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Solar-Altitude Nomogram

The height of an object can be determined from the length of its shadow on an aerial photo if other parameters are also known.

INTRODUCTION

AIR PHOTOS are valuable sources of direct information on areal distributions and the plan dimensions of objects and surface features. Information about the third dimension, the heights of objects, is less easily obtained, yet it is important in many types of studies; for example, in forestry, urban plan-

sively in making stand volume estimates. A thorough review of the method with sample calculations was presented by Seely in 1942,[2] and by S. H. Spurr and C. T. Brown in 1946.[3] Further refinements were made in 1948 with the introduction of the *pole scale*,[4] which permitted precise measurements of shadow length, and the *shadow height calculator*,[5]

ABSTRACT: *Parallax procedures are most frequently employed in making height measurements on air photos. However, there are situations where parallax procedures cannot be applied and the only alternative is the shadow-length method which requires the calculation of the solar altitude. The procedures currently used to obtain this value are tedious and time-consuming, and for this reason graphical solutions are preferable. By careful scaling of the axes, the shadow-height nomogram described in this article incorporates all the variables within a single, conveniently sized diagram, thus permitting a solution for the solar altitude using either the date and time of photography, or the date and azimuth of the object shadow. The nomogram provides a means of obtaining the solar altitude rapidly with a minimum of calculation and within an error margin of ±0.5 of a degree. With a little practice the solution for the solar altitude can be obtained in approximately two minutes for any given set of conditions, thus providing a considerable economy of time and effort if compared with conventional methods.*

ning, and military intelligence. There are two basic procedures for obtaining object heights from air photos—by calculation from the shadow length and by calculations based on differential parallax.

The shadow factor method was first applied in 1929 for the measurement of tree heights in eastern Canada,[1] and during the 1930's H. E. Seely and his associates with the Dominion Forestry Service used the procedure exten-

which simplified the calculations by providing a mechanical solution of the astronomical triangle, thus permitting a rapid determination of the solar altitude. For some years this calculator could be obtained through the Canadian Forest Service. It is no longer available; however, a similar device known as a *solarmeter* has been constructed by the Satellite Data Laboratory, Canadian Meteorological Service, Department of Transport.

The shadow-height method, as it came to be called, was extensively used in Canada, largely as the result of the demonstration of its accuracy in controlled tests by A. J. Nash,[6] conducted under the ideal contrast conditions of the Canadian winter with long, clear sha-

dows etched on a snow background. The method was introduced into American forest practice by E. J. Rogers at the Northeastern Forest Experiment Station,[7] and it received detailed treatment in Spurr's textbook, *Aerial Photographs in Forestry*,[8] and G. T. McNeil's *Photographic Measurements*.[9] The only subsequent contribution to the methodology has been the design of special computing forms to simplify the calculations,[10] and the development of general purpose solar altitude nomograms.[11]

During the 1950's, the shadow-height method declined in popularity, and parallax procedures came to be more generally used for height measurement.[12] This change occurred in part because the parallax method is basically more accurate, but also because the calculations are much simpler, and can be completed directly with the aid of tables or an instrument such as the Zeiss *parallax converter*. However, there are situations where parallax methods cannot be applied, and the only alternative is the shadow-height method. An interesting example of this kind is found in a recent investigation of the feasibility of measuring the height of cumulonimbus clouds from satellite photos.[13] Parallax procedures could not be applied because the photos do not provide an adequate stereoscopic image; however, the shadow-height method was readily adapted to the situation and yielded results of acceptable accuracy. In such studies, a shadow-height nomogram of the type discussed in the following pages, can make a valuable contribution by providing a direct and rapid solution of the solar altitude equation.
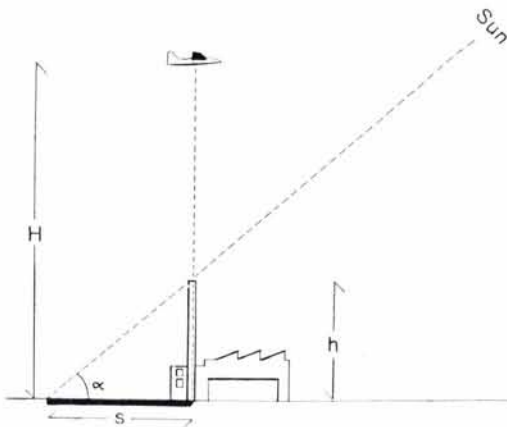

FIG. 1. Basic relationship between object height, shadow length, photo scale and solar altitude: $h = Hs (\tan \alpha)/f$.

