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A Tensor Approach to Block Triangulation

A program has been written to perform all computations required for the simultaneous adjustment of strips and blocks of 24 photographs or less using a 32K computer and operating entirely within its central memory.

INTRODUCTION

THE UNITED STATES Department of the Interior, Bureau of Land Management (BLM), Division of Cadastral Surveys, has legal responsibility for the survey and resurvey of the public lands of the United States for purposes of identification and description. In support of this mission the BLM is continuously engaged in the improvement and modernization of cadastral survey methods to obtain greater precision, improved efficiency, and savings in time, money, and manpower.

Analytical aerotriangulation methods have been among the most promising of the new techniques adopted by the BLM for this purpose in recent years. Following an evolutionary period during which analytical systems developed by the Coast and Geodetic Survey and the National Research Council of Canada were used, the specific requirements of the BLM were determined and a new analytical system was developed which combined many of the best features of the two existing systems plus a number of new features designed to satisfy the particular needs of the BLM.

ABSTRACT: *A computer program was written to perform all computations required for the simultaneous adjustment of small strips or blocks of photographs using a 32K computer. Vector algebra was used extensively in the analytical development of the program, with vector equations being expressed in tensor form using the summation convention. The simultaneous adjustment method adopted is based on the principle of collinearity. It utilizes least-squares methods and sparse matrix-compression techniques.*

Foremost among BLM requirements was the development of a relatively small strip and block adjustment program which could be used on small to medium sized (32K) computers; and from which greater precision could be obtained than was possible using existing polynomial methods.[4]

For this purpose a simultaneous adjustment method based on the principle of collinearity was adopted, and a computer program was written to perform all computations required for the adjustment of a strip or block of up to 24 photographs arranged in not more than four strips.

Vector algebra was used extensively in the analytical development of the computer program, with vector equations being expressed in tensor form, using Einstein's summation convention. Derivations of the equations used, and the analytical procedures employed are described herein.

OBJECTIVE OF STRIP OR BLOCK AEROTRIANGULATION

The primary objective of strip or block aerotriangulation is the precise determination of the ground coordinates of a set of points whose images appear on a strip or block of near-vertical aerial photographs using, as a basis for determination, the true ground coordinates of a relatively small subset of the point set, the approximate ground coordinates of the remainder of the point set, and a well-defined, relative-position relationship among all points in the set.

A BRIEF DESCRIPTION OF THE ADJUSTMENT PROCESS

The simultaneous adjustment of a strip or block of photographs, as performed by BLM, can be described from a physical point of view as a simultaneous, least-squares adjustment of the aerotriangulation network to agree as closely as possible with established ground control.

The aerotriangulation network consists of all lens-image-object rays through all control points, pass points and tie points in the block. It imposes a constraining influence which tends to maintain the proper relative-position relationship among these points, as established by photogrammetry, in such a way that, as the network is adjusted to fit ground control, all other points approach their true ground positions.

The adjustment of the aerotriangulation network is accomplished mathematically by holding unchanged the known ground coordinates of all control points, and applying position corrections to all other points consistent with the relative-position constraints imposed by the network.

Corrections are computed for a set of approximate ground coordinates of all pass points and tie points imaged on the block of photos, and for a set of approximate coordinates and orientation angles of the camera lens at each exposure station.

A set of observation equations (linearized equations of collinearity) which define the mathematical relationship between selected ground objects, their photographic images, and the camera lens, the positions and orientations of which are, for the most part, only approximately known, are formed and solved by least-squares methods to obtain the set of corrections for the approximate values that most nearly satisfy all of the observation equations (in the least-squares sense).

An iterative procedure is employed by which the corrections are computed, applied to the initial approximations, and a new set of corrections computed. This process is continued until the magnitude of the corrections diminishes to a predetermined level.

DATA REQUIREMENTS FOR BLOCK ADJUSTMENT

The data required for a simultaneous strip or block adjustment, as performed by BLM, includes the following:

- Precise three dimensional rectangular Cartesian ground (object space) coordinates of all control points,
- Approximate ground coordinates of all pass points and tie points needed for block formation,
- Refined photographic image coordinates of all control points, pass points, and tie points needed for block formation, and
- Approximate ground (object space) coordinates and orientation angles of the camera lens at the time of exposure of each photograph.

SOURCES OF DATA

Ground coordinates of control points are obtained by survey or from published descriptions of triangulation stations and bench marks. Approximate ground coordinates of pass points and tie points are obtained from preliminary absolute orientation computations or from other strip adjustments. Refined image coordinates are photo coordinates of image points (referred to the photo center of the photographic plate) which have been measured with a comparator and corrected for film shrinkage, atmospheric refraction, and lens distortion.

