

Twelfth Congress of the
International Society of Photogrammetry
Ottawa, Canada, July 23–August 5, 1972
Presented Paper, Commission III

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Linear Least-Squares Interpolation

With filtering, the method has numerous applications in computational photogrammetry.

ABSTRACT: Linear interpolation and filtering by least squares is a general and flexible method which may be used in many problems of computational photogrammetry. This is a concise, but complete, presentation including an interpretive photogrammetric demonstration, evaluation and analysis. Consideration is given to the effects of theoretical assumptions in practice, of the number of reference points and of the choice of trend surface and covariance function, with simulated test results. Some of the areas considered include: compensation for film and lens distortion, external horizontal and vertical block adjustment, problems in digital terrain model (DTM), and use of Side Looking Radar (SLAR) imagery. Further research is presently continuing on DTM, SLAR and other remote sensing records.

INTRODUCTION

PROBLEMS OF interpolation, and to a lesser extent extrapolation, exist in many a field of applied science and engineering. For such problems, an effective linear method using least squares has been frequently used. Although it has been known for many years and has been considerably applied to problems in geodesy,⁷ its use in photogrammetry has been rather limited. It was not until recently^{2,6} that this method was applied to some photogrammetric problems. The English-speaking photogrammetric community, however, remains largely either unaware of, or unfamiliar with, the method. Because of the general nature of the method, its excellent potential applications to several phases of computational photogrammetry, and the expressed desire of many for an exposé, the authors have endeavoured here to meet this need. They wish to emphasize that the basic concepts and relationships involved in the method are not new and have been long known in the field of stochastic processes.¹ However, they have attempted to make the presentation in a form appealing to practicing photogrammetrists using their own interpretive demonstration mostly from photogrammetry.

Interpolation problems exist in many aspects of computational photogrammetry. For example, after strip and block triangulation, correlated residuals (or in conventional terminology, residuals with *systematic* behaviour) may exist at the control points, thus requiring interpolation at the pass points. Preprocessing photo-coordinates for *systematic* effects, particularly lens distortion and film deformation, is

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another operation where interpolation is necessary to estimate values at points other than those used to control the systematic effect in question.

Problems of contouring and profiling in the digital terrain model also apply interpolation techniques. Interpolation may also be utilized in cadastral photogrammetry in detecting and isolating gross errors in block adjustment, in improving methods of field calibration of cameras, and in several phases of remote sensing. At the present time many of the above-mentioned situations are handled by using polynomials of varying degrees. Although the use of polynomials may be adequate for some problems, they are neither general nor flexible enough to adapt for all situations. If polynomials, or other mathematical equations, are not used (for example for external block adjustment), one is often forced to use relatively more involved procedures in the block triangulation phase. With a general and flexible method of interpolation one can use less complex techniques of triangulation and treat the strip or block afterwards.

Although least squares interpolation has many advantages, we must point out, however, that for some situations other interpolation methods, such as sliding-average, piece-wise polynomials,⁶ may yield good results. The method to be presented here is statistical (non-parametric) and not functional (parametric). Therefore, aside from the regular considerations of sample size, there are no restrictions on it, and its form does not change from one problem to another. To have an appreciation of the method, we first introduce some basic assumptions and definitions which are necessary for the derivation that follows.

BASIC DEFINITIONS AND ASSUMPTIONS

The problem of linear interpolation addressed here begins by a given field of reference points (stochastic field), at each of which certain information is known. The solution of the problem is obtained by *interpolating* the information at points other than reference points. The information at each reference point is considered to be composed of two classes of *random variables*. The first class which we shall call the *s-components*, is composed of *correlated* stochastic variables (Figure 1). This is an important characteristic because interpolation would not be meaningful if the *s-components* at different reference points were uncorrelated. For a given *experiment* (or one realization of the stochastic field) the *s-component* at any one point is fixed and remains fixed as measurements are repeated at that point. However, if the experiment is itself repeated (many realizations of the stochastic field) the *s-component* would vary *randomly* from experiment to another (Figure 2).

It is this component which represents the rather misleading term called *systematic effect*. (Fortunately the letter *s* stands for both terms to assist the reader in associating it with what has conventionally been used). The second group which will be termed the *r-components*, represents the conventionally known *observational error* and will vary randomly as measurements are repeated whether at the same point, at different points, or as the entire experiment is repeated. Figures 1 and 2 show the two components and their relationship, on which more elaboration will be given subsequently. To aid in visualizing the difference between the two components defined above we use as a conceptual model the problem of film deformation (Figure 3).