TABLE 1. BASIC NOTATION

Celestial Parameters		Object Parameters	
α	Solar altitude	h	Object height
δ	Solar declination	l	Latitude of object
τ	Solar hour angle	s	Length of shadow cast by object
System Parameters		a	Azimuth of the shadow.
f	Focal length of camera		
H	Flight of orbital altitude.		

BASIC TRIGONOMETRY OF SHADOWS

At any given time the length of an object's shadow as cast on a horizontal surface is directly proportional to its height, and to the angular elevation of the source illumination. These relationships remain unchanged if the object shadow is imaged on an air photo, the only effect is the introduction of a scaling factor as a function of flying height and camera focal length (Figure 1 and Table 1):

$$h = Hs(\tan \alpha)/f. \qquad (1)$$

Apart from the object height, which is the required parameter, the only unknown factor in this equation is the solar altitude, because the shadow length can be measured on the air photo with a scale magnifier, and the flying height and camera focal length can be read from the data block, or the flight report.

The solar altitude can be calculated in terms of the astronomical triangle which requires that three other parameters be known: the latitude of the observation, the solar declination for the particular day on which the air photo was obtained, and the hour angle at the actual time of the photography:

$$\sin \alpha = (\cos l)(\cos \delta)(\cos \tau) + (\sin l)(\sin \delta). \qquad (2)$$

The solar declination, or latitude, can be obtained from a solar ephemeris for the particular date required, and it is positive or negative depending on whether the sun is on the same, or opposite side of the equator from the observer. The latitude of the observation can be obtained from topographic maps, and the hour angle can be calculated as long as the time of day at which the air photo was taken has been recorded on the data block or the flight report.

Unfortunately, many air photos do not provide information on the time of photography, and so a more complex equation has to

be used in which the azimuth of the cast shadow replaces the hour angle as the unknown parameter:

$$\sin \alpha = \frac{(\cos l)(\cos a)\sqrt{(\cos^2 \delta)-(\cos^2 l)(\sin^2 a)}+(\sin \delta)(\sin l)}{1-(\cos^2 l)(\sin^2 a)} . \tag{3}$$

Equation 3 is derived from the relationships in the astronomical triangle as given in Equation 2, and the equation for the shadow azimuth (Equation 4) which can be stated in terms of the same parameters:[14]

$$\sin a = \frac{(-\cos \delta)(\sin \tau)}{(\cos \alpha)} . \tag{4}$$

Equations 2 and 3, although providing a precise solution for the solar altitude, are cumbersome and tedious to work through because it is necessary to repeat the calculations for any change in the time of day (and hence shadow azimuth), and this occurs even between adjacent air photos in the same flight line. Equation 3 poses an additional complication, because if the solar declination is greater than the latitude of the observation, it is possible for the functions within the square root to become negative, thus precluding any useful solution.

For these reasons, graphical solutions for obtaining the solar altitude are to be preferred. Although several such presentations have been made,[15] none of them incorporates all the variables within a single diagram, thus reducing the accuracy obtainable. However, by careful scaling of the axes, it is possible to produce a nomogram which permits the graphical solution of all the variables within a single plot. This was accomplished with the aid of the CALCOMP Plotter at the McGill University Computing Centre, and is presented in Figure 2.

THE SOLAR ALTITUDE NOMOGRAM

The nomogram consists of four separate but linked plots, which are scaled so as to permit the direct transfer of points by their rectangular coordinates from one plot to another.

