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# Film Deformation Correction with Least Squares Interpolation

**The method is effective with 25 to 50 réseau points, but 4 or 8 fiducial marks are not sufficient.**

ABSTRACT: *A new method for the correction of film deformation applying least-squares interpolation is considered to be a most powerful scheme. The use of 25 to 50 réseau points correct for a considerable portion of the systematic deformation, but 4 or 8 fiducial marks are not sufficient. The method is independent of the systematic character of the deformation, is also independent of the number and distribution of the réseau points, and is well suited for automation.*

## INTRODUCTION

THE ACCURACY OF photogrammetric restitution is determined mainly by our knowledge of the physical reality of taking aerial photographs, in particular by our knowledge of the geometry of the photographs. Especially, by refining the mathematical methods as applied to numerical restitution, a high degree of approximation can be obtained. A new method is presented in this article for the mathematical description and correction of film deformation. The method is based on the assumption that the errors at some image points—réseau points—are known. It can generally be stated that the differences between the actual locations of the image points and their error-free positions are composed essentially of the following three components:

- Regular or systematic film deformation within certain areas.
- Irregular or random film deformation where the magnitudes and signs change over very short distances.
- Errors committed in measuring the réseau points.

It is, in the first instance, the systematic film deformation, where we must determine and correct the errors mathematically in order to improve the accuracy of numerical restitution of aerial photographs. Film deformation altering in magnitude and direction within a space of a few millimeters is classified as irregular, and is not considered by the mathematical correction method.

The errors at the réseau points form a two-dimensional vector field which is composed partly of systematic and partly of irregular components (Figure 1).

The next section of this article shows how the characteristic features of such a vector field can be described mathematically with very few parameters. The analysis of the vector field leads to a reasonable and practicable definition of systematic and irregular film deformations. Using such a definition, a new interpolation method, which is called *least-squares interpolation*, can be explained. It is a most appropriate method for describing the systematic parts of the total film deformation. The effectiveness of this interpolation method will finally be demonstrated by a few examples.

## MATHEMATICAL DESCRIPTION OF TWO-DIMENSIONAL VECTOR FIELDS

In the description of a vector field the following items are of particular interest:

- Absolute magnitude of the vectors,
- Separation of systematic and irregular components, and
- The range of systematic effects and their variation as a function of distance.

It is assumed that no correlation occurs between the $x$ and $y$ components of the two-

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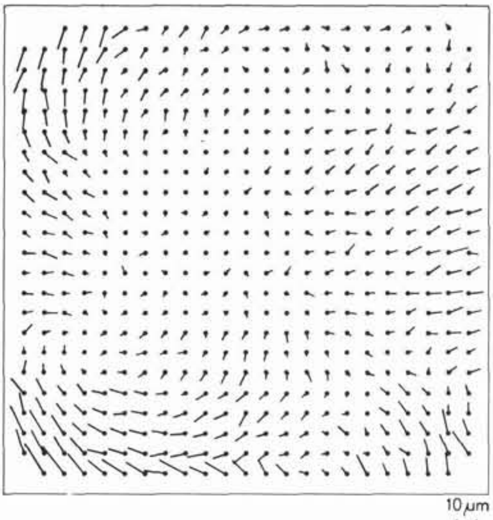


FIG. 1. Measured values of réseau points of Plate 358.

dimensional vectors. In that instance the original two-dimensional problems can be treated as two independent one-dimensional situations. We then search for concepts or terms describing mathematically the three items, separately for $x$- and $y$-directions. A comparison of such terms, or rather of the numerical values assigned to them, will give information about a possibly differing behaviour of the image errors in the $x$- and $y$-directions.

The absolute magnitude of the réseau image errors $l_i$ in the $x$- or the $y$-direction can be described by the variance

$$V = M[l_i l_i]$$

where $M$ means mean-square value. The variance contains the irregular as well as the systematic components of the vector field.

We also introduce the covariance

$$C(s_{ik}) = M[l_i l_k],$$

which is the mean-square value of the cross product of the errors $l_i l_k$ of all réseau points $P_i$ and $P_k$. The covariance contains the systematic part of the vector field, the irregular part being filtered out. With increasing distance between the points $P_i$ and $P_k$, the common systematic part will decrease, and therefore also the covariance $C$ tends to decrease. Thus the empirically determined covariances are a perfect indicator on how the common systematic effects between the points decrease with increasing distance between them. We call the covariances presented as a function of the distances between the points, the covariance function.

