = (\bar{A}_0A_0^{-1}A_1 - \bar{A}_1)g_1. \qquad (15)$$

Similar formulas can be derived with the aid of a least-squares solution if redundant observations are applied to set the basic equation system. In order to distinguish formally between this and the previous case, the rectangular matrix of the transformation system is denoted B. The essential equations for both cases are listed in the following Table 1. The final Equations 16 (shown in Table 1) are adopted from 15 using the definition of an auxiliary matrix P with the dimensions n_0, n_1, which denote the number of parameters in vectors g_0, g_1 respectively. This matrix P contains all the information about the original errors due to neglected transformation terms, and about their redistribution in the course of the specific solution, with respect to a definite configuration of the fiducial marks used.

Taking into consideration the dimensions of matrices and vectors in Formula 16,

$$P(n_0, n_1), \quad \bar{A}_1(2, n_1), \quad g(n_1, 1),$$

one can alternatively express the vector of residual errors as a sum

$$dx = \sum_{j=1}^{n1} (\bar{A}_0p_j - \bar{a}_{1_j})g_{1_j}, \qquad (17)$$

where p_j, $\bar{a}_{1j}$ represent individual columns of the relevant matrices and g_{1j} are relevant

TABLE 1. REDISTRIBUTION OF SYSTEMATIC ERRORS.

(*No Redundancy*)		(*Redundancy*)	
	Substitute transformation		
$A\hat{g}_0 = Ag$		$v = Bg - B_0\hat{g}_0$	(11)
	False parameters		
	$\hat{g}_0 = g_0 + dg$		(13)
	Errors in parameters		
$dg = A_0^{-1}A_1g_1$		$dx = (B_0'B_0)^{-1}B_0'B_1g_1$	(14)
	Residual errors		
$dx = (\bar{A}_0A_0^{-1}A_1 - \bar{A}_1)g_1$		$dx = (\bar{B}_0(B_0'B_0)^{-1}B_0'B_1 - \bar{B}_1)g_1$	(15)
	or		
$dx = (\bar{A}_0P - \bar{A}_1)g_1$		$dx = (\bar{B}_0P - \bar{B}_1)g_1$	
	where		(16)
$P = A_0^{-1}A_1$		$P = (B_0'B_0)^{-1}B_0'B_1$	

vector components. This arrangement makes it possible to perform the analysis separately for individual terms or groups of terms in the neglected part of the transformation $A_1 g_1$, e.g., in groups of missing or redundant terms according to Figure 1.

PRACTICAL APPLICATIONS

The theoretical analysis leading to Equations 15 or 16 can be practically utilized in two ways: to analyze and compare various formulations of *incomplete* transformations; and to assess the effect of different number and configuration of fiducial marks used to control the transformation. To demonstrate this, a few examples of such an analysis are presented here without any intention to prefer some of the applied formulations before the others.

COMPARISON OF BILINEAR AND PROJECTIVE TRANSFORMATIONS

Bilinear and projective transformations represented by Equations 4 and 5, respectively, have very similar structures. The only difference consists of two additional quadratic terms in the projective formulation which increase its potential and partially correlate x- and y-changes. The number of parameters is the same but their geometric interpretation is different.

In recognition of its more general form let us consider the projective transformation as *ideal* whereas the other could be regarded as *incomplete*. In accordance with the above used notation, one defines matrices

$$\bar{A}_0 = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 & xy & 0 \\ 0 & 1 & 0 & x & 0 & y & 0 & xy \end{bmatrix}$$

and

$$\bar{A}_1 = \begin{bmatrix} 0 & x^2 \\ y^2 & 0 \end{bmatrix}.$$

With reference to the classification in Figure 1, there are only *additional* terms to be considered for the matrix $\bar{A}_1$. Four fiducial marks should be used to form an invertable matrix A_0. To simplify the analysis the location of corner fiducial marks P_i can be defined in a right-handed system by unit coordinates, as follows:

$$P_1(-1, 1),\ P_2(1, 1),\ P_3(-1, -1),\ P_4(1, -1).$$

Consequently, the matrix A_0 of the incomplete transformation and its inverse A_0^{-1} are

$$A_0 = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$A_0^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Following Equation 16 one derives

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad P = A_0^{-1}A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{A}_0 P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and finally, the residual errors are

$$\begin{aligned} dx &= (1 + x)(1 - x)g_8 \\ dy &= (1 + y)(1 - y)g_7. \end{aligned} \tag{18}$$

The resulting Equation 18 can be interpreted by stating that the bilinear transformation differs from the projective one in leaving quadratic discrepancies along lines parallel to the x and y axes.