Plot A is constructed with the solar declination on the abscissa, positive or summer values to the right, and negative, winter values to the left. As the seasons change directly with the solar declination, it is possible to include a calendar on the same axis as indicated on the base line. The ordinate is scaled as the sine function of solar altitude, and the plot contains two sets of curves. The upper set of curves gives the plot of the cosine function of the latitude minus the declination for various values of latitude:

$$y = (\cos l)(\cos \delta) + (\sin l)(\sin \delta) = \cos (l - \delta). \tag{5}$$

The lower set gives the plot of the sine function of latitude and declination for various values of latitude:

$$y = (\sin l)(\sin \delta). \tag{6}$$

As explained in the next section, the curves serve to provide the coordinates of the *time line* which is constructed in Plot C. For a particular solar declination (which occurs on two dates in the year), the curves fix the upper limit of the solar altitude, the maximum value for the day being provided by the intersection of the *date line* with the latitude of the observer as read from the upper set of curves. The absolute minimum value of the solar altitude which would occur at midnight is of little practical value, and so the "effective" lower point is obtained for 0600 hours and 1800 hours by the intersection of the *date line* with the latitude of the observer as read from the lower set of curves.

Plot B is constructed with the same abscissa scale as Plot A, which can be read either as a calendar or as the solar declination. On the ordinate a non-linear scale of z *factors* has been used which is similar to the abscissa of Plot C:

$$y = \frac{\sqrt{z^2 + (x^2) - 1}}{z} , \tag{7}$$

thus when $(y)=0$, $z^2+(x^2)-1=0$, or $(x) = \sqrt{(1-z^2)}$. If $z>1$, the equation $(x) = \sqrt{(z^2-1)}$ is used. The z-*factor* is an arbitrary term for the sine function of the shadow azimuth divided by the cosine function of the solar declination, and the values given on the ordinate have been multiplied by a hundred for convenience in handling:

$$y = \frac{\sin a}{\cos \delta} \times 100 = \textit{the z factor} \tag{8}$$

By substitution in Equation 4, the z *factor* can be stated in terms of the hour angle and the solar altitude.

$$\frac{z\text{-factor}}{100} = \frac{-\sin \tau}{\cos \alpha} \tag{9}$$

The plot contains a series of curves for differ-

ent values of the shadow azimuth, and the intersection of the curve for a particular azimuth value with the *date line* permits the reading of the *z factor* on the right hand ordinate scale. The *z factor* has a positive or negative value depending on the shadow azimuth. With an azimuth of between 0 and 90°, as measured in either direction from the local noon position of the shadow (north in the northern hemisphere), the curve corresponding to the angle is used directly, and the *z factor* obtained has a positive value. If an azimuth exceeds 90° the *z factor* is obtained from the curve corresponding to 180° minus the azimuth and has a negative value.

Plot C has the solar altitude on the ordinate, scaled in the same manner as in Plot A, whereas the abscissa is scaled as the cosine function of the hour angle:

$$x = \cos \tau \quad \text{and} \quad y = \sin \alpha. \tag{10}$$

If the effective limits of the solar altitude, as determined from Plot A, are transferred to Plot C, it is possible to generate a *time line* by joining the maximum solar elevation recorded on the left margin of the plot with the effective lower point recorded on the central axis. The gradient of this line is given by the equation of a straight line,

$$y = aX + B. \tag{11}$$

The gradient a is the intercept A when $x = 1$, minus the intercept B when $x = 0$.

$$a = A - B = (\cos l)(\cos \delta). \tag{12}$$

The equation of the time line can be derived from Equations 5, 6, 9 and 10:

$$y = (\cos l \cos \delta)x + (\sin l \sin \delta) \tag{13}$$

Stated in terms of the scaled axes of Plot B, this equation becomes the same as Equation 2.

$$\sin \alpha = (\cos l)(\cos \delta)(\cos \tau) + (\sin l)(\sin \delta). \tag{2}$$

The *time line*, in fact, represents the movement of the sun from dawn, when the solar altitude is zero, through to midday, when the hour angle is zero, and then back along the same line to sunset, when the solar altitude becomes zero again.

As the hour angle is directly related to the true solar time, it is possible to include the latter on the abscissa of Plot C as a secondary scale. Thus, if the time of photography is indicated in the data block or flight report, this can be corrected using the equation of time as explained in the next section to give the true solar time, and Plot C can be entered with this value, permitting the direct reading of the

solar altitude for the particular date and time of day by using the coordinate from the intersection of the true solar time with the *time line*.