For a given aerial film one may have several frames of exposure *m* for which *correction* for film deformation is to be performed. Suppose that we have a good number *n* of reseau points which will be used to control film deformation. Given *one* aerial photograph the position of the reseau points on the film are fixed. The set of residuals at these points, being one realization of a stochastic field, are correlated from one reseau point to another (and are in a sense deterministic). But they vary *from one photograph to another* and this variation is in fact stochastic. This is what has been termed the *s-component*. To illustrate the *r-component*, we take one photograph and re-

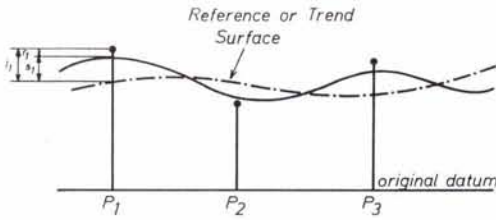


FIG. 1. The s - and r -components.

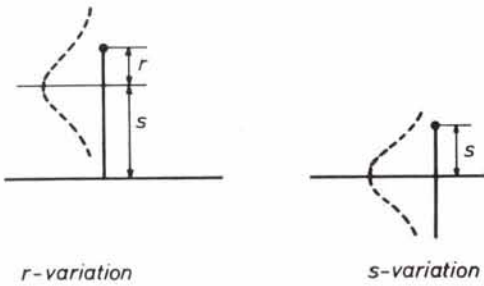


FIG. 2. Variation of the s - and r -components.

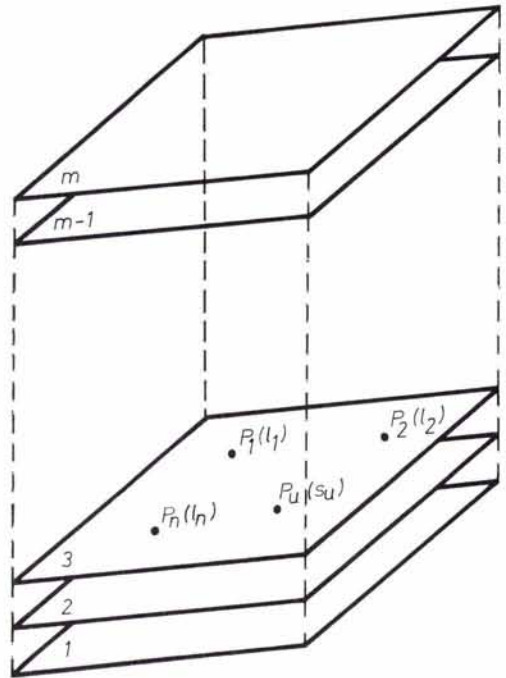


FIG. 3. Conceptual model: film deformation in m photographs. (Theoretical limit $m \rightarrow \infty$)

measure the coordinates of *one* point many times. If we take, for simplicity, the *vector* lengths of the differences between the measured positions and calibrated position we will find that they vary randomly. This variation is the r -component. It also changes from point to point on the same frame and from frame to frame.

In order to make the application of this method to photogrammetric problems possible we must recognize a few *assumptions*:

- (1) The r - and s -components are independent random variables with zero means, i.e., $E(r) = 0$ and $E(s) = 0$; ($E(\cdot)$ means expectation or mean).
- (2) The s -component in a given field of points forms regions of varying sign relative to (i.e., above and below) a *reference surface*. This is illustrated in Figure 1 where a reference surface is determined to replace the original datum with respect to which the given values were *all* of the same sign. The reference surface represents the trend in the stochastic field. It is an important step to shift to this surface, otherwise the method would not be efficient.
- (3) The regions described in (2) above need not be of specified or known form or extent. What is important is that both positive and negative regions exist in a more or less random fashion.
- (4) While the s -component may be of the same sign in any finite region the r -component is strictly random. (Any two points no matter how close are as likely to have two r -components of different sign as of the same sign).
- (5) The number of repeated experiments, m , (such as the number of photographs in the conceptual model, Figure 3) is assumed to be infinitely large to allow for theoretical derivation.

Because both s and r are two independent random variables, then their sum is also a random variable. Thus,

$$l = s + r \quad \text{with } E(l) = 0 \tag{1}$$

is the random value at each point of the given field. From this value, we first define the variance at any point p as

$$\sigma_p^2 = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m l_{pk}^2 \tag{2}$$

and the covariance between two points p and q as

$$\sigma_{pq} = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m l_{pk} l_{qk} \tag{3}$$

where m represents the number of repeated experiments. In the following derivation we shall assume that necessary variances and covariances for the random variables involved are available. This is, however, one of the most important practical aspects of the method and a detailed discussion of how these variances and covariances are obtained will follow the derivation.