Figure 2 shows first the variance $V$ and the empirical covariances $C(s_{ik})$ for a number of distances $s$ of the $y$-components of the vector field of Figure 1. The empirical covariances can be approximated by a continuous function, the covariance function, for which the well-known Gaussian curve is the most appropriate:

$$C(s) = CO \cdot \mathrm{Exp}\left\{-k^2 \cdot s^2\right\}.$$

The maximum value $CO$ is the covariance between points which are infinitely close together ($s=0$). The constant $k$ is responsible for the rate of decrease of the covariance function with increasing distance. In other words, it describes whether the common systematic effects in the vector field decrease slowly ($k$ small), or rapidly ($k$ large) with increasing distance.

The conclusion is that three constants are sufficient to describe the behaviour of the components of a vector field. These are: the variance $V$, the maximum value $CO$ of the covariance function, and the constant $k$. For the $y$-components of the vector field of Figure 1 we found the values

$$V = 18.4\,\mu\mathrm{m}^2; \qquad CO = 12.2\,\mu\mathrm{m}^2;$$
$$k = 0.0173\,\mathrm{mm}^{-2}.$$

From these parameters the numerical values for the three terms by which we describe the

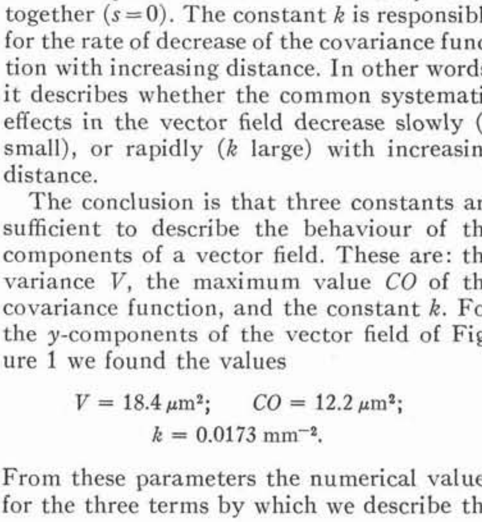


FIG. 2. The variance $V$, the empirical covariances $C(s_{ik})$ and the covariance function of the $y$-components of Plate 358.

three basic features of a vector field can be derived directly:

The mean-square value of the absolute magnitude of the $y$-components was

$$\sigma_{ya} = \sqrt{V} = \pm 4.3\ \mu\text{m}.$$

The mean-square value of the systematic component was

$$\sigma_{ys} = \sqrt{CO} = \pm 3.5\ \mu\text{m}.$$

The mean-square value of the irregular component was

$$\sigma_{yu} = \sqrt{V - CO} = \pm 2.5\ \mu\text{m}.$$

The constant $k = 0.0173$ is a representative value for the behavior of the systematic effects depending on the distances. It would be possible to replace the constant $k$ by an expression which would permit geometrical interpretation, such as the abscissa of the point of inflection of the Gaussian curve.

With the concepts which we have just defined relating to the description of a vector field, we have now a most practicable definition of the irregular and the systematic film deformations. The mean-square value $\sigma_u$ represents the irregular film deformation. The mean-square value $\sigma_s$ represents the magnitude of the systematic film deformation. The constant $k$ represents the decrease of the systematic film deformation with increasing distance.

The new definition is more powerful and more objective than the operational definitions used so far. Up to now, irregular and systematic parts of film deformation used to be analyzed by subdividing a photograph into smaller units within each of which were computed the arithmetic mean representing the systematic film deformation and the standard deviation of the mean representing the irregular film deformation. Contrary to this operation definition the new definition does not contain subjective elements. In addition, the constant $k$ reveals the functional dependency of the systematic errors of points on the distance between them.

At the Photogrammetric Institute of Stuttgart University, three plates Nos. 302, 358 and 412 of the OEEPE test block "Oberschwaben" have been investigated. Table 1 shows the results. It displays for the three plates the random component $\sigma_u$ and the systematic component $\sigma_s$ of the image deformation and the constant $k$ of the covariance function, separately for $x$- and $y$-directions. It is sufficient to mention the consistency of the results and to note that the film deformation is more pronounced in the $y$-direction than in the $x$-direction, an effect which is well known.