Approximate ground coordinates and orientation angles of the camera lens at the time of exposure of each photo are computed by resection using the ground coordinates and image coordinates described above. Resection computations are included in the block aerotriangulation program.

DATA REQUIRED FOR SUBSEQUENT COMPUTATIONS

During the block-adjustment phase of the computations the only points considered are the control points, pass points and tie points required for block formation. After completion of the adjustment phase, ground coordinates are computed for all other points of interest. For this purpose refined image coordinates are needed for all the other required points whose images appear on two or more photographs.

COMPUTATIONAL RESULTS

Upon completion of the block-adjustment phase, the ground coordinates of all pass points and tie points used to form the block have been determined. Also the ground (object space) coordinates and orientation angles of the camera lens have been determined for all exposure stations. Subsequent computations include the determination of ground coordinates of all other points of interest not considered in the adjustment phase. For an evaluation of precision, the final results include ground coordinate residual values for all control points, image coordinate residual values for all image points, and the root-mean-square values (*RMS*) of all image coordinate residual values.

An outline of the aerotriangulation routine is presented next and is followed by derivations of equations used and descriptions of various analytical procedures employed throughout the computational processes. Tensor notation and Einstein's summation convention appear throughout. The range of values of free indices and repeated indices are $k=1,2$ and $i,j=1,2,3$ unless otherwise indicated.

Tensor equations are presented in expanded form in Appendix I, and partial derivatives are evaluated in Appendix II.*

OUTLINE OF AEROTRIANGULATION ROUTINE

DATA INPUT PHASE

The aerotriangulation process is initiated with the input of ground coordinates, x_i^P $(i=1,2,3)$, and refined photographic image coordinates, y_k^P $(k=1,2)$, of all control points, pass points and tie points needed for block formation, in addition to the focal length of the camera lens f and the various weight factors to be applied.

Also needed are the approximate ground coordinates x_i^C and the orientation angles α_i of the camera at each exposure station. These values are computed for each photograph by resection in the manner described below, using the ground coordinates and refined image coordinates of six selected pass points on each photo.

RESECTION PHASE

Resection describes the mathematical process used to compute the angular orientation of a photo coordinate system relative to the ground coordinate system, and the ground coordinates of the spatial point of intersection of the bundle of lens-image-object rays at the focal point of the camera lens.

A set of initial approximations for the position of the camera lens, in terms of ground coordinates, and the angular orientation of the lens axis, in terms of orienta-

* Appendices included in the original paper have been omitted for reasons of brevity: Tensor equations expanded; Rotation matrix and its partial derivatives; and Notation. Interested readers may obtain this material from the author at USDI Bureau of Land Management, P.O. Box 3861, Portland, Oregon 97208.—*Editor.*

tion angles, are computed for each photograph using Equations 1–6:

$$x_1^C = x_1^3 \tag{1}$$

$$x_2^C = x_2^3 \tag{2}$$

$$x_3^C = \frac{fD}{d} + x_3^3 \tag{3}$$

$$\alpha_1 = 0 \tag{4}$$

$$\alpha_2 = 0 \tag{5}$$

$$\alpha_3 = \text{arc cos}\left[\frac{(x_k^2 - x_k^1)(y_k^2 - y_k^1)}{Dd}\right] \tag{6}$$

in which

$$d = \sqrt{y_k^1 y_k^1 - 2y_k^1 y_k^2 + y_k^2 y_k^2} \tag{7}$$

$$D = \sqrt{x_k^1 x_k^1 - 2x_k^1 x_k^2 + x_k^2 x_k^2} \qquad (k=1,2) \tag{8}$$

f = focal length of camera lens,
x_i^1 = ground coordinates of point 1, the center pass point along the trailing edge of the photograph,
x_i^2 = ground coordinates of point 2, the center pass point along the leading edge of the photograph,
x_i^3 = ground coordinates of the pass point nearest the photo center,
y_i^1 = refined photographic image coordinates of point 1,
y_i^2 = refined photographic image coordinates of point 2.