LINEAR LEAST-SQUARES INTERPOLATION AND FILTERING

STATEMENT OF THE PROBLEM

There is given a stochastic field of reference points, P_1, P_2, \dots, P_n at each of which is given a vector of random values l_1, l_2, \dots, l_r ,* respectively (Figure 3). The size of each vector l_i depends on the dimensionality of the interpolation problem (i.e., one- two- or multidimensional). Each element l of any vector is composed of the two r and s components as given by Equation 1.

At a point P_u , other than the reference points, an estimate of the s -components s_u , is required such that all $l_i, i = 1, 2, \dots, n$, of the reference points contribute to that estimate. Although s_u can be written as a general function of all l_i , we shall concern ourselves with linear functions, hence linear interpolation. As we are seeking a value for s only, we are in fact filtering the r -component.

From a practical viewpoint, each l_i can be made of one, two, three or more elements depending on the dimension of the interpolation problem. Without much loss in generality, and for ease in derivation, we shall concern ourselves with one-dimensional case first then extend the results to others.

DERIVATION FOR ONE-DIMENSIONAL INTERPOLATION AND FILTERING

In this case s_u (at point P_u) is a scalar which may be expressed by a linear function of l_i as

$$s_u = a_1 l_1 + a_2 l_2 + \dots + a_n l_n \tag{4}$$

where a_i are as yet unknown coefficients. In matrix form,

$$s_u = \mathbf{a}^t \mathbf{l} \tag{5}$$

where \mathbf{a} and \mathbf{l} are two $n \times 1$ vectors. There are obviously infinite estimates s_u depending on the choice of \mathbf{a} . Of all these we choose a particular estimate for which the variance is minimum. If the actual value is denoted by s_u (which is also called hypothetical measurement⁸), the deviation of s_u would be (see Figure 4)

$$v_u = s_u - \hat{s}_u = s_u - \mathbf{a}^t \mathbf{l} = [1 - \mathbf{a}^t] \begin{bmatrix} s_u \\ \mathbf{l} \end{bmatrix}. \tag{6}$$

This deviation v_u is a random variable with a zero mean. If the covariance matrix of the vector $[s_u \ \mathbf{l}^t]^t$ is denoted symbolically by

$$\Sigma = \begin{bmatrix} \sigma_{s_u}^2 & c_{s_u \mathbf{l}^t} \\ c_{\mathbf{l} s_u} & C_{\mathbf{l} \mathbf{l}^t} \end{bmatrix} \tag{7}$$

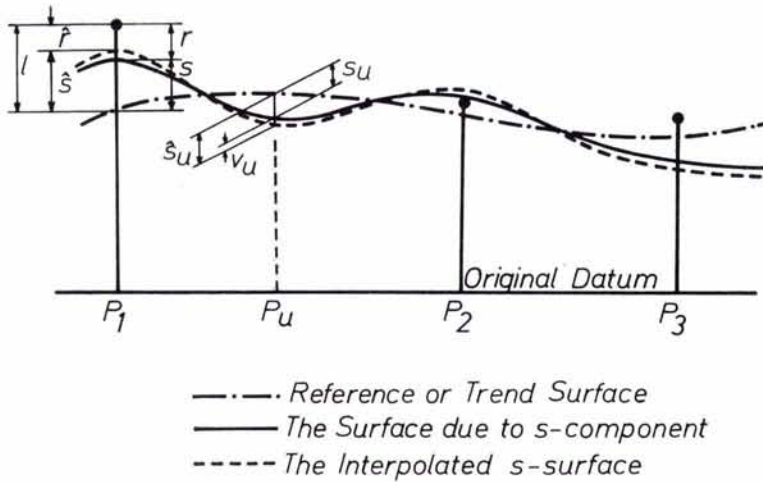


FIG. 4. The interpolated surface.

where $\sigma_{s_u}^2$ is the variance of the s -component of l_u , $c_{ls_u} = c_{s_u l}^t$ is an $n \times 1$ covariance vector relating the elements of the vector l to s_u , and C_{ll}^t is an $n \times n$ covariance matrix for the elements of l .

These matrices are assumed, for the moment, to be defined and that c and C are known. Following the derivation, we shall discuss how these matrices can be obtained in practice. Applying the law of propagation of covariance to Equation 6,

$$\sigma_{v_u}^2 = [1 - a^t] \Sigma \begin{bmatrix} 1 \\ -a \end{bmatrix}$$

which, in view of Equation 7, becomes

$$\sigma_{v_u}^2 = \sigma_{s_n}^2 - 2a^t c_{ls_u} + a^t C_{ll}^t a. \quad (8)$$

The value $\sigma_{v_u}^2$ is the variance due to getting an estimate \hat{s}_u instead of s_u , and which is to be minimized. To get its minimum we equate its derivative with respect to a to zero. Dropping the subscripts for convenience (and without ambiguity) we have

$$Ca - c = 0 \quad (9)$$

or

$$a = C^{-1}c. \quad (10)$$

Substituting Equation 10 into 5 we get the desired estimate

$$\hat{s}_u = c^t C^{-1} l. \quad (11)$$

Equation 11 is the basic relationship for one-dimensional interpolation problems. Note that l is simply the vector of values at the reference points which *need not be* in one-dimensional space. In other words, although Equation 11 applies to one-dimensional interpolation problems, the field of reference points can be one, two, or multi-dimensional as long as *only one* value l_i is attached to each point i . Thus, equations of type 11, and extended versions to follow, apply to reference points (stochastic fields) of any dimension.