TABLE 1. RANDOM COMPONENT OF FILM DEFORMATION $\delta_u$, SYSTEMATIC COMPONENT OF FILM DEFORMATION $\delta_s$, CONSTANT $k$ OF THE COVARIANCE FUNCTION

| Plate | *x-component* | | | *y-component* | | |
|---|---|---|---|---|---|---|
| | $\delta_u$ $\mu m$ | $\delta_s$ $\mu m$ | $k$ $mm^{-2}$ | $\delta_u$ $\mu m$ | $\delta_s$ $\mu m$ | $k$ $mm^{-2}$ |
| 302 | ±1.7 | ±2.8 | 0.017 | ±2.8 | ±3.9 | 0.017 |
| 358 | ±1.8 | ±3.3 | 0.014 | ±2.5 | ±3.5 | 0.017 |
| 412 | ±2.1 | ±3.9 | 0.015 | ±3.0 | ±4.9 | 0.021 |

## LEAST-SQUARES INTERPOLATION FOR THE CORRECTION OF SYSTEMATIC IMAGE DEFORMATION

As mentioned before, the assumption was made that at least several réseau points have been measured. For them, the difference between the actual location of the réseau point and the error-free position is known. Those points are called reference points. With the help of them, corrections are to be derived for other observed points of the photograph. In deriving corrections for image deformation based on the measurements of a few reference points, two problems have to be solved simultaneously:

- The irregular film deformation and the irregular measuring error at the reference points have both to be filtered out.
- The remaining systematic film deformation at the reference points has to be transferred to all the other image points by interpolation.

The interpolation method should work with as few reference points as possible, and should not need any assumptions about the geometry, or rather the mathematical structure, of the systematic film deformation. We assume only that the three constants derived in the previous section are known.

A similar problem of interpolation is known in the field of geodesy: based on measurements at a number of gravity stations, gravity values have to be interpolated for other points on the surface of the earth. For this problem Professor Moritz[4] of Berlin has proposed an optimum interpolation method which was originally derived by N. Wiener and A. Kolmogorov, two well-known pioneers in the field of cybernetics. About a year ago the author[1] proposed the introduction of this method into photogrammetry. Subsequently, the author has programmed the method in Algol and Fortran and has successfully applied it for several problems.[2] It was shown that the least-squares interpolation is most appropriate in eliminating the systematic components of residual errors at the control points of block adjustment in aerial triangulation. Also the systematic effects of the residual errors between photogrammetric models after block adjustment can be successfully eliminated.

The method is based on a linear relationship between the value $u$ to be interpolated and the $n$ known reference values $l_i$:

$$u = \alpha_1 l_1 + \alpha_2 l_2 + \cdots + \alpha_n l_n.$$

The coefficients $\alpha_i$ are determined in such a way that the standard error, or rather the variance, of the value $u$ is minimized (hence the name least-squares interpolation). Working out the derivation, one obtains the final formula

$$u = (C(\overline{PP_1})\ C(\overline{PP_2}), \cdots, C(\overline{PP_n}))$$

$$\cdot \begin{vmatrix} C(0) & C(\overline{P_1P_2}) & \cdots & C(\overline{P_1P_n}) \\ C(\overline{P_1P_2}) & C(0) & \cdots & C(\overline{P_2P_n}) \\ \vdots & \vdots & & \vdots \\ C(\overline{P_1P_n}) & C(\overline{P_2P_n}) & \cdots & C(0) \end{vmatrix}^{-1} \begin{vmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{vmatrix} = cC^{-1}l.$$

The row vector $\boldsymbol{c}$ contains the covariances between the point $P$ to be interpolated and the $n$ reference points $P_i$. After calculating the distances $s$ between the points, those covariances can be determined from the covariance function. In the $\boldsymbol{C}$-matrix, the covariances between the reference points $P_i$ are obtained in the same way from the covariance function using the distances $s$. On the main diagonal, however, the covariance $C(0)$ is replaced by the variance $V$. It is exactly this replacement that gives the desired filtering of the irregular errors at the reference points. The following examples will explain the effect.