Using these initial approximations, a system of equations is formed and solved for a set of corrections which are applied to the initial values to obtain improved approximations. Two equations, (Equation 9), in three linear corrections Δx_i^c and three angular corrections $\Delta\alpha_i$ are written for each of the six selected pass points:

$$\frac{\partial F_k}{\partial x_i^C}\,\Delta x_i^C + \frac{\partial F_k}{\partial \alpha_i}\,\Delta\alpha_i = -F_k \qquad (k=1,2) \tag{9}$$

in which

$$F_k = y_k^P[t_{3j}(x_j^P - x_j^C)] + f[t_{kj}(x_j^P - x_j^C)] \tag{10}$$

$$\frac{\partial F_k}{\partial x_i^C} = -y_k^P t_{3i} - f t_{ki} \tag{11}$$

$$\frac{\partial F_k}{\partial \alpha_i} = y_k^P\left[(x_j^P - x_j^C)\,\frac{\partial t_{3j}}{\partial \alpha_i}\right] + f\left[(x_j^P - x_j^C)\,\frac{\partial t_{kj}}{\partial \alpha_i}\right] \tag{12}$$

$$t_{ij} = \begin{bmatrix} c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\ -c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix} \tag{13}$$

$$c_i = \cos\,\alpha_i, \text{ and } s_i = \sin\,\alpha_i.$$

A system of 12 equations is thus formed and solved (by least-squares methods) for these six corrections, which are then applied to the initial approximations after which the computations are repeated. This process is continued until the magnitude of the corrections diminishes to a specified value at which time a reasonably accurate set of values for the position and orientation of the camera lens has been obtained.

The resection computations are continued until a complete set of approximate values has been determined for all photographs in the strip or block. Upon completion of the resection phase the adjustment phase is begun.

ADJUSTMENT PHASE FORMATION OF SYSTEM OF OBSERVATION EQUATIONS FOR BLOCK ADJUSTMENT

The adjustment phase begins with the formation of a system of observation equations (linearized equations of collinearity) which describe the collinear condition of all lens-image-object rays through all of the control points, pass points and tie points needed for block formation. Using the approximate values obtained by resection, plus the ground coordinates and refined image coordinates of all ground points which were input initially, two equations (Equation 14) in six linear corrections, Δx_i^P and Δx_i^C, and three angular corrections, $\Delta \alpha_i$, are written for every lens-image-object ray:

$$\frac{\partial F_k}{\partial x_i^P} \Delta x_i^P + \frac{\partial F_k}{\partial x_i^C} \Delta x_i^C + \frac{\partial F_k}{\partial \alpha_i} \Delta \alpha_i = -F_k \qquad (k=1,2) \tag{14}$$

in which

$$\frac{\partial F_k}{\partial x_i^P} = y_k^P t_{3i} + f t_{ki} \tag{15}$$

and y_k^P represents the refined image coordinates of point P on photograph C. This usually results in the formation of a very large system of equations, two equations in nine unknowns for every selected image on every photograph.

COLLINEARITY WEIGHT APPLIED TO OBSERVATION EQUATIONS

Because control points are nearly always paneled, their photographic images are generally more sharply defined than non-targeted images; hence, their image coordinates can be more precisely measured. Also, ground coordinates of control points are more precisely known than ground coordinates of non-control points. It follows, therefore, that the subset of observation equations describing the collinear condition of lens-image-object rays through control points tend to be more accurate than the observation equations that describe the collinear condition of rays through other points.

The potentially beneficial effects of control-point equations upon the adjustment are minimized, however, by the disproportionately greater number of non-control-point equations. Coefficients of control-point equations are therefore multiplied by a weight factor which increases the influence of control-point collineation upon the adjustment of the aerotriangulation network.

FORMATION OF SYSTEM OF NORMAL EQUATIONS

As an object is usually imaged on several photographs, the number of observation equations invariably exceeds the number of unknown corrections. Least-squares methods are therefore employed to reduce the system of observation equations to a system of normal equations which can be solved for the set of corrections that will most nearly satisfy all of the observation equations (in the least-squares sense).

POSITION WEIGHT APPLIED TO NORMAL EQUATIONS

The primary objective of block aerotriangulation, as was previously stated, is the precise determination of the ground coordinates of a large number of points whose positions are only approximately known. This is achieved through block adjustment by adjusting the aerotriangulation network to fit ground control as closely as possible. Simultaneously, through the constraining influence of the aerotriangulation network that maintains a proper relative position relationship among points, all other points are moved into their proper positions in relation to the control points and to each other.

The adjustment of the aerotriangulation network to fit ground control is accomplished here mathematically by multiplying* the normal equation matrix diagonal coefficients of control point ground coordinate correction terms by a weight factor to minimize the magnitude of control point position corrections. Position corrections are therefore largely limited to non-control points. These corrections are equal to the changes in position necessary to move the non-control points into their proper positions, as defined by the system of equations.