EXTENSION TO TWO- AND THREE-DIMENSIONAL INTERPOLATION AND FILTERING

Equations similar to 11 can be derived for two- and three-dimensional cases which

we give here without derivation. For the two-dimensional case,

$$\begin{bmatrix} \hat{s}_{ux} \\ \hat{s}_{uy} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_{xx}^t & \mathbf{c}_{yy}^t \\ \mathbf{c}_{yx}^t & \mathbf{c}_{yy}^t \end{bmatrix} \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{bmatrix}^{-1} \begin{bmatrix} l_x \\ l_y \end{bmatrix} \tag{12}$$

and for three-dimensional case,

$$\begin{bmatrix} \hat{s}_{ux} \\ \hat{s}_{uy} \\ \hat{s}_{uz} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_{xx}^t & \mathbf{c}_{xy}^t & \mathbf{c}_{xz}^t \\ \mathbf{c}_{yx}^t & \mathbf{c}_{yy}^t & \mathbf{c}_{yz}^t \\ \mathbf{c}_{zx}^t & \mathbf{c}_{zy}^t & \mathbf{c}_{zz}^t \end{bmatrix} \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} & \mathbf{C}_{xz} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} & \mathbf{C}_{yz} \\ \mathbf{C}_{zx} & \mathbf{C}_{zy} & \mathbf{C}_{zz} \end{bmatrix}^{-1} \begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} \tag{13}$$

In the above two equations, \hat{s}_{ux} , \hat{s}_{uy} , \hat{s}_{uz} are the estimates of the x , y , and z s -components of l at point u , \mathbf{c}_{xx} , \mathbf{c}_{yy} , \mathbf{c}_{zz} are the autocovariance vectors between the components of the point of interest u and the corresponding components of all n known points, \mathbf{c}_{xy} , \mathbf{c}_{yz} , \mathbf{c}_{zx} are the cross-covariance vectors between the components of the point of interest u and the other components of all n points (e.g., \mathbf{c}_{xy} is the cross-covariance vector between the x -component of u and the y -components of all n points), \mathbf{C}_{xx} , \mathbf{C}_{yy} , \mathbf{C}_{zz} are autocovariance matrices for the components of the given n points, \mathbf{C}_{xy} , \mathbf{C}_{yz} , \mathbf{C}_{zx} are cross-covariance matrices between pairs of components of the given n points, and l_x , l_y , l_z are the vectors of quantities at the reference points, for the x -, y -, and z -components, respectively.

The use of Equations 11, 12, and 13 is rather straight-forward once the elements of covariance matrices \mathbf{c} and \mathbf{C} are known.

DETERMINATION OF NECESSARY VARIANCES AND COVARIANCES

The elements of the matrices \mathbf{c} and \mathbf{C} may be evaluated, either from theoretically known functions, or by estimation from the sample, or by empirical means. In any event, a so-called *covariance function* is used. Let us first discuss the construction of a covariance function from the given data.

The Covariance Function. For the one-dimensional case, using the given l_i at the reference points a sample, variance may be computed by (assuming that the ergodic* property applies)

$$V = \frac{1}{n} \sum_{i=1}^n l_i^2 \tag{14}$$

Two points may be noted here; first, all l_i are independent, and second, the larger the value of n the closer is the sample variance V to the population variance σ^2 (defined by Equation 2) with the limit being reached when $n \rightarrow \infty$. Using the two parts of l_i from Equation 1, Equation 14 becomes

$$V = \frac{1}{n} \sum_{i=1}^n s_i^2 + \frac{1}{n} \sum_{i=1}^n r_i^2 + \frac{2}{n} \sum_{i=1}^n s_i r_i \tag{15}$$

Because of assuming that s and r are independent, and n large, the last term may be dropped and Equation 15 may be written, with obvious correspondence in terms, as

$$V = V_s + V_r \tag{16}$$

In a manner similar to the definition of covariance in Equation 3, we introduce here a sample covariance between two reference points P_i and P_j . As we have only one set of n observations ($l_i, i=1, \dots, n$) one at each of n points, covariances as

* Ergodic property allows the use of values at different points in a field in place of many values at one point which requires many realizations.