## Some Examples

Professor Ackermann has suggested the testing of the correction of image deformation by the least-squares interpolation method with the help of some réseau photographs of the OEEPE test block Oberschwaben. The investigation concerned the first, the middle and the last photograph of strip 10. Strip 10 has 26 photographs in all, taken with a Zeiss wide-angle réseau camera. The 524 réseau points of each photograph were measured with a Wild Stereocomparator StK1 at the ITC in Delft. The ITC also processed the original data by performing a similarity transformation over 81 identical points. After this transformation the differences between the measured points of the 1-cm réseau grid and the calibrated values were determined. Those differences of plate 358 are shown in Figure 1.

A report from ITC by Kure and Rijsdijk[3] states that there is no significant difference in the results whether the similarity transformation was based on 524, 81 or only 25 identical points. Therefore, in varying the number of reference points in the investigation, one can always refer to those points as reference points for the least-squares interpolation.

In the first test only 147 réseau points (or a 2-cm grid) were used. Based on those 147 reference points the image deformation values were interpolated for all the points of the 1-cm grid. The results of Plate 358 are given in Figure 3.

It can be seen from the structure of the vector field that the systematic effects of the original measurements of all 524 points are approximated very well. The mean-squares values of the systematic error components are shown in Table 2, columns 3 and 4. From the original measurements of all 524 points the corresponding values were 3.3 $\mu$m and 3.5 $\mu$m (Table 1, columns 3 and 6). This agreement confirms the practicability of the new definition of random and systematic image deformation. It also proves the effectiveness of the least-squares interpolation method. Encouraged by the good results, we continued the investigation reducing gradually the number of reference points.

Before continuing, it is worthwhile to show also the results of the filtering process on the 147 reference points. The random components contain both the random part of the film deformation and the random measuring

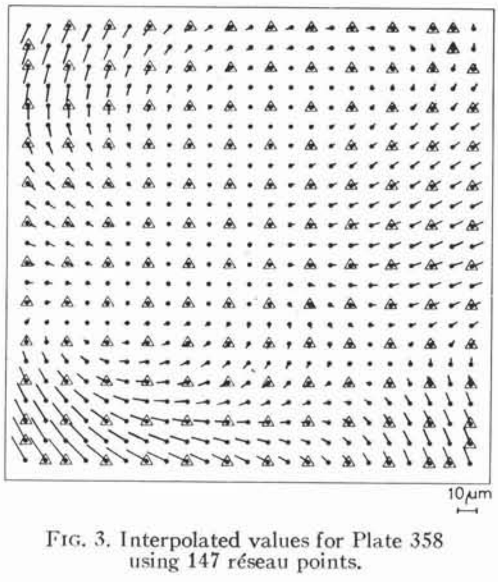


Fig. 3. Interpolated values for Plate 358 using 147 réseau points.

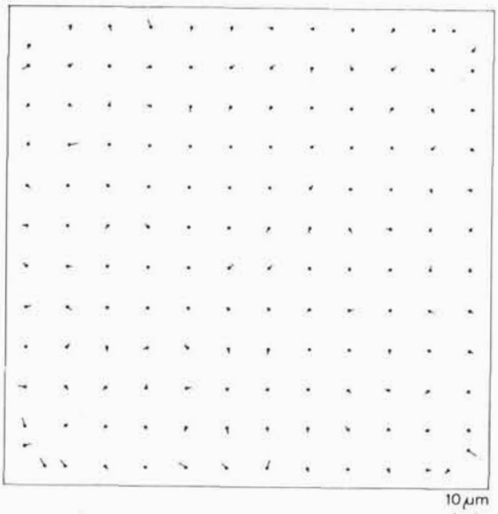


FIG. 4. The random components for Plate 358.

errors at the reference points. The expected irregular distribution of the vectors of plate 358 is shown in Figure 4.

If the random measuring errors dominate over the random film deformation, those vectors can be interpreted as showing, in the first instance, the measuring errors. It is important to keep this in mind as some other methods for correcting film deformation do not filter out the measuring errors. One of them is the method by which the measured value of the closest réseau point is directly transferred to the image point to be corrected. Here the measuring error at the réseau point is fully imposed upon the correction of the image point. For a small film deformation, such use of the réseau for correction does have a deteriorating effect on the results which may even come out worse than with no correction at all. It is supposed that some of the results of réseau corrections are strongly influenced by this effect.

We now return to the problem of how many reference points are needed for correcting the systematic component of film deformation. In Figure 5, the interpolation values using only 52 reference points also represent the typical pattern of the systematic film deformation of Plate 358.