SYSTEM OF NORMAL EQUATIONS SOLVED TO OBTAIN CORRECTIONS

The system of weighted normal equations is now solved to obtain corrections to be applied to all approximate values used in the formation of the observation equations. This includes corrections for the approximate ground coordinates of all pass points, tie points, and exposure stations, and corrections for the approximate orientation angles of the camera lens at all exposure stations.

The corrections are then applied to obtain an improved set of approximations and the computations are repeated. This process is continued until the magnitude of the corrections diminishes to a predetermined level, at which time the corrected values are accepted as the final results of the block adjustment.

INTERSECTION PHASE

During the adjustment phase the only points considered are control points, pass points, and tie points needed for block formation. The ground coordinates of all other points of interest whose images appear on two or more photographs are obtained from subsequent computations. Each point is considered independently. The refined photographic image coordinates of a point on all photos on which its image appears are needed for these computations, and are input at this time.

Using this set of image coordinates in addition to the ground coordinates and orientation angles of the camera lens at each exposure station (computed during the adjustment phase), a set of equations in terms of the unknown ground coordinates of the point of interest are formed and solved by least-squares methods as described next. Two collinearity equations (Equation 16) in the unknown ground coordinates x_i^P are written for each image:

$$x_i^P (y_k^P t_{3i} + f t_{ki}) = x_i^C (y_k^P t_{3i} + f t_{ki}) \qquad (k=1,2). \tag{16}$$

A system of 4 to 18 equations in the three unknowns are thus formed as an object can appear on two to nine photographs. Employing least-squares methods, the system of collinearity equations is reduced to three normal equations and solved for the three unknown ground coordinates that most nearly satisfy all of the collinearity equations (in the least-squares sense). The ground coordinates of an unlimited number of points can be computed in this way.

PRECISION EVALUATION PHASE

An indication of the degree of agreement or lack of agreement between various computed values and known or measured values is desirable for assistance in evaluating the computational precision obtained. Three indicators are computed for this purpose. They are: the ground coordinate residual values for all control points, the image coordinate residual values for all image points, and the *RMS* values of all image coordinate residual values of all points used for block formation.

Ground coordinate residual values are differences between computed ground coordinates and surveyed ground coordinates of control points. The computed

* Although a value is *added* in least-squares theory, a corresponding result is achieved here by *multiplication* with some computational simplification.—*Editor.*

ground coordinates used for this purpose are obtained by intersection using the camera orientation angles and the exposure station positions computed during the adjustment phase, and are compared with the true ground coordinates which were input initially.

Image-coordinate residual values are differences between computed image coordinates and refined image coordinates of an object on a particular photograph. Computed image coordinates y_k^P are obtained using the following equations:

$$y_k^P = \frac{t_{kj}(x_j^P - x_j^C)}{t_{3i}(x_i^P - x_i^C)} f \qquad (k=1,2). \tag{17}$$

Two image coordinates are computed for each image of an object on each photo on which the object appears. These values are compared with the corresponding refined image coordinates which were input initially.

Ground-coordinate residual values and image-coordinate residual values of control points are sensitive to the weight factors applied to enforce the collinearity and position of control points. Large collinearity weight factors reduces image-coordinate residual values of control points and, if combined with a large position weight factor, not only forces agreement with ground control but also reduces ground-coordinate residual values, whether or not the ground coordinates are correct.

Image-coordinate residual values are recorded and the root-mean-square residual value is computed for the entire block. This computation concludes the block aerotriangulation process.

Derivations of Equations and Descriptions of Various Analytical Procedures

EQUATIONS OF COLLINEARITY

The analytical aerotriangulation system adopted by the Bureau of Land Management is based on the projective-geometry principle of collinearity. Equations of collinearity are applied in analytical aerotriangulation to enforce the condition that the camera lens C, a ground object P, and its photographic image I, are collinear; i.e., that all three points lie on the same straight line (Figure 1). This condition is illustrated in Figure 1. The equations of collinearity appear in various forms throughout the analytical processes, and derivations of these equations are considered next.

The collinearity equations in the most general form can be derived using vector algebra in the following manner. The coordinates of point C and point P in a rectangular Cartesian ground coordinate system (object space coordinate system) are x_i^C and x_i^P (i=1,2,3), respectively. The coordinates of points C, P, and I are y_i^C, y_i^P, and y_i^I in a rectangular, Cartesian, photographic-image coordinate system (image-space coordinate system).