defined in Equation 3 cannot be computed. Therefore, for any two reference points P_i and P_j which are separated by a distance d , we take the mean of all possible product pairs $l_i l_j$ with distances in the interval $d \pm \Delta d$, as the covariance between l_i and l_j . The increment Δd is rather arbitrary as it simply fixes the interval for which d is a representative distance. Thus the covariance as a function of distance may be defined as (again assuming that ergodic property applies)

$$C(d) = C(\overline{P_i P_j}) = \frac{1}{n_{ij}} \sum_{i < j} l_i l_j. \quad (17)$$

It should be noted that the value of the covariance would approach zero as d increases indefinitely, and will do so *only* after introducing the reference or trend surface. We should also say that this is true under the assumption that the covariance function depends only on distance.

Using Equation 1, we expand Equation 17:

$$C(d) = \frac{1}{n_{ij}} \left[\sum_{i < j} s_i s_j + \sum_{i < j} s_i r_j + \sum_{i < j} r_i s_j + \sum_{i < j} r_i r_j \right]. \quad (18)$$

Under the assumptions given earlier and large n_{ij} , one may neglect the last three terms of Equation 18, as they approach zero when n approaches ∞ . Thus, Equation 18 reduces to

$$C(d) = \frac{1}{n_{ij}} \sum_{i < j} s_i s_j. \quad (19)$$

Using Equation 19, a number of covariances corresponding to different distances may be computed from the data associated with the reference points. The results can be represented schematically by a graph as shown in Figure 5.

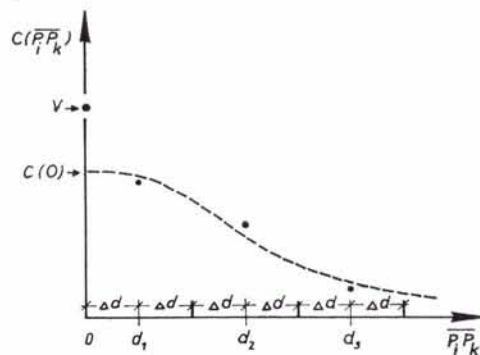


FIG. 5. The covariance function.

An appropriate equation, such as the Gaussian function shown in the figure, may be fitted to the data. If a Gaussian curve is used, its equation will be:

$$C(d) = C(0) \exp(-k^2 d^2) \quad (20)$$

with two coefficients, $C(0)$ and k , to be evaluated. Such an Equation 20 is called the *Covariance function*. Of importance is the term $C(0)$ because it represents the covariance of pairs of points which are infinitely close (with the limit of zero distance). If the data l_i contain only s -components, $C(0)$ will be equal to V . If, however, both s - and r -components exist, then $C(0)$ will be equal to V_s only, and from Equation 16

one may then compute an *a priori* variance of the *r*-component as

$$V_r = V - C(O). \tag{21}$$

In Equation 21, *V* would be computed directly from the data *l_i* whereas *C*(*O*) would be determined when the covariance function is evaluated. It is of fundamental importance in estimating the covariance function from the given data that the number of reference points must be reasonably large. If this is not true, then it would be better to use an empirical covariance function. Such an empirical function may be obtained from experience with previous similar problems, or from simply knowing something about the behaviour of the problem.

Once the covariance function is available, the elements of *c* and *C* can be computed from it. For *c*, the distances from the point to be interpolated *P_u* and each reference point *P_i* are computed and each used in the covariance function to compute the respective covariances. For *C*, the variance *V* as computed from Equation 14 is used along the diagonal, and the covariance between any two reference points is evaluated from the covariance function using the distance between the two points as argument.

THEORETICAL EVALUATION OF THE METHOD

An attractive feature of the method is that each reference point can be used as an unknown point and an estimate *s_i*, with a minimum variance, can be computed. If all *s_i* are computed and collected in a vector *s-hat*, a vector of estimates, *r-hat*, for the *r*-component may be computed from

$$\hat{r} = l - \hat{s}. \tag{22}$$

Referring to Equation 11, the *i*-th element of *s-hat* is

$$\hat{s}_i = c_i' C^{-1} l$$

in which

$$c_i = \begin{bmatrix} C(\overline{P_1 P_i}) \\ \vdots \\ C(\overline{P_i P_i}) = C(0) \\ \vdots \\ C(\overline{P_n P_i}) \end{bmatrix}, \quad C = \begin{bmatrix} V & C(\overline{P_1 P_2}) & \cdots & C(\overline{P_1 P_n}) \\ & V & \cdots & C(\overline{P_2 P_n}) \\ & & \cdots & \\ \text{(symmetric)} & & & V \end{bmatrix}.$$