Compared with the test using 147 reference points, which serves as a basis of comparison here, we have mean-squares differences of only 0.8 $\mu$m in $x$ and 0.9 $\mu$m in the $y$-directions. That means that using these 52 réseau points, one can still catch 76 and 73 percent, respectively, of the systematic film deformation. All these values and those from the other two plates are shown in Table 2.

Figure 6 shows the results of plate 358 using 25 réseau points only. The differences against the results from the 2-cm réseau grid amount to 1.2 $\mu$m in both $x$ and $y$. In this instance we catch 64 and 65 percent, respectively, of the systematic film deformation (see Table 2).

The results of another test, based also on approximately the same number essentially of reference points which are distributed along the edge of the photograph, are shown in Figure 7. The results here are somewhat inferior to the previous one with a more

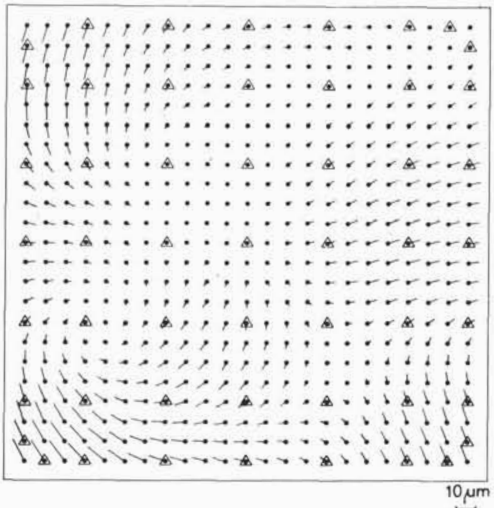


FIG. 5. Interpolated values for Plate 358 using 52 réseau points.

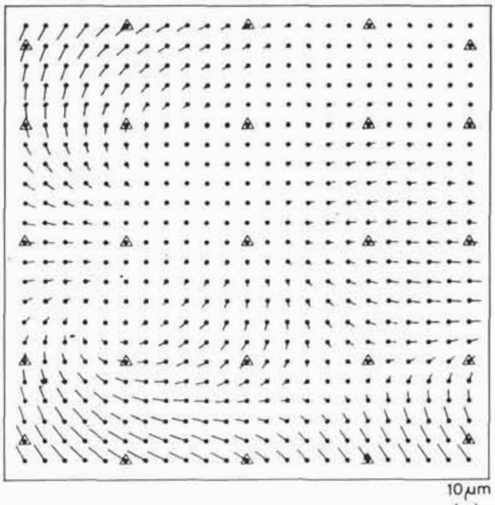


FIG. 6. Interpolated values for Plate 358 using 25 réseau points.

TABLE 2. RÉSEAU CORRECTION WITH DIFFERENT NUMBER OF RÉSEAU POINTS

| *Plate* | *No. of réseau points* | *Systematic component of film deformation* | | *RMS (root-mean-square) value of the differences against full réseau correction* | | *Effectiveness of correction* | |
|---|---|---|---|---|---|---|---|
| | | *x-component μm* | *y-component μm* | *x-component μm* | *y-component μm* | *x-component Percent* | *y-component Percent* |
| 302 | 147 | ±2.8 | ±3.8 | — | — | 100 | 100 |
| | 52 | | | ±0.7 | ±0.9 | 75 | 76 |
| | 25 | | | ±1.5 | ±1.3 | 46 | 66 |
| | 27 (R) | | | ±1.5 | ±1.5 | 46 | 60 |
| | 8 (R) | | | ±1.8 | ±2.1 | 26 | 45 |
| 358 | 147 | ±3.3 | ±3.4 | — | — | 100 | 100 |
| | 52 | | | ±0.8 | ±0.9 | 76 | 73 |
| | 25 | | | ±1.2 | ±1.2 | 64 | 65 |
| | 27 (R) | | | ±1.0 | ±1.6 | 70 | 53 |
| 412 | 147 | ±3.8 | ±4.8 | — | — | 100 | 100 |
| | 52 | | | ±0.7 | ±1.0 | 82 | 79 |
| | 25 | | | ±1.1 | ±1.9 | 71 | 60 |
| | 27 (R) | | | ±1.4 | ±2.6 | 63 | 46 |

regular distribution of the réseau points (see Table 2).