The solar altitude can be obtained from Plot C even if the time of photography is not known. This is made possible by transforming each *z factor* value into a set of coordinates, thus generating a set of *z-factor* curves. It would have been most convenient to have had $\sin \tau$ and $\cos \alpha$ as the axes (see Equation 9), but the plot is already scaled for $\sin \alpha$ and $\cos \tau$. The difficulty can be overcome by using the general function in Equation 14 and the relationship in Equation 15, which is derived from Equation 9 above:

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{14}$$

$$\sin \tau = \frac{z \cos \alpha}{100} \tag{15}$$

hence

$$\sqrt{(1 - \cos^2 \tau)} = z\sqrt{(1 - \sin^2 \alpha)} \tag{16}$$

or

$$1 - \cos^2 \tau = z^2 - z^2 \sin^2 \alpha, \tag{17}$$

substituting Equation 10;

$$1 - x^2 = z^2 - z^2 y^2, \tag{18}$$

or

$$y = \frac{\sqrt{z^2 + x^2 - 1}}{z}. \tag{19}$$

The *z factor* curves in Plot C were obtained by substituting different *z*-values in Equation 19. If the time of photography is not known, the *z factor* curve corresponding to the value for *z* obtained in Plot B is identified or interpolated, and the intersection of this curve with the *time line* provides the rectangular coordinate for reading the solar altitude on the central axis. In the great majority of cases the *z factor* obtained in Plot B has a positive value, and the *z-factor* curve intersects the time line in Plot C at only one location.

However, in certain circumstances it is quite possible for the *z-factor* curve to intersect the *time line* twice, and it should be noted that when the *z-factor* obtained in Plot B has a positive value, the intersection nearest to the left hand axis (1200 hours) should be used, whereas if the *z factor* has a negative value, the intersection closest to the right hand axis (2400 hours) should be used. In most instances this latter point will be found to occur in the right hand quadrant.