If we construct a matrix *C-bar* whose columns are *c_i*, we get

$$\bar{C} = \begin{bmatrix} C(0) & C(\overline{P_1 P_2}) & \cdots & C(\overline{P_1 P_n}) \\ & C(0) & \cdots & C(\overline{P_2 P_n}) \\ & & \cdots & \\ \text{(symmetric)} & & & C(0) \end{bmatrix}$$

indicating that the difference between *C* and *C-bar* is only in the diagonal elements. On the basis of this analysis, Equation 22 becomes:

$$\hat{r} = l - \bar{C}' C^{-1} l = (C - \bar{C}) C^{-1} l$$

Remembering that $V - C(O) = V_r$, thus $(C - \bar{C}) = V_r \cdot I$ and

$$\hat{r} = V_r C^{-1} l \quad (23)$$

which is the vector of estimates for the r -components at the reference points. This vector represents the values at reference points which are filtered out while interpolating the s -component.

It is interesting at this point to carry the analysis further. Assuming that r_i have zero mean, a variance σ_r^2 may be computed from \hat{r} by

$$\sigma_r^2 = \frac{1}{n} (\hat{r}' \hat{r}). \quad (24)$$

This variance is an *a posteriori* estimate for V_r which is known *a priori* from the covariance function. If there are no errors in estimation (i.e., getting s for s) then σ_r^2 would be equal to V_r . As this is usually not the case, the relative magnitudes of σ_r^2 and V may be considered as an indication of how well filtering was done.

As an example, with a one-dimensional interpolation problem, the *a priori* variance V_r (from the covariance function) was 0.0026, whereas the estimate after interpolation was $\sigma_r^2 = 0.0019$. This implies that the variance of the r -components filtered out during interpolation is about 73 percent of the *a priori* value. Told in terms of standard deviation, the ratio would be 86 percent, a value which is rather good. It should be mentioned that such evaluation indicates the overall adequacy of the method *including* the covariance function used. It is in a way similar to using the reference variance in least squares adjustment to evaluate both the model and the given data.

ANALYSIS OF THE METHOD

In this section we shall discuss the practical significance of the different assumptions made, using numerical examples.

THE NEGLECTED TERMS

Equations 15 and 18 include terms which were neglected in the process of deriving the final relationships, Equations 16 and 19. It is worthwhile to evaluate the relative magnitudes of these terms to give some indication of their effect. Of course, the s - and r -components are not known *a priori* and can only be approximated by \hat{s} and \hat{r} , which *are only estimates*. In one particular application, the following numerical values were obtained (see Equation 15):

$$V = 0.0093; \quad (1/n) \sum s^2 = 0.0052; \quad (1/n) \sum r^2 = 0.0019;$$

and $(2/n) \sum \hat{s} \hat{r} = 0.0020$, which is more or less a representation of the term dropped from Equation 15. This is about 20 percent of V and is not insignificant. For the same interpolation problem the covariance for the distance interval $0 < d < 600$ m was computed from Equation 17 as $C(d) = 0.0057$. After computing the \hat{s} and \hat{r} , the last three terms of Equation 18 were evaluated, using these estimates, as:

$$(1/n_{ij}) \sum_{i < j} \hat{r}_i \hat{s}_j = 0.0008; \quad (1/n_{ij}) \sum_{i < j} \hat{s}_i \hat{r}_j = 0.0003; \quad (1/n_{ij}) \sum_{i < j} \hat{r}_i \hat{r}_j = -0.0004.$$

These are approximations for the terms neglected while deriving Equation 19 from 18. The largest of these is about 14 percent of the value of the covariance.

We have purposely chosen one of the extreme cases in order to point out the importance of remembering the basic assumptions of the method. In the above example we had a relatively small sample of (reference) points and, more importantly, the

variation in the field relative to the chosen reference surface did not adequately represent random variation. Stated another way, perhaps the trend surface was not well accounted for. And we must not forget that we used the *a posteriori* computed \hat{s} and \hat{r} , which include errors of estimation.

EFFECT OF THE NUMBER OF REFERENCE POINTS

To investigate how the interpolated *surface* changes as the number of reference points is altered, we performed a small simulation test of a one-dimensional problem. Figure 6a shows the result of interpolation when a field of 112 reference points are used. Figures 6b and 6c show the corresponding results when one-half and one-fourth,