In terms of image-coordinate components, the vector $\vec{V}$ from point C to point P can be written in the form,

$$\vec{V} = (y_i^P - y_i^C)\hat{e}_i \tag{18}$$

and the vector $\vec{v}$ from point C to point I in the form,

$$\vec{v} = (y_i^I - y_i^C)\hat{e}_i \tag{19}$$

in which the $\hat{e}_i$ are unit vectors in the directions of the Y_i (image coordinate) axes.

If the three points C, I, and P are collinear, the vectors $\vec{V}$ and $\vec{v}$ are related by the equation

$$\vec{V} = \lambda\vec{v} \tag{20}$$

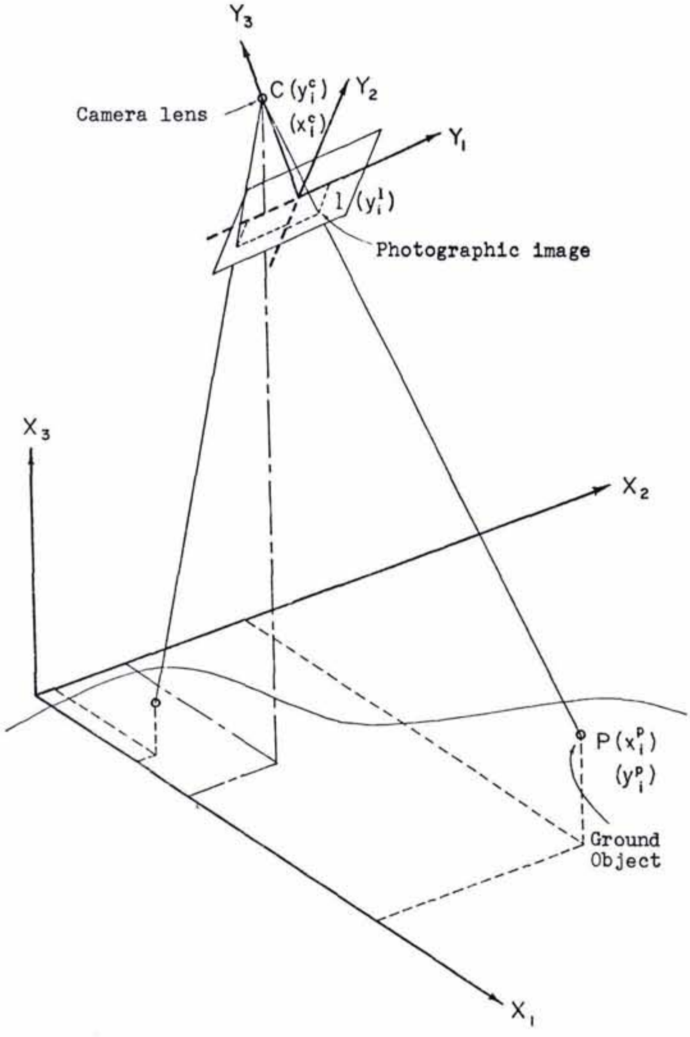

FIG. 1. Collinearity of a ground object P, its photographic image I and the camera lens C.

in which λ is a constant multiplier. In terms of components, this equation can be written in the form

$$(y_i^P - y_i^C)\hat{e}_i = \lambda(y_i^I - y_i^C)\hat{e}_i. \tag{21}$$

The ground-coordinate system and the image-coordinate system are related by the transformation equations

$$y_i = t_{ij}x_j + d_i \tag{22}$$

in which the t_{ij} are direction cosines of the angles measured from each of the coordinate axes of the ground-coordinate system to all three coordinate axes of the image-coordinate system, and the d_i are translation terms representing the displacement of the origin of the X_i coordinate system with respect to the origin of the Y_i coordinate system in terms of y_i coordinates.

Applying this transformation, the vector $\overrightarrow{V}$ can be expressed in terms of ground coordinate components,

$$V = t_{ij}(x_j^P - x_j^C)\hat{e}_i. \tag{23}$$

Substituting the right hand side of Equation 23 for the left hand side of Equation 21, the equations

$$ti_j(x_j^P - x_j^C)\hat{e}_i = \lambda(y_i^I - y_i^C)\hat{e}_i \tag{24}$$

are obtained. Equating corresponding components:

$$t_{1j}(x_j^P - x_j^C) = \lambda(y_1^I - y_1^C) \tag{25a}$$

$$t_{2j}(x_j^P - x_j^C) = \lambda(y_2^I - y_2^C) \tag{25b}$$

$$t_{3j}(x_j^P - x_j^C) = \lambda(y_3^I - y_3^C). \tag{25c}$$

If the left side of Equation 25*a* is now divided by the left side of Equation 25*c*, and the right side of Equation 25*a* is divided by the right side of Equation 25*c*, we obtain the equation:

$$\frac{{}_t t_{1j}(x_j^P - x_j^C)}{t_{3i}(x_i^P - x_i^C)} = \frac{(y_1^I - y_1^C)}{(y_3^I - y_3^C)}. \tag{26}$$