Plot D contains an analemma in the central portion, which permits a graphical solution
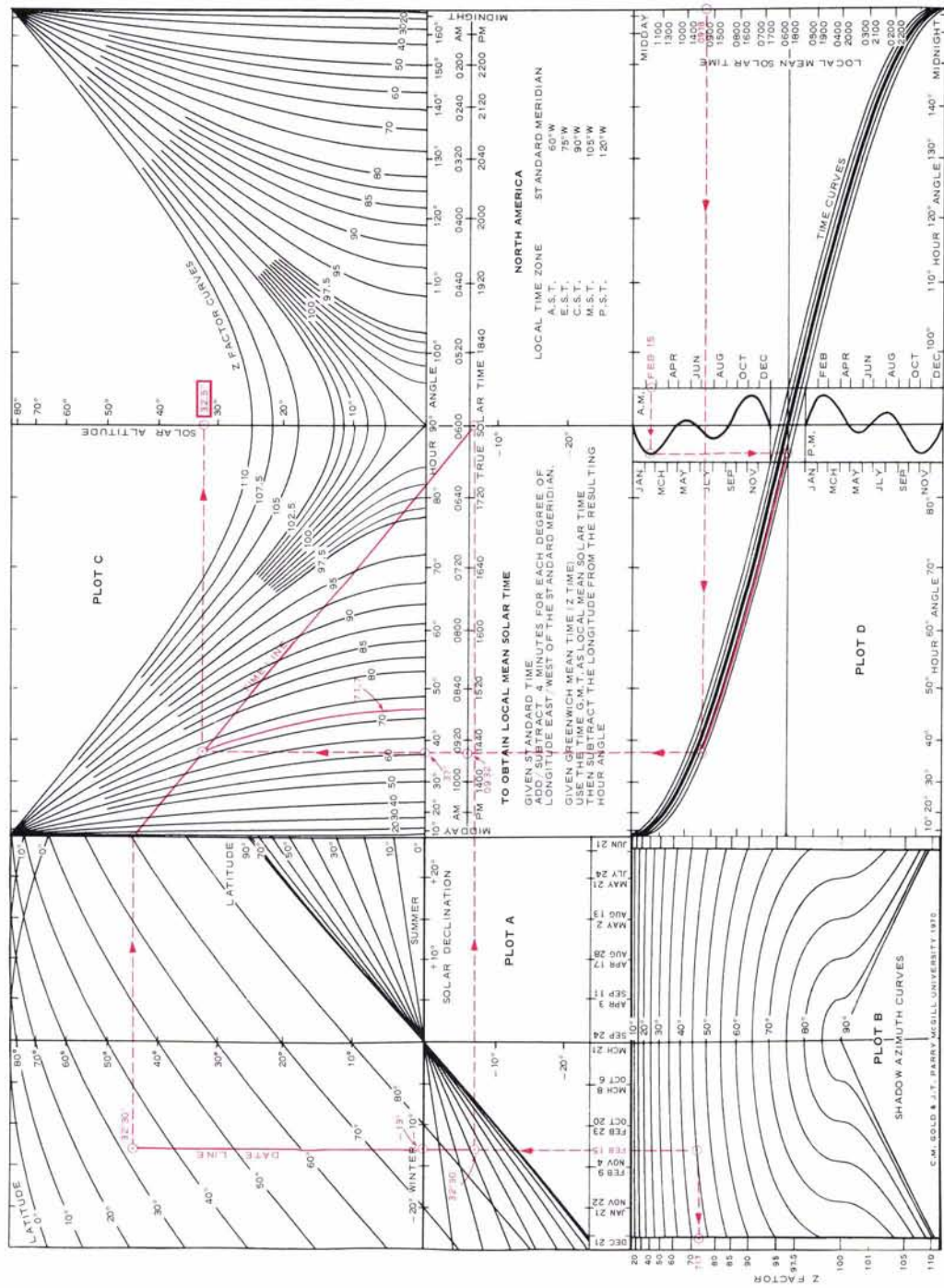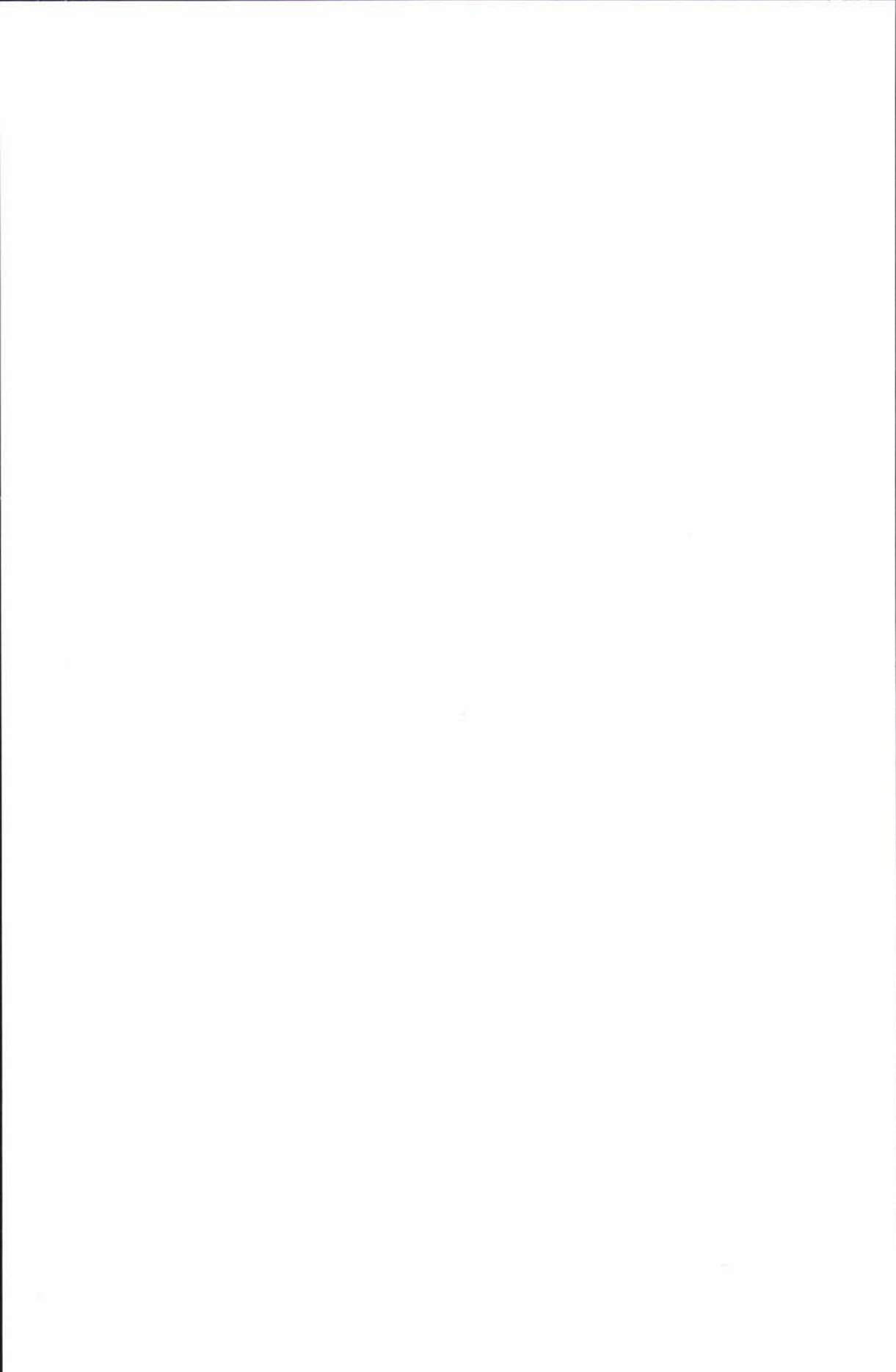
Fig. 2. Shadow-height nomogram.

