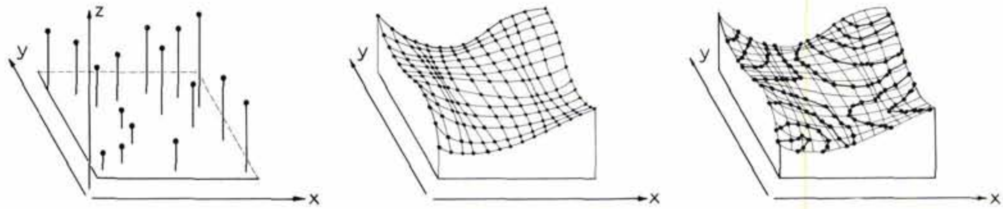

FIG. 1. "Grid" model of surfaces. *Left* – r real Z-data points. *Center* – Fine grid of discrete Z-estimates. *Right* – Linear interpolation between grid points for points along contours.

J. R. JANCAITIS
PROF. J. L. JUNKINS
University of Virginia
Charlottesville, Va. 22901

Modeling Irregular Surfaces*

The functional surface-modeling technique, an efficient basis for automated contouring, allows any prescribed standard deviation.

(*Abstract on next page*)

INTRODUCTION

THE CONTOUR MAP has long been recognized as one of the best methods for accurate visual representation of discretely measured three-dimensional surfaces. Hand production of contour maps, however, is a costly and time-consuming process. The utilization of computer graphics capabilities for automated cartography is more accurate, less subjective, and less costly than the hand-production methods. The development and use of the currently employed methods for automated cartography are recorded in Light's excellent bibliography and the subsequent work of Light and Biggin.[1,2]

The problem considered in this paper is the development of a technique which improves the practicing photogrammetrist's surface modeling and automated contouring capabilities.

Up to this time automated cartography has required modeling of the surface by creation of a fine grid of equally-spaced, discrete elevation estimates from a set of randomly located elevation observations.[1,2,3] This grid of elevation estimates is called a digital terrain model (DTM). The contour lines are then extracted by searching and interpolating between the points in the DTM (see Figure 1).

An alternate approach for surface modeling has received considerable attention – a mathematical representation of surfaces. Fourier Series, Hardy's "multiquadric equations of topography," and others have all been investigated thoroughly.[4,5] For several reasons, none of these functions have been adopted in practice. The major reason that the functional approach has not been adopted in practice is that it has been employed as only a more sophisticated way to produce the DTM for contouring; and it has been found to have a much greater computational expense as com-

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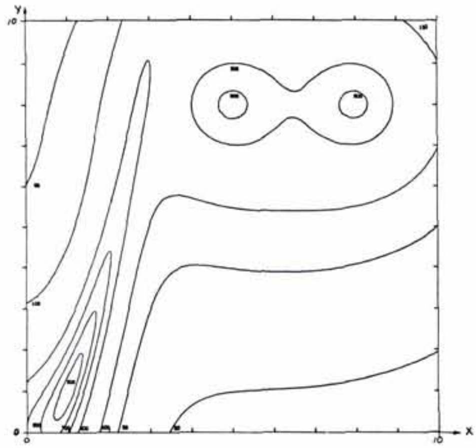

FIG. 2. Contours of test function extracted numerically.

pared to conventional methods. This reason and the associated problems that these surface function approaches: (1) were not applicable to arbitrarily large data sets, (2) did not give consistent results in elevation and partial derivatives along the boundaries of adjacent data sets, and (3) usually fit the data exactly, giving rise to an unrealistically rough surface, have made these previously investigated surface function methods unacceptable as the basis for an automated contouring program.

In light of these difficulties a new sequential continuous function surface modeling technique was developed. The surface function model is a sequential two-independent variable interpolation procedure which is an extension of a sequential one-independent variable interpolation procedure developed by the authors for production of continuous lines in a two dimensional space.[6] Contour lines can be extracted from the function surface model by using a new technique developed by the authors, called CONTUR.[7] CONTUR is based on analytically or numerically solving the functions representing the surface for contour lines of interest (see Figure 2).