DISTRIBUTION OF SYSTEMATIC ERRORS

The following set of examples shows the performance of selected typical transformations as to the compensation and redistribution of distortions not covered directly by the formulas. General third-degree Polynomial 8 is considered as an *ideal* transformation with reference to above adopted convention. Matrix $\bar{A}$ is arranged by composing A_x, A_y from Equation 8 in the following way:

$$\bar{A} = \begin{bmatrix} 1 & 0 & x & 0 \cdots x^3 & 0 & y^3 & 0 \\ 0 & 1 & 0 & x \cdots 0 & x^3 & 0 & y^3 \end{bmatrix}. \tag{19}$$

First, the effect of different configurations of four fiducial marks is discussed as used for the projective Transformation 5. The *incomplete* transformation matrix

$$\bar{A}_0 = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 & xy & x^2 \\ 0 & 1 & 0 & x & 0 & y & y^2 & xy \end{bmatrix}$$

compared with the *ideal* one shows up some redundant terms which should be removed using the auxiliary matrix

$$\bar{A}_1 = \begin{bmatrix} 0 & -x^2 \\ -y^2 & 0 \end{bmatrix}$$

in connection with parameters, g_7 and g_8. In addition, a series of missing terms should be hypothetically supplemented by the matrix

$$\bar{A}_1 = \begin{bmatrix} x^2 & 0 & y^2 & 0 & x^2y & 0 & xy^2 & 0 & x^3 & 0 & y^3 & 0 \\ 0 & x^2 & 0 & y^2 & 0 & x^2y & 0 & xy^2 & 0 & x^3 & 0 & y^3 \end{bmatrix}$$

applied to relevant parameters, $g_9, g_{10}, \cdots$ to g_{20}. Further extension of the analysis is different for the solution based on the use of either four corner fiducials (*C*-variant) or four middle side fiducials (*S*-variant). The results are condensed in Table 2.

Equations 20, 21 (Table 2) could easily be interpreted in individual terms to show where maximum residuals are left with respect to each parameter, but this would not be very instructive. Instead, the composite effect of all errors can be assessed by the analysis of error propagation in Equations 20, 21. Under the statistical assumption that the long-term occurrence of the parameters has a random character, the second-order parameters $(g_7 \cdots g_{12})$ and the third-order parameters $(g_{13} \cdots g_{20})$ are substituted by root-mean-square estimates m_2, m_3 respectively. Applying the quadratic law of error propagation one gets for Equation 20,

$$\begin{aligned} m_x^2 &= (2(x^2-1)^2+(y^2-1)^2)m_2^2 \\ &\quad +(x^2+y^2)((x^2-1)^2+(y^2-1)^2)m_3^2 \\ m_y^2 &= ((x^2-1)^2+2(y^2-1)^2)m_2^2 \\ &\quad +(x^2+y^2)((x^2-1)^2+(y^2-1)^2)m_3^2. \end{aligned} \qquad (20a)$$

Similarly, the other Equation 21 could be adapted into

$$\begin{aligned} m_x^2 = m_y^2 &= (3x^2y^2 + (1 - x^2 - y^2)^2)m_2^2 \\ &\quad + (x^4y^2 + x^2y^4 + x^2(1 - x^2)^2 \\ &\quad + y^2(1 - y^2)^2)m_3^2. \end{aligned} \qquad (21a)$$

TABLE 2. THEORETICAL DISTRIBUTION OF RESIDUAL SYSTEMATIC ERRORS IN THE PROJECTIVE TRANSFORMATION

C-variant: where the fiducial marks lie in the **Corners** of the photo.

Distribution matrix:

$$P = \begin{bmatrix} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$g_7 \quad g_8 \quad g_9 \quad g_{10} \quad g_{11} \quad g_{12} \quad g_{13} \quad g_{14} \quad g_{15} \quad g_{16} \quad g_{17} \quad g_{18} \quad g_{19} \quad g_{20}$$

Residual errors:

$$\left.\begin{aligned} &\boldsymbol{dx} = \boldsymbol{Dg}_x \qquad \boldsymbol{dy} = \boldsymbol{Dg}_y \\ &\boldsymbol{D} = [(1-x^2) \quad (1-y^2) \quad y(1-x^2) \quad x(1-y^2) \quad x(1-x^2) \quad y(1-y^2)] \\ &\boldsymbol{g}_x' = ((g_9-g_8) \quad g_{11} \quad g_{13} \quad g_{15} \quad g_{17} \quad g_{19}) \\ &\boldsymbol{g}_y' = \quad (g_{10} \quad (g_{12}-g_7) \quad g_{14} \quad g_{16} \quad g_{18} \quad g_{20}) \end{aligned}\right\} \qquad (20)$$

S-variant: where the fiducial marks lie in the **Sides** of the photo.