of the equation of time for different dates through the year. The correction obtained from the analemma is projected onto the central horizontal axis, which intersects a series of cosine curves covering the total range for the correction to the equation of time (approximately ± 20 minues). The ordinate (local mean solar time) is scaled in regular hourly intervals, and the abscissa (true solar time) is the same as that in Plot C—the cosine function of the hour angle. To convert local mean solar time to true solar time, the appropriate *time curve* is identified by dropping a perpendicular from the intersection of the analemma with the date line onto the central axis. This *time curve* is then followed to the point where it intersects the value for the local mean solar time. The perpendicular from this intersection to the abscissa provides a value for the true solar time which is used in Plot C to obtain the solar altitude.

Application of the Nomogram

The value of the nomogram is best appreciated from actual practice, and the example chosen to illustrate its application was originally presented in a conventional manner in the article by E. W. Johnson.[19] A comparison of the two methods will demonstrate the ease and rapidity of the nomogram solution, which is presented in Figure 2 by means of the red overprint.

The data given in Johnson's shadow-height computing sheet are as follows: date February 15, 1952, latitude 32°30′ N., longitude 85°30′ W., local time 0900 hours. Entering Plot A for the date February 15, the *date line* is constructed between the latitude intercepts at 32°30′ on the upper and lower curves. The solar declination can then be read directly from the center axis as −13° (the actual figure appearing in the solar ephemeris for that date is −13° 06.6′).

Entering Plot C, the intersections of the upper and lower limits of the date line with the left margin of the plot and the central ordinate are used to construct the *time line*. This permits direct reading of the solar altitude on the central ordinate for any time of day, so long as true solar time is used.

The conversion of local time to true regular time is accomplished in Plot D. The time of photography in the example is given as 0900 hours Central Standard Time, which represents the local time along the 90° W meridian, which is the standard for the central time zone. However, the area of photography was 85°30′ W., and so the local time is ahead of the standard. The difference, four minutes

per degree of longitude, amounts to 18 minutes, giving a local mean solar time of 0918 hours. The correction for the equation of time is made in the center of the plot by obtaining the intersection of the date line with the analemma in the upper part, which indicates the variation for the period before midday. The amount of the correction can be interpolated on the central portion of the abscissa as −14 minutes (the actual figure appearing in the solar ephemeris for that date is −14.285 minutes); however, there is no need to determine this figure because it is taken into account by following the appropriate *time curve* from the intersection on the central portion of the abscissa up to the left to the intersection with the previously determined value of 0918 hours for the local mean solar time.

Returning to Plot C on the perpendicular from this intercept, the true solar time and the hour angle can be read directly on the abscissa as 0932 hours and 37° respectively, (the actual figure obtained by calculation is 36°56′). Following the same perpendicular to the intersection with the *time line*, and thence on the rectangular coordinate to the central ordinate, gives a value for the solar altitude of 32.5° (the actual figure obtained by calculation is 32°32′).