ABSTRACT: *A two-independent variable interpolation procedure has been developed for modeling irregular functions of two variables. This method represents the surface as a family of locally valid mathematical functions which join together continuously. This surface modeling technique has been found to be an efficient and accurate mathematical basis for automated contouring.*

CONTINUOUS FUNCTION SURFACE MODEL

The continuous-function surface model[8] is based on a set of locally valid polynomials of the form

$$Z = \sum_{ij} C_{ij}\, x^i\, y^j. \tag{1}$$

The base plane of the original elevation observations is divided into square regions (see Figure 3). Without loss of generality, the length of the sides of the square regions can be chosen as unity because simple scaling can produce this result. A *best-estimate* of the elevation and slopes at the corners of each of the unit-square

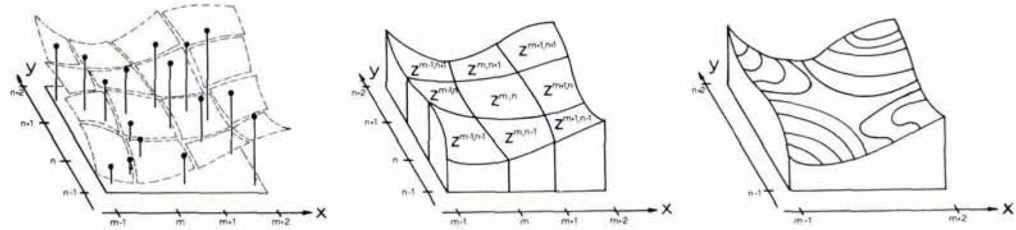

FIG. 3. Continuous-function surface model. *Left*—Least-squares fits to local subsets of the original data. *Center*—(x, y) unit-square regions of validity for local polynomial surface models. *Right*—Contour lines from the surface function $Z = \sum_{m,n}^{M,N} \delta_{m,n} Z^{m,n}$.

regions is calculated by least-squares fitting a local subset of the original data with a plane, e.g., utilizing the standard weighting least-squares procedure:

$$\begin{bmatrix} \bar{Z} \\ \bar{Z}_x \\ \bar{Z}_y \end{bmatrix} = [\mathbf{A}^T \mathbf{W} \mathbf{A}]^{-1} \, \mathbf{A}^T \mathbf{W} \, \mathbf{O} \qquad (2)$$

where $\bar{Z}$, $\bar{Z}_x$, and $\bar{Z}_y$ are best estimate values for the elevation and slopes at a corner of a unit square region. **O** is a $n \times 1$ vector of a local subset of the original data, **A** is a matrix of partial derivatives defined as

$$\begin{matrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & x_n & y_n \end{matrix}$$

x_i, y_i is the x,y location of the i-th elevation data point, **W** is a $n \times n$ weighting matrix defined as

$$\begin{matrix} w_1^2 & 0 & 0 & \ldots & 0 \\ 0 & w_2^2 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \ldots & w_n^2 \end{matrix}$$

where w_i is a relative weight assigned to the i-th elevation data point.