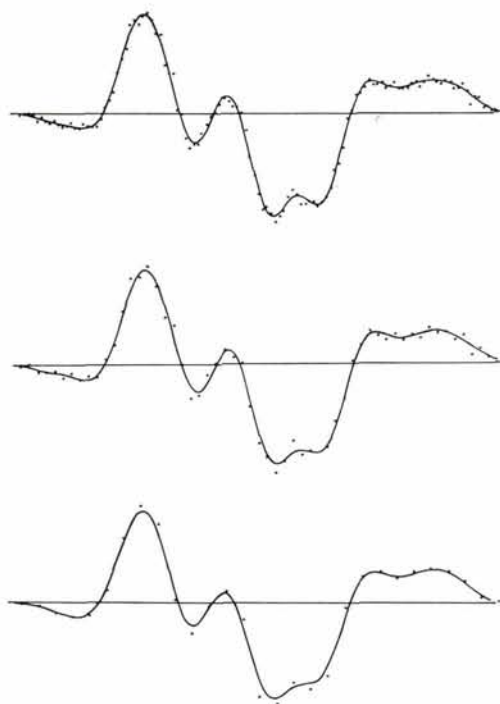


FIG. 6. *Top (6a)*—Interpolation with n points. *Center (6b)*—Interpolation with $n/2$ points. *Bottom (6c)*—Interpolation with $n/4$ points.

respectively, of the points were used. A comparison of these figures indicates that the method works well and is flexible at least for this one set of data.

CHOICE OF TREND OR REFERENCE SURFACE

We said earlier that it is important to shift the data to the trend surface before interpolation and filtering (see Figures 1 and 4). The choice of a function for that surface is not essentially difficult in practice, once one studies the nature of the interpolation problem at hand. In general, the function should have relatively few terms or coefficients. Furthermore, if prior knowledge is available about a particular type of deformation in the given field, one can choose the trend surface function to account, at least partly, for that deformation. For example, film deformation often causes affine distortion of a triangulated block, and therefore one may choose an affine transformation for the trend surface in performing external block adjustment

by the present method of interpolation. We realize of course that a particular trend surface would probably be best suited for a particular problem. However, a study on the effect of different references surfaces is considered beyond the scope of this paper.

CHOICE OF COVARIANCE FUNCTION

This is another important aspect of the estimation problem as all the variances and covariances depend on the proper choice of the covariance function. The function therefore must reflect the characteristics of the problem to be solved. For example, the Gaussian function is often used where the field (and the surface to be estimated) is continuous. It is not, however, suitable for use if the field has discontinuous first derivatives (i.e., with singularities). In such instances, an exponential function would probably be more suitable ($C(d) = C(0) \exp(-d)$). The coefficients of these functions are determined by least squares fitting to the given data. Of course, in other fields (e.g., geodesy) theoretical covariance functions are known.

To have an appreciation of the importance of the covariance function, a simulated one-dimensional interpolation problem was set-up. Figure 7 shows the empirical covariance functions used, limiting consideration to Gaussian and exponential functions only. Figure 8 shows several interpolated profiles corresponding to different

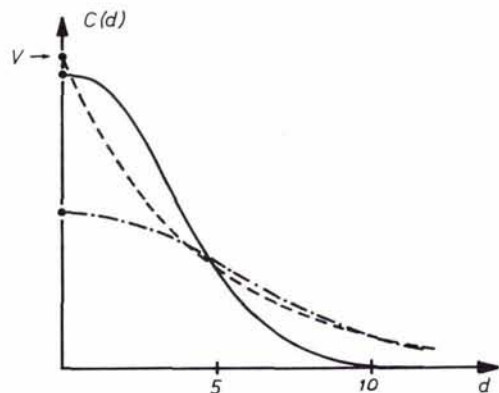


FIG. 7. Experimentation with different covariance functions. The solid line is Gaussian with $C(0) = 0.9 V$, the dotted line is Gaussian with $C(0) = 0.5 V$ and the dashed line is Exponential with $C(0) = V$.

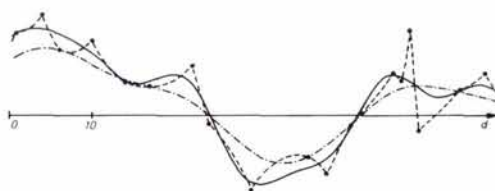


FIG. 8. Interpolated profiles corresponding to Figure 7.

covariance functions with different ratios between V and $C(0)$. Points to note here are:

- (1) Gaussian function produces continuous estimated profiles while the exponential function leads to profiles which exhibit slope changes (sharp peaks) at the locations of the reference points.
- (2) If $C(0) = 1$ the estimated profiles pass through the reference points (i.e., no filtering).
- (3) As $C(0)$ gets smaller relative to V , the profiles in general get farther away from the reference points (i.e., more filtering).

EXAMPLES OF APPLICATIONS IN PHOTOGRAMMETRY

The method of least squares interpolation and filtering has been applied to a number of photogrammetric problems during the past two or three years. We shall give some examples and refer the reader to open literature for others. The objective here is to give an indication of actual results obtained using the method.