In the situation where the time of photography is not known, the nomogram can be used by measuring the azimuth of the cast shadow from the air photo. This measurement must be made from true north, and so reference to a map and to the local magnetic declination may also be necessary.

On the air photos of Auburn, Alabama, which were used by E. W. Johnson in his study, the azimuth of the shadows, which were imaged soon after 0900 hours Central Standard Time, is approximately 45°. Entering Plot B for the date February 15, the intersection with the 45° shadow angle curve gives a *z-factor* value of 71.7 on the ordinate. This is interpolated on the *z-factor* curves in Plot C to the intersection with the *time line*, and thence on the coordinate to the central ordinate, giving a value for the solar altitude of 32.5° as obtained previously.

If the reader goes through the same procedures with the aid of the overprint in Figure 2, the advantages of the nomogram solution will soon become apparent, and it will be found that with a little practice the solution for the solar altitude can be obtained in approximately two minutes for any given set of conditions. This represents a considerable saving of time compared to the conventional procedures, and in addition, it should be

noted that once the time line has been constructed, it is possible to obtain the solar altitude for any time of day without further calculations. This provides a considerable economy of time and effort as compared with the procedures presented by Spurr, Rogers and Johnson[16] for plotting the solar altitude curve for a particular day.

The accuracy of the nomogram was checked by comparing the results with those obtained using the solar ephemeris for the same set of conditions. More than a hundred such comparisons were made, and the errors were found to be less than half-a-degree except in the following situations:

i. If the solar declination has the same value as the latitude of the photograph, it is found that the maximum solar altitude (Plot A) is 90° which fixes the *time line* in the upper left hand corner of Plot C where the *z-factor* curves converge. In this situation, it is impossible to determine the locus of intersection between the *time line* and a particular *z-factor* curve at the upper limit of the *time line*. The same problems arise if there is a difference of only a few degrees between the solar declination and the latitude of the observer. However, it should be remembered that the whole shadow-height method is unreliable in these circumstances because the sun is overhead at noon, with the result that shadows are either absent, or so short as to preclude their accurate measurement, and so a solution to Equation 1 is not possible.

ii. If the area photographed is at the equator, and the solar declination is zero $(l = \delta = 0)$, the *time line* coincides with the *z-factor* curve for 100 (Plot C). Although at first sight this seems to be anomalous, further reflection will show that it is in fact correct, because on two dates in the year March 21st and September 21st the sun follows an east-west arc through the zenith and the morning shadows are cast to the west at all times, contracting to extinction at noon, and then lengthening to the east during the afternoon. As with the previous example, it is very difficult to use Plot C for small hour angles because of the convergence of the *z-factor* curves, and the same difficulty is experienced if the latitude and the solar declination are within a few degrees of zero.

iii. If the solar altitude approaches zero, and the hour angle is close to 90° (Plot C), it is only possible to read the solar altitude values to within one or two degrees.

With these exceptions the solar altitude nomogram has consistently given results which are accurate within half a degree. This is quite acceptable for use in the shadow height calculation (Equation 1), because of the impossibility of obtaining an absolutely accurate measurement of the other factor in the equation—the shadow length that has a much greater effect on the result.

Conclusion

- In certain circumstances, parallax methods cannot be used in measuring the heights of objects on air photographs. The most common situation of this type results from variable overlap between photos along the flight line which makes stereoscopic viewing difficult or impossible. In such circumstances, the shadow-height method is the only alternative.
- The procedures currently used in the shadow-height method are tedious and time-consuming, and have contributed to the declining popularity of the method.
- The solar-altitude nomogram presented in this study provides a means of obtaining the solar altitude rapidly with a minimum of calculation, and within a error margin of half a degree.
- The only parameters required in using the nomogram are the geographical coordinates of the area, and *either* the date and time of photography, *or* the date and azimuth of the object shadow. The procedures are equally rapid in either circumstance.
- The nomogram incorporates all the variables in the solar-altitude problem as it relates to the shadow-height method within a single conveniently sized diagram, thus permitting a graphic solution.

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