The terrain above each of the unit-square regions is represented by a surface function of the Form 1. The coefficients C_{ij}, for the polynomial valid over each unit-square region are determined from the best estimate values so that neighboring surface functions join continuously. The continuity in elevation and slope along the mutual boundary of validity between two neighboring unit-square regions is assured by mathematically requiring that the neighboring polynomial surface functions reduce to exactly the same function along that mutual boundary. The difference in slope perpendicular to the mutual boundary of validity between the two neighboring unit square regions is minimized by requiring the neighboring polynomial surface functions to agree exactly in slope perpendicular to the boundary at the end points of the boundary. Considering Figure 3 and the definitions: $\mathbf{Z}^{m,n}$ is the polynomial surface function valid over the unit square region whose lower left corner is located at $x = m$, $y = n$, $\mathbf{Z}^{m,n}|_{y=n}$ is the one dimensional polynomial in x found by putting $y = n$ in the two-dimensional surface function $\mathbf{Z}^{m,n}$, $\mathbf{Z}_x^{m,n}|_{(m,n)}$ is the partial differential of $Z^{m,n}$ with respect to x evaluated at the point $x = m$, $y = n$, e.g., the x-slope of the surface function at the point x,y. Then along the mutual boundary of validity between $\mathbf{Z}^{m,n}$ and $\mathbf{Z}^{m,n+1}$, e.g., the line $y = n + 1$ from $x = m$ to $x = m + 1$, the conditions stated above can be interpreted as

$$\mathbf{Z}^{m,n}|_{y=n+1} = \mathbf{Z}^{m,n+1}|_{y=n+1} \tag{1}$$

for elevation continuity.

Note that continuity of slope along the boundary

$$\mathbf{Z}_x^{m,n}|_{y=n+1} = \mathbf{Z}_x^{m,n+1}|_{y=n+1}$$

follows directly from the elevation continuity.

$$\mathbf{Z}_y^{m,n}|_{(m,n+1)} = \mathbf{Z}_y^{m,n+1}|_{(m,n+1)}$$

$$\mathbf{Z}_y^{m,n}|_{(m+1,n+1)} = \mathbf{Z}_y^{m,n+1}|_{(m+1,n+1)} \tag{2}$$

for agreement in slope perpendicular to the boundary at the end points of the boundary.

One of the lowest order polynomials which can satisfy these two conditions along all four boundaries of a unit square region is:

$$\begin{aligned}\mathbf{Z} = C_{00} &+ C_{01}y + C_{02}y^2 + C_{03}y^3 + C_{10}x \\ &+ C_{11}xy + C_{12}xy^2 + C_{13}xy^3 + C_{20}x^2 \\ &+ C_{21}x^2y + C_{30}x^3 + C_{31}x^3y.\end{aligned} \tag{3}$$

The choice of the last term is somewhat arbitrary because the (3,2), (1,3), (2,3) or (3,3) term could have been utilized. The total number of coefficients is the important parameter. The two conditions stated above applied at all four boundaries of a unit square region result in twelve conditions on an equations of the Form 1. In order to satisfy these 12 conditions, 12 constants are necessary in the final polynomial surface function chosen. Obviously, the choice of the 12th term is arbitrary because any of the five possibilities is acceptable.

The 12 conditions which must be imposed on the surface Function 3 in order to meet the two stated conditions along all four boundaries are simply that the surface function agree in elevation and slopes with the best estimate values at all four corners. Calculation of the coefficients C_{ij} for a given unit-square region, from the elevation and slopes at its four corners is accomplished by:

$$\mathbf{C} = \mathbf{A}^{-1}\,\mathbf{Z} \tag{4}$$

where $\mathbf{C}$ is a 1×12 vector of the coefficients C_{ij}, $\mathbf{A}$ is a 12×12 matrix containing elements depending upon the x,y positions of the four corners of this unit-square region, consistent with Equation 3, $\mathbf{Z}$ is a 12×1 vector of the best estimate values of elevation and slopes at the four corners.

The matrix $\mathbf{A}$ contains only the relative x,y location of the four corners of the x,y unit-square region of validity. By defining a local roving coordinate system, centered at the lower left corner of the x,y unit-square region under consideration, the matrix $\mathbf{A}$ is the same for all of the regions over the entire data set. This fact means that the matrix $\mathbf{A}$ need only be computed and inverted once and then stored. The final coefficients for each locally valid polynomial can be found by simply multiplying the stored matrix times the appropriate *best estimate* values. Having to invert the $\mathbf{A}$ matrix only once results in a large savings in a computer run time.