Distribution matrix:

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Residual errors:

$$\left.\begin{aligned} &\boldsymbol{dx} = \boldsymbol{Dg}_x \qquad \boldsymbol{dy} = \boldsymbol{Dg}_y \\ &\boldsymbol{D} = \quad [xy \quad (1-x^2-y^2) \quad -x^2y \quad -xy^2 \quad x(1-x^2) \quad y(1-y^2)] \\ &\boldsymbol{g}_x' = ((g_{12}-g_{10}-g_7) \quad g_{11} \quad g_{13} \quad g_{15} \quad g_{17} \quad g_{19}) \\ &\boldsymbol{g}_y' = ((g_9-g_{11}-g_8) \quad g_{10} \quad g_{14} \quad g_{16} \quad g_{18} \quad g_{20}) \end{aligned}\right\} \qquad (21)$$

The average variance for the whole image area is derived by solving a double integral for Functions 20a and 21a in the range of one image quadrant $(0<x, y<1)$. It follows that

for C-variant,

$$S_x^2 = S_y^2 = 1.60m_2^2 + 0.51m_3^2, \quad (22)$$

for S-variant,

$$S_x^2 = S_y^2 = 0.62m_2^2 + 0.29m_3^2. \quad (23)$$

A significant conclusion can be drawn from comparison of Equations 22 and 23 indicating that middle side fiducial marks ensure a better statistical distribution of second- and third-order systematic errors, left beyond the potential of projective transformation, than corner fiducials. This holds true if one considers strictly systematic errors only and their typical occurrence, disregarding any random local discrepancies. It is interesting that the same observation was made and quoted in Reference 1 as a result of extensive experiments based on processing real data.

The potential of the bilinear transformation can be tested only for the corner point configuration because the other version defaults due to singularity of the solution matrix. Inasmuch as the bilinear transformation does not contain any x, y-correlations, the x and y coordinates are treated separately. The *incomplete* matrix

$$\bar{A}_0 = [1 \quad x \quad y \quad xy]$$

is to be supplemented by adding

$$\bar{A}_1 = [x^2 \quad y^2 \quad x^2y \quad xy^2 \quad x^3 \quad y^3].$$

For the corner point solution the distribution matrix $\boldsymbol{P}_x = \boldsymbol{P}_y$ is derived as

$$\boldsymbol{P}_x = \boldsymbol{P}_y = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and final residual errors are

$$dx = \boldsymbol{D}_x\boldsymbol{g}_x, \qquad dy = \boldsymbol{D}_y\boldsymbol{g}_y$$

where

$$\begin{aligned} \boldsymbol{D}_x = \boldsymbol{D}_y &= [(1 - x^2)\,(1 - y^2)\,y(1 - x^2) \\ &\quad x(1 - y^2)\,x(1 - x^2)\,y(1 - y^2)], \\ \boldsymbol{g}_x' &= (g_9 \quad g_{11} \quad g_{13} \quad g_{15} \quad g_{17} \quad g_{19}), \\ \boldsymbol{g}_y' &= (g_{10} \quad g_{12} \quad g_{14} \quad g_{16} \quad g_{18} \quad g_{20}). \end{aligned} \quad (24)$$

The root-mean-square value of coordinate residuals becomes

$$\begin{aligned} m_x^2 = m_y^2 &= ((x^2 - 1)^2 + (y^2 - 1)^2)m_2^2 \\ &= (x^2 + y^2)((x^2 - 1)^2 + (y^2 - 1)^2)m_3^2 \end{aligned}$$

and finally, the average variance is

$$S_x^2 = S_y^2 = 1.07m_2^2 + 0.51m_3^2. \quad (25)$$

From a comparison of Equations 22 and 25 it follows that the bilinear transformation could generally perform better than the projective transformation, anticipating the same configuration of fiducial marks. At first sight this seems surprising and contradictory as the bilinear transformation looks inferior, not containing quadratic terms. The observation is, however, justified by realizing that the available x^2 or y^2 coefficients in Equation 5 are correlated with the xy-terms and, therefore, exercise influence in definite direction without having necessary freedom. On the other hand, the general third-degree transformation, which is considered *ideal* for the purpose of analysis, does not recognize any correlation whatsoever.