A similar division of Equation 25*b* by Equation 25*c* poduces the equation:

$$\frac{t_{2j}(x_j^P - x_j^C)}{t_{3i}(x_i^P - x_i^C)} = \frac{y_2^I - y_2^C}{y_3^I - y_3^C}. \tag{27}$$

If the photocenter is selected as the origin of the image-space coordinate system, the coordinates y_1^C, y_2^C and y_3^I can all be set equal to zero and $y_3^C = f$, the focal length of the camera. It is also convenient at this point to use the superscript P instead of I to represent the image coordinates of point P, i.e., $y_i^P = y_i^I$.

Substituting these values for the corresponding coordinates in Equations 26 and 27, we obtain:

$$y_k^P = \frac{t_{kj}(x_j^P - x_j^C)}{t_{3i}(x_i^P - x_i^C)}(-f); \quad (k=1,2). \tag{28}$$

In partially expanded form,

$$y_k^P = \frac{t_{k1}(x_1^P - x_1^C) + t_{k2}(x_2^P - x_2^C) + t_{k3}(x_3^P - x_3^C)}{t_{31}(x_1^P - x_1^C) + t_{32}(x_2^P - x_2^C) + t_{33}(x_3^P - x_3^C)}(-f). \tag{29}$$

Equation 28 can also be expressed in the form:

$$y_k^P[t_{3j}(x_j^P - x_j^C)] + f[t_{kj}(x_j^P - x_j^C)] = 0 \tag{30}$$

and

$$x_j^P(y_k^P t_{3j} + f t_{kj}) = x_j^C(y_k^P t_{3j} + f t_{kj}) \tag{31}$$

which is the form used for intersection. Equation 28–31 are variations of the familiar equations of collinearity.

COORDINATE TRANSFORMATIONS

Equations 28–31 seem to be nonlinear functions of the 17 variables x_i^C, x_i^P, y_k^P and $t_{ij}(k=1,2)$. Actually only 11 independent variables are in these equations, however, because the nine t_{ij} terms are themselves functions of only three variables—the angles α_i through which the X_i coordinate axes would have to be rotated about themselves if they were to be oriented parallel to the Y_i coordinate axes.

The t_{ij} have been previously defined as direction cosines. They first appear in Equation 22, the transformation equations used to express the image space coordinates y_i^P of a ground point P in terms of ground coordinates x_i^P.

The t_{ij} can also be described as elements of a rotation matrix $[t_{ij}]$. They can be evaluated in terms of the angular rotations α_i by taking the product of the three rotation matrices that would transform ground coordinates into image coordinates by increments, comparable to rotating pairs of X-coordinate axes about the third X axis through the angles α_i, one rotation at a time, until the X_i axes are parallel to the Y_i axes (Figure 2).

After rotation the transformed x_i coordinates x_i' can be equated to corresponding y_i coordinates:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{32}$$

in which $c_i = \cos\alpha_i$ and $s_i = \sin\alpha_i$. The product of the three rotation matrices is the rotation matrix $[t_{ij}]$

$$[t_{ij}] = \begin{bmatrix} c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\ -c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\ s_2 & -s_1c_2 & c_1c_2 \end{bmatrix}. \tag{33}$$

The angles α_i correspond to the angles ω, ϕ, κ in more conventional notation.

LINEARIZATION OF COLLINEARITY EQUATIONS

Two collinearity condition equations (Equation 30) are written for each selected photographic image. This will obviously produce a very large system of equations upon being multiplied by the number of images in a block of photographs. The