The total functional relationship representing the surface would be:

$$Z = \sum_{m,n}^{M,N} \left(\delta_{mn} \sum_{ij} C_{ij}^{mn}\, x^i\, y^j \right) \tag{5}$$

where

$$\delta_{mn} = \begin{cases} 1 \text{ for } m < x < m+1 \text{ and } n < y < n+1 \\ 0 \text{ for all other } x \text{ and } y. \end{cases}$$

The final but most important requirement of the surface model is that

$$\left[\frac{\sum_{i=1}^{r} [Z(x_i, y_i) - Z_i]^2}{r}\right]^{1/2} < \sigma \tag{6}$$

where r is the total number of measurements and σ is the Z_i observation's measurement error. Equation 6 insures that the standard deviation of the functional model from the original observations is within their measurement error. By choosing the size of the regions of validity smaller or larger, so that correspondingly less or more area is covered by the same mathematical function, the agreement between the total surface function and the original elevation data will increase or decrease. Therefore the choice of the size of the unit-square regions of validity determines the standard deviation of the functional model from the original data.

The contour lines of altitudes of interest can be extracted from the surface model utilizing the CONTUR method previously mentioned. Points along contours of interest can be found by simply analytically or numerically solving for them. The specific logic involved in finding and plotting these points is not the subject of this paper; the interested reader can find a complete description in Junkins and Jancaitis, 1971(a).

Application of the Method

A prototype computer program was written in Fortran IV for a CDC 6400, based on the continuous-function surface model. The program was written to handle the special case of equally spaced elevation data. It should be noted that the *method* is not restricted to equally spaced data, only *this program* is. Equally spaced original data allows the use of a local roving coordinate system in the calculation of the least-squares fits Equation 2. As in Equation 4 above, the matrix inverted in a least-squares solution is a function of only the x,y positioning of the data. For equally spaced data the x,y positioning of the original data relative to the local roving coordinate system is always the same, so that only one matrix was computed, inverted and stored. The best estimate values were calculated for each region by simply multiplying the stored matrix times the appropriate observation column. For this application two least-squares fits were calculated, each to a different subset of five local elevation observations, and the best estimate values were calculated by averaging the results of the two least-squares fits.

Points along contours of interest were found by root solving the functional model using a two variable Newton's Method. A savings in the amount of core storage required was obtained by contouring and plotting each region as its coefficients were found. In this way only a small subset of the entire data array needed to be in core at any given time. Also, as the contour lines are plotted as they are calculated, neither the coefficients C_{ij} nor the points along the contours had to be saved.

The program was run using a set of 1,600 equally spaced UNAMACE data points supplied by TOPOCOM. The elevation data ranged from 20 to 340 meters with an estimated noise level of 5 meters.

The data was divided into 324 and then 121 x,y unit-square regions of validity. Figures 4 and 5 are the computer output for these two applications respectively, contoured every 20 meters. Figure 6 is the program output using the lower left subset of 400 data points, divided into 81 x,y unit square regions of validity, contoured every 5 meters. The computed standard deviations of the three cases were 5.4, 7.2 and 5.4 meters, respectively.

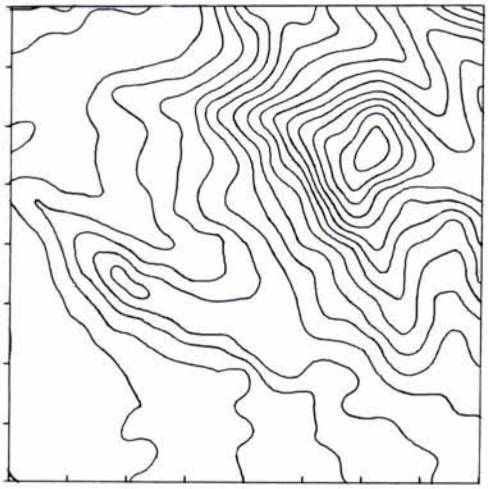

FIG. 4. Function-surface model program. 324 (x, y) unit-square regions of validity, 1,600 data-point set, 0.55 minute, 20-m contour interval.