The final example shows the redistribution of systematic errors for a more complex eight-point transformation usually expressed separately for x and y by the matrix

$$\bar{A}_0 = [1 \quad x \quad y \quad xy \quad x^2 \quad y^2 \quad x^2y \quad xy^2].$$

The only missing cubic terms can be supplemented by

$$\bar{A}_1 = [x^3 \quad y^3].$$

The necessary construction of the distribution matrix $\boldsymbol{P}$ is based on the knowledge of four corner and four middle side points, with the result

$$\boldsymbol{P}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With the use of $\boldsymbol{P}$, the final residual errors are derived as

$$\begin{aligned} dx &= x(1 - x^2)g_{17} + y(1 - y^2)g_{19}, \\ dy &= x(1 - x^2)g_{18} + y(1 - y^2)g_{20}. \end{aligned} \quad (26)$$

These formulas give rise to the root-mean-square estimate

$$m_x^2 = m_y^2 = (x^2(1 - x^2)^2 + y^2(1 - y^2)^2)m_3^2$$

and are generally represented by the average variance

$$S_x^2 = S_y^2 = 0.15m_3^2. \quad (27)$$

Quite logically, the eight-point transformation seems to be superior to the other analyzed formulations.

Distribution of Random Errors

So far, the analysis has been concerned entirely with the effect of the compensation and redistribution of systematic errors inherent in the image. As a matter of fact, in performing the image transformation one is unwillingly and inevitably treating also *random changes*. These changes include not only the random errors of measurements but also irregular local distortions caused by imaging system errors and deficiencies. It is to be realized that systematic errors usually prevail in larger areas subjected to a transformation, whereas the treatment of the image deformation in smaller areas is more sensitive to the effect of random changes. It can be stated that the internal distribution of random errors, which happen to affect decisive observations at control points (fiducial marks), is completely independent of the size of the control figure. However, it is considerably dependent on the number and configuration of these points.

Using any transformation, the parameters of which are determined from measurements at certain discrete points, one introduces some unavoidable changes back into the image field. These resulting changes can be characterized by means of the variance-covariance matrix $\boldsymbol{Q}_t$,

$$\boldsymbol{Q}_t = \begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix}, \tag{28}$$

typical for the effect of the relevant transformation

$$x = \bar{\boldsymbol{A}}\boldsymbol{g}.$$

It can be formally written

$$\boldsymbol{Q}_t = \bar{\boldsymbol{A}}\boldsymbol{Q}_g\bar{\boldsymbol{A}}'.$$

Because the variance-covariance matrix of the vector of transformation parameters is given by

$$\boldsymbol{Q}_g = (\boldsymbol{A}'\boldsymbol{A})^{-1}, \tag{29}$$

the final expression for $\boldsymbol{Q}_t$ is

$$\boldsymbol{Q}_t = \bar{\boldsymbol{A}}(\boldsymbol{A}'\boldsymbol{A})^{-1}\bar{\boldsymbol{A}}'. \tag{30}$$

Some of the above used transformation will now be analyzed in order to find the matrix $\boldsymbol{Q}_t$ and assess the distribution of random errors.

BILINEAR TRANSFORMATION

The matrix Equation 4 is reduced to the form

$$\bar{\boldsymbol{A}} = [1 \quad x \quad y \quad xy]$$

and used for the transformation based on the fit at four corner points. The transformation yields the parameters $\boldsymbol{g}$ and relevant variance-covariance matrix

$$\boldsymbol{Q}_g = \tfrac{1}{4}\boldsymbol{I}$$

where $\boldsymbol{I}$ represents the unit matrix. In accordance with Equation 30, one derives

$$\boldsymbol{Q}_t = \frac{1}{4}\begin{bmatrix} (1+x^2)(1+y^2) & 0 \\ 0 & (1+x^2)(1+y^2) \end{bmatrix}. \tag{31}$$

As the transformation is performed independently for the x and y coordinates, it holds true that

$$Q_{xx} = Q_{yy}, \qquad Q_{xy} = Q_{yx} = 0.$$

The distribution of weight coefficients Q_{xx} in the range of the transformed image is illustrated in the following scheme

1.00	0.62	0.50	0.62	1.00
0.62	0.39	0.31	0.39	0.62
0.50	0.31	0.25	0.31	0.50
0.62	0.39	0.31	0.39	0.62
1.00	0.62	0.50	0.62	1.00

The average variance representing the uncertainty of the transformation over the whole image is derived by double integration of Q_{xx} in the area of the picture and yields the value

$$S_x^2 = S_y^2 = 0.44 S_0^2 \tag{32}$$

where S_0^2 is the variance factor.