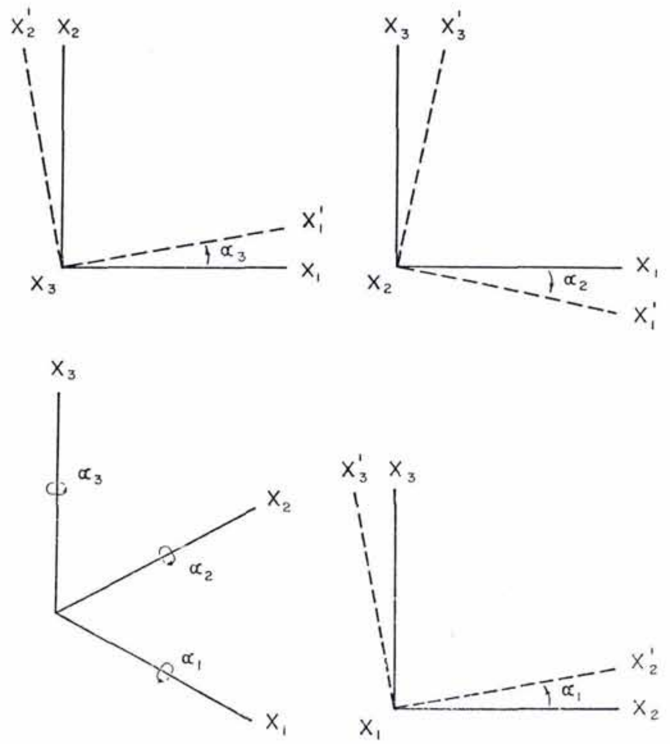

FIG. 2. Positive directions of the rotations of the axes.

solution of this system of equations is the primary objective of subsequent mathematical processes.

In order to solve the system of equations most efficiently, Equations 30, which are nonlinear, must first be linearized so that the methods of linear algebra developed for handling large systems of linear equations can be applied.

If the functions of Equations 30 have continuous partial derivatives in a sufficiently large neighborhood of points x_i^C, x_i^P, y_i^P, α_i to include a reasonable range of values $x_i^C + \Delta x_i^C$, $x_i^P + \Delta x_i^P$, $y_i^P + \Delta y_i^P$, $\alpha_i + \Delta\alpha_i$, linear approximations can be derived by a Taylor series expansion:

$$F_k(x_i^C, x_i^P, y_i^P, \alpha_i) + dF_k(x_i^C, x_i^P, y_i^C, \alpha_i) \approx 0 \tag{34}$$

in which

$$dF_k = \frac{\partial F_k}{\partial x_i^C}\,\Delta x_i^C + \frac{\partial F_k}{\partial x_i^P}\,\Delta x_i^P + \frac{\partial F_k}{\partial y_i^P}\,\Delta y_i^P + \frac{\partial F_k}{\partial \alpha_i}\,\Delta\alpha_i \tag{35}$$

Equations 34 are linear in the corrections Δx_i^C, Δx_i^P, Δy_i^P, $\Delta\alpha_i$. The constant term F_k and the coefficients of the corrections terms are given by Equations 10 to 13 and Equation 15.

For each image on each photograph two linear equations (Equation 34) can be written in the unknown corrections Δx_i^C, Δx_i^P, Δy_i^P, $\Delta\alpha_i$, with constant coefficients evaluated using the approximations x_i^C, x_i^P, y_i^P, α_i. Equations 34 and 35 express the collinear condition of all lens-image-object rays in a block of photographs in most general terms. For special purposes these equations are modified by evaluating those variables that assume predictable values.

RESECTION

If the coordinates of three or more ground points appearing on a photograph are known, and if the refined image coordinates of the corresponding photographic images are known, the position in terms of ground coordinates x_i^C (object-space coordinates) and the orientation angles α_i of the camera at the time of exposure can be computed by resection using a special forms of Equations 34 and 35 (Figure 3).

For resection purposes it is assumed that all ground coordinates x_i^P and all image coordinates y_i^P remain unchanged, whereupon the corrections $\Delta x_i^P = 0$ and $\Delta y_i^P = 0$. If these values are substituted into Equations 34 and 35, the resection equations are obtained:

$$\frac{\partial F_k}{\partial x_i^C}\,\Delta x_i^C + \frac{\partial F_k}{\partial \alpha_i}\,\Delta\alpha_i = -F_k. \tag{36}$$

EQUATIONS FOR BLOCK ADJUSTMENT

For strip and block adjustment purposes it is assumed that only the image coordinates y_i^P do not change. Hence, only the image-coordinate correction terms Δy_i^P of Equations 34 and 35 are set equal to zero. In this way a set of observation equations is obtained in the form desired for the adjustment phase of the computations:

$$\frac{\partial F_k}{\partial x_i^C}\,\Delta x_i^C + \frac{\partial F_k}{\partial x_i^P}\,\Delta x_i^P + \frac{\partial F_k}{\partial \alpha_i}\,\Delta\alpha_i = -F_k. \tag{37}$$

Two linear observation equations (Equation 37) in the nine unknown corrections Δx_i^C, Δx_i^P, $\Delta\alpha_i$ are written for each selected image on each photograph.