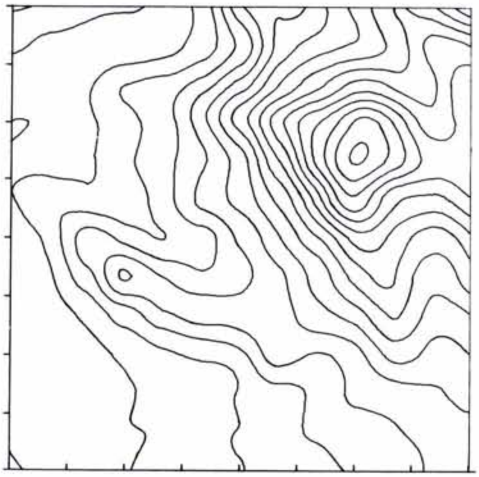

FIG. 5. Function-surface model program. 121 (x, y) unit-square regions of validity, 1,600 data-point set, 0.35 minute, 20-m contour interval.

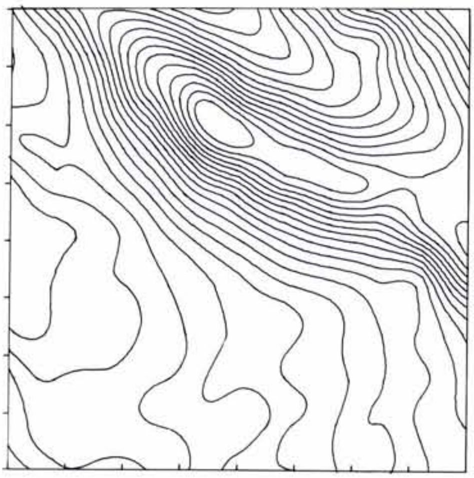

FIG. 6. Function-surface model program. 64 (x, y) unit-square regions of validity, 400 data-point set, 0.2 minute, 5-m contour interval.

All three of these plots exhibit the following properties:

- A lack of high-frequency oscillations usually found in digital contour plots of noisy-data;
- Excellent trend definition due to the high degree of contour parallelism along the ridges and drainage patterns,
- A lack of unrealistic "small peaks and depressions" which are often evident in computer plots of noisy data,
- Excellent agreement in elevation and slope between neighboring regions of validity, demonstrating that this method will give consistent results along the boundaries of adjacent data sets and,
- Smooth, continuous slope contours without the use of any contour smoothing routine.

The computer run times for these three cases were 0.55, 0.35 and 0.20 minutes,

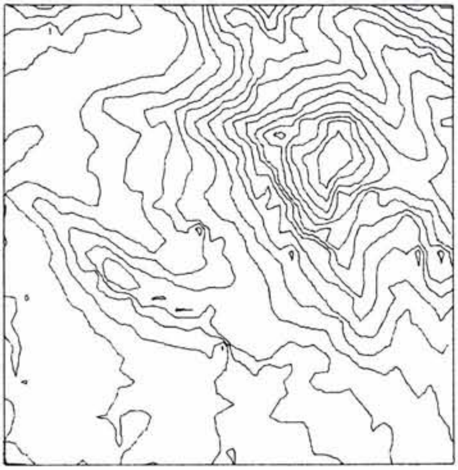

FIG. 7. TOPOCOM contouring program. 1,600 data-point set, 0.75 minute, 20-m contour interval.