CORRECTION FOR FILM AND LENS DISTORTION

Photo-images undergo positional shifts due to the distortion characteristics of the taking lens, and the deformation of the emulsion and film. As relatively simplified functional models are used (e.g., applying collinearity equations), the photo-coordinates are therefore modified to account for these image shifts. The present method of interpolation can be effectively used for this purpose. Because the results of this application have already appeared in available literature we shall not duplicate it here. For lens distortion correction one may consult Reference 5, whereas Reference 3 deals with the question of film deformation.

EXTERNAL HORIZONTAL (AND VERTICAL) BLOCK ADJUSTMENT

After block triangulation, particularly by relatively simplified methods (such as Anblock), residual vectors remain at the ground control points. Residuals at different points invariably include portions which are correlated (s -components), while the remaining parts are usually uncorrelated (r -components). In order to account for these correlated residuals and to compute corresponding quantities at the pass points (\hat{s} -components) without the r -component, least-squares filtering proved to be an excellent procedure. As an example, Figure 9 shows a photogrammetric block with the residuals after triangulation at 42 horizontal control (reference) points. Using a



FIG. 9. External block adjustment. Residuals given at $n=42$ horizontal control points.

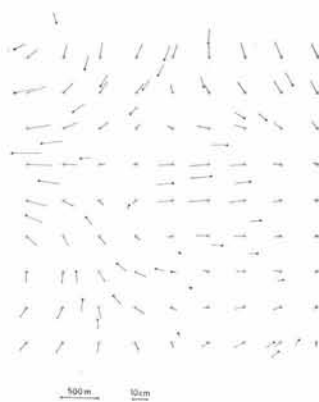


FIG. 10. \hat{s} -Components at control points are shown by solid spots and pass points by open circles.

Gaussian covariance function (see Equation 20) the values $C(0)=0.72$ $V=1.00$ and $k=0.00086$ m^2 were estimated from the data. Both X and Y components were estimated independently and separately and then combined to plot horizontal vectors. Figure 10 shows the estimated \hat{s} for both the control (reference) points (designated by solid circles) and the interpolated pass points (open circles). Figure 11 shows the remaining $\hat{r}=l-\hat{s}$ component of the residuals at the control points. It can be easily ascertained by comparing Figures 10 and 11 how the \hat{s} - and \hat{r} -components vary in appearance and behavior. Although the \hat{s} -components are quite correlated, the \hat{r} -components are obviously random.

APPLICATION TO DIGITAL TERRAIN MODEL

The method appears to have potential for application to problems in digital terrain model. Contouring is one such problem which is under investigation at the present time. Some results have already been achieved and the reader may consult reference (4) for details.



FIG. 11. ϕ -Components at control points.

APPLICATION TO NUMERICAL RECTIFICATION OF SIDE LOOKING RADAR (SLAR) IMAGERY

A very interesting and recent⁶ application has been attempted on a strip of SLAR imagery. For a flat area with a number of horizontal control points the positions of additional points were to be computed by interpolation using SLAR imagery. However, as of the time of this writing, the results obtained were not conclusive. The SLAR imagery exhibited certain characteristics, such that questions regarding stationarity of the field and the choice of proper covariance functions arose. The answers to these questions require further study which is presently continuing.

CONCLUSIONS

The method of linear least-squares interpolation is a useful procedure to apply for a variety of problems in photogrammetry. It possesses some desirable features which include

- (1) It is rather versatile and does not generally depend on the problem to be solved;
- (2) It is a statistical estimation process which is general enough to be applied to different problems of interpolation and filtering;
- (3) It readily accounts for the random measuring portion by filtering it out and interpolating only the component of interest;
- (4) Although a reasonable size of the sample of reference points is advisable, there are no particular restrictions on the disposition of these points provided the covariance function is known;
- (5) In areas of no reference points, the method yields zero interpolated values which is quite logical. This is unlike interpolation by fitting of mathematical functions where one may obtain grossly unrealistic values in such areas of no control.

The introduction of this interpolation method to photogrammetry is rather recent and therefore more study and experimentation would be fruitful. Of the many possibilities, we mention the writing of the covariance function in terms of point positions, in two and eventually in three dimensions, instead of the current practice of using only distances. Actually, the whole subject of proper covariance functions for different photogrammetric problems remains to be one of the fertile areas of investigation. The authors are aware of several institutions currently applying and experimenting with the method.

ACKNOWLEDGEMENTS

The authors wish to thank Prof. Ackermann, of Stuttgart, for his keen interest and encouragement and valuable suggestions made during many sessions of discussion, and to Mr. E. Assmüs for contributions to section D. Thanks are also due to

Dr. F. Leberl, and Mr. K. Tempfli, of ITC, for critically reading the text and for general discussions and contributions.

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