It is of interest to study the effects of further reduction in the number of réseau points used. Réseau points located at the edge of the photograph are equivalent to fiducial marks. Therefore this investigation concerns the problem of to what extent systematic film deformation can be corrected, if the corrections are derived from fiducial marks only.

Figure 8 shows the results of the least squares interpolation of Plate 302 based on 8 fiducial marks. The mean-square values of the differences against the case of the 2-cm grid amount to 1.8 $\mu$m in $x$ and 2.1 $\mu$m in $y$. Thus only about 35 percent of the systematic film deformation is corrected. This cannot be

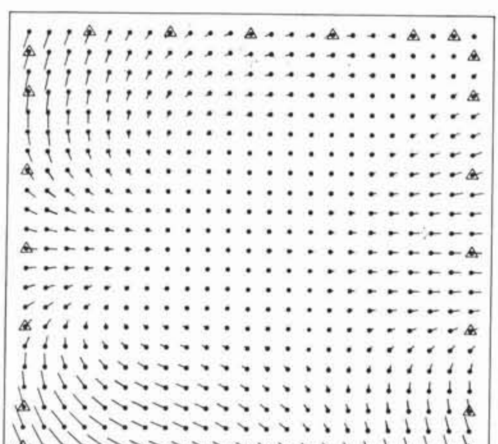


FIG. 7. Interpolated values for Plate 358 using 27 reference points distributed along the edge of the photograph.

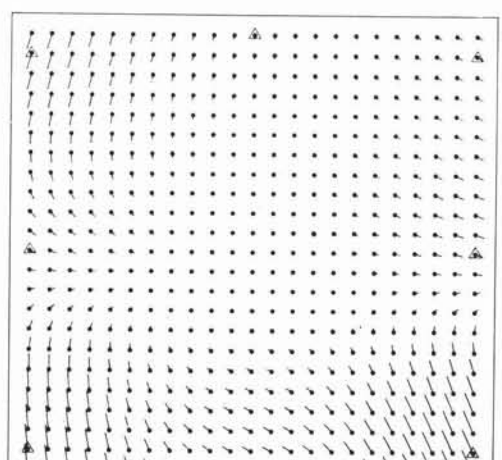


FIG. 8. Interpolated values for Plate 302 using 8 fiducial marks.

considered satisfactory. A further reduction of the reference points to only 4 fiducial marks was also treated, with even worse results. The conclusion is justified that the conventional fiducial marks in air survey cameras are not suited to correct systematic film deformation.

## Summary

Summarizing, we can state that with 52 réseau points the correction is 70 percent effective, with 25 réseau points 60 percent, with 27 fiducial marks 56 percent and with 8 fiducial marks about 35 percent Those results are derived from three photographs only. Admittedly, three photographs are too small a sample to consider the obtained values generally significant. The qualitative conclusions, however, are significant: firstly, with a rather limited number of réseau points, a considerable portion of the systematic film deformation can be corrected; secondly, 4 or 8 fiducial marks are not sufficient to derive significant corrections for systematic film deformation.

The main features of the new method are:

- ▲ An improved theoretical foundation as compared with all other interpolation procedures used so far.
- ▲ Filtering of the random components at the measured réseau points, with only the systematic components entering into the interpolation.
- ▲ Conventional interpolation methods assume knowledge of the type and structure of systematic film deformation: the corrections operate with formulas of predetermined type, such as affine transformations, projective transformations, polynomial transformations. Contrary to this, the least-squares interpolation method is independent of the type and structure of the systematic film deformation.
- ▲ The method is independent of the number and distribution of the réseau points, contrary to some conventional procedures where, for instance, the degree of polynomials is chosen depending on the number and distribution of réseau points used.
- ▲ The method is perfectly suited for full automation.

The least-squares interpolation method must be considered a most effective and powerful interpolation method. We propose, therefore, to reinvestigate the problem of réseau corrections with this method. We believe that some of the results of réseau investigations obtained in the past are unduly influenced by the unfavorable filtering property of the methods applied. It is presumed that the improvement of accuracy obtainable by réseau corrections is more effective than the results of known tests would suggest.

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## Articles for Next Month

*D. D. Egbert* et al, Effect of angles on reflectivity.
*J. L. Junkins*, et al, Smooth irrecular curves.
*D. L. Light*, Photo geodesy for the moon.
*P. N. Slater*, Multiband cameras.
Abstracts of March 1972 Convention Papers.