PROJECTIVE TRANSFORMATION

The eight-by-eight matrix $\boldsymbol{A}$ necessary for the solution is compiled according to Equation 5, either with the use of four corner points (C-variant) or of four middle side points (S-variant). The relevant variance-covariance matrices $\boldsymbol{Q}_g$ and $\boldsymbol{Q}_t$, as well as the distribution of weight coefficients and the resulting estimate of coordinate variances, are presented by Table 3. The difference between C- and S-variants is very clear from the comparison of weight coefficients. The expected errors in corners for the S-variant are 1.6 times larger than the standard error expressing the magnitude of the inherent random disturbance, whereas all the errors for the C-variant are smaller than the standard error.

From the standpoint of applying the projective transformation to the image where the *irregular random* and local changes considerably contribute to the overall distortion, the projective transformation based on the use of *corner points* is significantly superior. This is quite an opposite conclusion from the previous consideration about the distribution of *systematic errors* for this type of transformation.

CONCLUSIONS

A suitable image transformation represents an important means to improve image quality and to increase the accuracy of analytical photogrammetric processing. As no mathematical transformation can entirely cope with the physical nature of the image changes, the inherent errors are not completely eliminated. One usually succeeds in supressing the

TABLE 3. DISTRIBUTION OF RANDOM ERRORS IN THE PROJECTIVE TRANSFORMATION.

C-variant: where the fiducial marks lie in the **Corners** of the photo.	*S*-variant: where the fiducial marks lie in the **Sides** of the photo.
Variance-covariance matrix Q_g for transformation parameters:	
$Q_g=\frac{1}{4}\begin{bmatrix} 2&0&0&0&0&0&0&-1\\ 0&2&0&0&0&0&-1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&-1&0&0&0&0&1&0\\ -1&0&0&0&0&0&0&1 \end{bmatrix}$	$Q_g=\frac{1}{2}\begin{bmatrix} 1&0&0&0&0&0&0&-1\\ 0&1&0&0&0&0&-1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&-1&0&0&0&0&2&0\\ -1&0&0&0&0&0&0&2 \end{bmatrix}$
Variance-covariance matrix Q_t for transformation changes:	
$\frac{1}{4}\begin{bmatrix} x^4+x^2y^2-x^2+y^2+2 & x^3y+xy^3-2xy \\ x^3y+xy^3-2xy & y^4+x^2y^2+x^2-y^2+2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 2x^4+2x^2y^2-x^2+y^2+1 & 2x^3y+2xy^3-2xy \\ 2x^3y+2xy^3-2xy & 2y^4+2x^2y^2+x^2-y^2+1 \end{bmatrix}$

Distribution of weight coefficients:

	C	C	C	C	C	S	S	S	S	S
Q_{xx}	1.00	0.77	0.75	0.77	1.00	2.50	1.19	1.00	1.19	2.50
Q_{xy}	0	0.09	0	−0.09	0	−1.00	−0.12	0	0.12	1.00
Q_{yy}	1.00	0.63	0.50	0.63	1.00	2.50	1.38	1.00	1.38	2.50
	0.63	0.53	0.56	0.53	0.63	1.38	0.62	0.62	0.62	1.38
	0.09	0.09	0	−0.09	−0.09	−0.12	0.12	0	−0.12	0.12
	0.77	0.53	0.45	0.53	0.77	1.19	0.62	0.44	0.62	1.19
	0.50	0.45	0.50	0.45	0.50	1.00	0.44	0.50	0.44	1.00
	0	0	0	0	0	0	0	0	0	0
	0.75	0.56	0.50	0.56	0.75	1.00	0.62	0.50	0.62	1.00
	0.63	0.53	0.56	0.53	0.63	1.38	0.62	0.62	0.62	1.38
	−0.09	−0.09	0	0.09	0.09	0.12	−0.12	0	0.12	−0.12
	0.77	0.53	0.45	0.53	0.77	1.19	0.62	0.44	0.62	1.19
	1.00	0.77	0.75	0.77	1.00	1.50	1.19	1.00	1.19	2.50
	0	−0.09	0	0.09	0	1.00	0.12	0	−0.12	−1.00
	1.00	0.63	0.50	0.63	1.00	2.50	1.38	1.00	1.38	2.50

Average coordinate variances:

C-variant	S-variant
$S_x^2=S_y^2=0.58\ S_0^2$	$S_x^2=S_y^2=0.81\ S_0^2$

effect of deformations, partially compensating and redistributing the errors. A theoretical study of this distributing mechanism can help to understand the existing internal correlations. Higher-order systematic changes are usually mixed with random local errors. The distribution scheme is entirely different for these two categories of distortions, and their practical separation is difficult. The effect of using various transformations or the effect of applying different setups for control fiducials can be analyzed separately, but the final assessment should be done only when knowing the magnitude level for the errors in these two basic categories.

REFERENCE

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