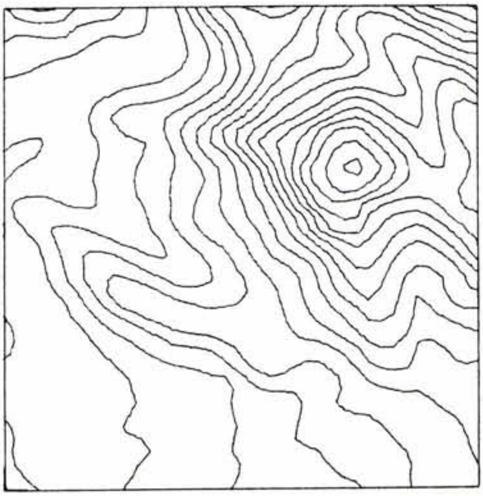

FIG. 8. TOPOCOM contouring program. Edited and smoothed 1,600 data-point set, 0.74 minute, 20-m contour interval.

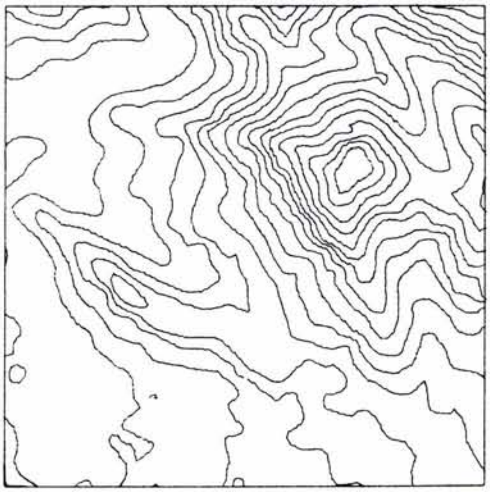

FIG. 9. TOPOCOM contouring program. Edited and smoothed 6,400 data-point set, 1.17 minutes, 20-m contour interval.

respectively, on the University of Virginia CDC 6400 computer. A CALCOMP 570 plotter was used to plot the contour maps. It is important to note that the mathematical discontinuity in slope perpendicular to the boundaries of the regions of validity is not visible in any of these plots, supporting the arguments that this difference is effectively minimized.

Figures 7, 8 and 9 are computer outputs of TOPOCOM's state-of-the-art Contouring Program.[2] Figure 7 is TOPOCOM's output using the same 1,600-point data set used to produce the continuous function results in Figures 4-6. Figure 8 is the result using a 1,600 point data set produced by editing and smoothing the original 1,600 point data set. Figure 9 is the result, on an edited and smoothed 6,400 data point set over the same region. The computer run times for these three figures were 0.75, 0.74, and 1.17 minutes, respectively, on an IBM 7094.

Comparison of these results shows that the mathematical surface modeling technique gives contours of superior quality to those determined using conventional techniques on the original data. This is evident not only in smoothness and parallelism of contours, but also in the ability of the modeling techniques to reliably predict the surface between data points. Comparison of Figures 4 and 9 reveals that the mathematical surface model reasonably predicts altitude and slope between the original data points.

Conclusion

The functional surface-modeling technique has been shown to be an efficient basis for automated mapping. The flexible nature of the method allows a given data set to be modeled to any prescribed standard deviation.

This method does not embody any of the difficulties associated with conventional methods for approximating surfaces using functions. The family of locally valid surfaces *fit together* in a smooth manner and interpolate between the original data to provide a realistic and accurate mathematical model of topography and are an efficient basis for contour determinations.

With the development and application of a functional surface model the authors offer an alternative method for the basis of automated contouring which provides an accurate and aesthetically pleasing product from noisy data.

The authors gratefully acknowledge the comments and criticisms of Messrs. Don Light and Merle Biggin of USATOPCOM.

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Proceedings of Canadian Symposium

The Proceedings of the *Discussion of Man-Machine Interface in Photogrammetry*, held at the Department of Surveying Engineering, University of New Brunswick, Fredericton, N.B., Canada, on 7-9 August, 1972 are now available. They may be ordered for Can. $5.00 (plus postage) through the above address. The 200 pages of the booklet contain eight up-to-date papers on topics related to the on-line use of computers in photogrammetry, as well as transcriptions of the lively discussions.