Only those points needed to form the strip or block are considered at this point in order to minimize the number of equations to be solved simultaneously. These include pass points, tie points, and control points. All other points are considered in subsequent computations.

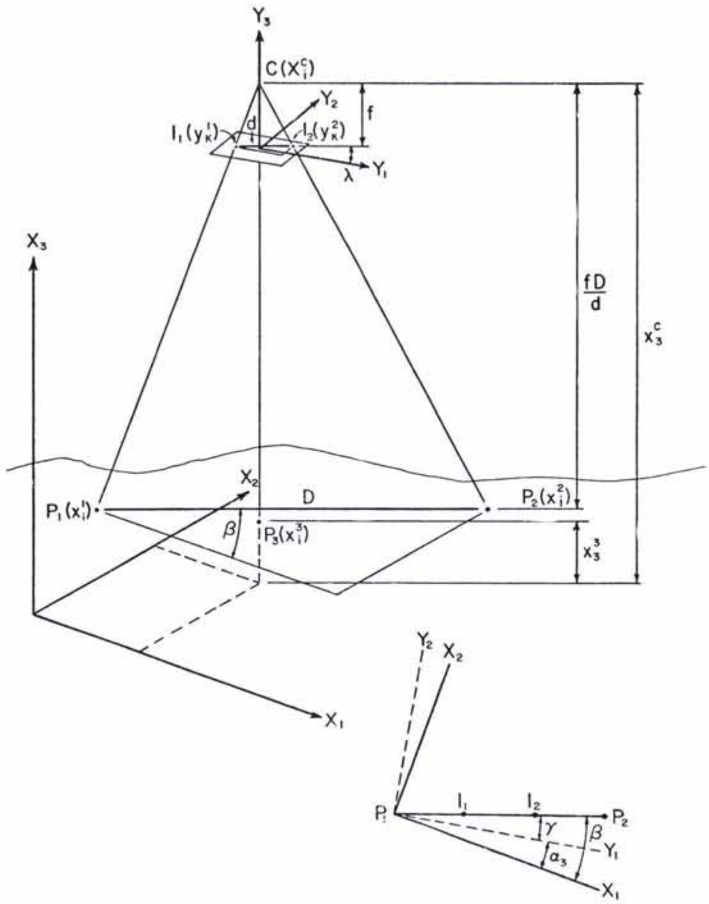

FIG. 3. Simplified geometric relationships used to obtain the initial resection approximations.

SOLUTION OF SYSTEM OF NORMAL EQUATIONS

The system of normal equations is solved for the corrections which most nearly satisfy all of the observation equations. This operation requires more computational effort than all other operations combined. Several methods for solving large systems of linear equations have been used. These include the Gauss-Jordan elimination method, the Cholesky elimination method, and a combination of partitioning, elimination and intersection. For small blocks (24 photos or less) the differences in numerical results obtained using these various methods apparently are not significant, as they might be for larger blocks where roundoff error can become a problem. Differences in processing time required by the various methods are also small.

TEST MODELS USED FOR VERIFICATION OF RESULTS

To test the precision of the analytical solution and the performance of the computer program, several topographic tests models were created on which strip and block adjustments were performed, as discussed next.

An area was selected on a topographic map and subdivided into a grid representing photographic coverage. For each photographic area, pass-point locations and exposure-station locations were selected and assigned object-space coordinates and orientation angles. A small number of control points were also selected, and object-space coordinates assigned. With these assigned coordinates and orientation angles, a set of image coordinates of all pass points and control points were computed using the equations of collinearity (Equation 28). Subsequently, random errors from 0 to 100 feet were introduced into the ground coordinates of all pass points producing a set of approximate ground coordinates.

Strip and block adjustments were then performed using as input the computed image coordinates, the approximate ground coordinates of pass points, and the assigned ground coordinates of control points. In every instance the assigned ground coordinates of all pass points were obtained from the strip or block adjustment with image coordinate residual values and ground coordinate residual values approaching zero. Also, the assigned exposure station positions and orientation angles were obtained.

A number of real strips and blocks with excess control have also been adjusted, withholding varying numbers of points of known position for the purpose of comparing computed coordinates of those points with surveyed coordinates. Excellent agreement has always been obtained.

SUMMARY

A computer program has been written to perform all computations required for the simultaneous adjustment of strips and blocks of 24 photographs or less using a 32K computer and operating entirely within central memory. Excellent results have been obtained from all tests of analytical precision and program performance. The average computational time required for the adjustment of a 24-photo block is approximately 30 seconds using a CDC 6400 computer.

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