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Lens Distortion and Close Range

The variation of symmetrical lens distortion with object distance can be estimated with improved accuracy for close-range applications.

ABSTRACT: *A. A. Magill[4] showed that, given a chief ray for small aperture opening, the symmetrical lens distortion changes linearily with the magnification. D. C. Brown[2] extended Magill's formula so that the symmetrical lens distortion polynomial at any principal distance can be calculated from known values of the lens distortion polynomial at any two principal distances. Analyzing the relations involved, it was found that, although Brown's formula gave good results for the principal distances within the range of the two known given principal distances, the results are not as good for an object distance beyond that range. This article gives the proof of a new formula by which the symmetrical lens distortion polynomial can be estimated with much higher accuracy than that obtained by Brown's formula. A comparison between the results obtained by using Brown's formula and the author's formula is also included.*

INTRODUCTION

D. C. BROWN [2] developed a new formula for the variation of symmetrical lens distortion for the points outside the focus plane. In his formula one needs to know the symmetrical lens distortion at any principal distance. For that reason Brown extended Magill's formula [4] so that one can estimate the symmetrical lens distortion at any principal distances if the symmetrical lens distortion at any two principal distances are given. Examining Brown's formula, the author found out that the formula gives good results for symmetrical lens distortion as long as the principal distance given is between the range of the two known principal distances. But the results obtained for any principal distances outside that range are not so good. This article gives the proof of a new formula developed by the author and published in 1971 [1] in his work with Bendix Research Laboratory. The same formula was published by Brown [3] in 1972.

A subsequent section gives the theoretical basis of the new formula—which is also an extension of Magill's formula. This formula gives a very good result for any principal distance outside, as well as inside, the range. The theoretical basis and the results of Brown's extension formula and the author's extension formula are given in the next sections.

MAGILL FORMULA

Magill proved that at any incident angle u, the radial lens distortion dr can be computed using the following formula:

$$dr = (p \tan u - f \tan u) - (p \tan u - f \tan u)m. \quad (1)$$

Referring to Figure 1, u and u are the angles between the chief ray and the optical axis in the object and the image space respectively. The terms p and p are the shift distances of the pupils from focus in the object and the image space respectively and x and x are the distances of the object and the image planes from their perspective fuci respectively. N and N are the nodal points and F and F are the focal points. The distance f is the equivalent focal length of the lens system. The magnification m can be computed as $m = x/f = f/x$.

It can be seen from Equation 1 that for

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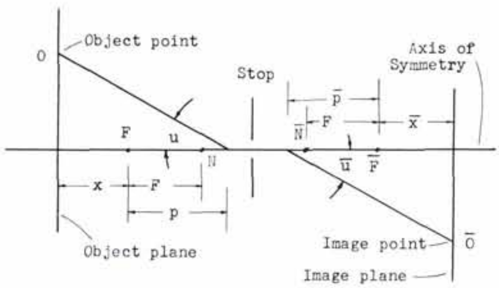

FIG. 1. Optical relationship of object and image.

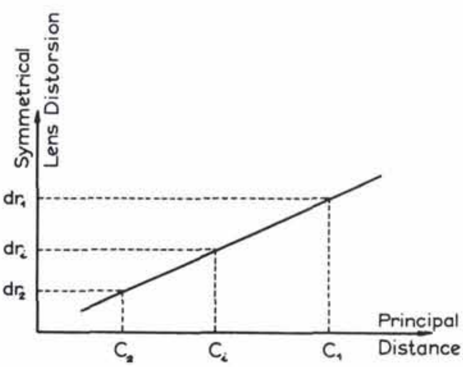

FIG. 2. Straight-line relationship of principal distance and symmetrical lens distortion.

any chief ray all the terms constant with the exception of *dr* and *m* which are linearly related as follows:

$$\mathrm{dr} = \mathrm{a} + \mathrm{bm} \qquad (2)$$

where

$$\mathrm{a} = (\mathrm{p} \tan \mathrm{u} - \mathrm{f} \tan \mathrm{u})$$

$$\mathrm{b} = (\mathrm{p} \tan \mathrm{u} - \mathrm{f} \tan \mathrm{u}).$$

Substituting x/f for m in Equation 2 gives

$$\mathrm{dr} = \mathrm{a} + \mathrm{b}(\mathrm{x}/\mathrm{f}).$$

The above equation can be reduced to

$$\begin{aligned}\mathrm{dr} &= \mathrm{a} + \mathrm{b}(\mathrm{x}/\mathrm{f}) + \mathrm{b} - \mathrm{b}\\ &= (\mathrm{a} - \mathrm{b}) + (\mathrm{b} + (\mathrm{x}/\mathrm{f})\mathrm{b})\\ &= (\mathrm{a} - \mathrm{b}) + \mathrm{b}(1 + (\mathrm{x}/\mathrm{f}))\\ &= (\mathrm{a} - \mathrm{b}) + \mathrm{b}(\mathrm{f} + \mathrm{x})/\mathrm{f}.\end{aligned}$$

Substituting principal distance C for $(f + x)$,

$$\begin{aligned}\mathrm{dr} &= (\mathrm{a} - \mathrm{b}) + \mathrm{b}(\mathrm{C}/\mathrm{f})\\ &= \mathrm{a} + \mathrm{bC} \qquad (3)\end{aligned}$$

where

$$\mathrm{a} = (\mathrm{a} - \mathrm{b})$$

$$\mathrm{b} = \mathrm{b}/\mathrm{f}.$$

Equation 3 states that for any chief ray there is a linear relation between *dr* and *C*, the principal distance.

ESTIMATION OF *dr* AT THE PRINCIPAL DISTANCE *C*

In order to estimate *dr* (the symmetrical radial lens distortion at the principal distance *C*) for a specific ray, one needs to know the values of symmetrical radial lens distortion *dr* for at least two points on the same ray. Let *dr* and *dr* be the values of symmetrical lens distortion for a specific ray at the principal distances C_1 and C_2 respectively.

Let *dr* be the value of symmetrical lens distortion of the same specific ray at the principal distance *C* . From Equation 3 one can see that the points *C* , *dr* , C_2, *dr* and *C* , *dr* can be represented as a straight line as in Figure 2.

From Figure 2 one can get the following relationship:

$$\frac{\mathrm{dr}_{s_1} - \mathrm{dr}_{s_2}}{\mathrm{C}_1 - \mathrm{C}_2} = \frac{\mathrm{dr}_{s_i} - \mathrm{dr}_{s_2}}{\mathrm{C}_i - \mathrm{C}_2}$$

$$\mathrm{dr}_{s_i} = \mathrm{dr}_{s_2} + \frac{\mathrm{C}_i - \mathrm{C}_2}{\mathrm{C}_1 - \mathrm{C}_2}(\mathrm{dr}_{s_1} - \mathrm{dr}_{s_2})$$

$$\mathrm{dr}_{s_i} = \frac{\mathrm{C}_i - \mathrm{C}_2}{\mathrm{C}_1 - \mathrm{C}_2}\mathrm{dr}_{s_1} + \left(1 - \frac{\mathrm{C}_i - \mathrm{C}_2}{\mathrm{C}_1 - \mathrm{C}_2}\right)\mathrm{dr}_{s_2}.$$

Letting

$$\mathrm{a}_{s_i} = \frac{\mathrm{C}_i - \mathrm{C}_2}{\mathrm{C}_1 - \mathrm{C}_2}, \qquad (4)$$

$$\mathrm{dr}_{s_i} = \mathrm{a}_{s_i}\,\mathrm{dr}_{s_1} + (1 - \mathrm{a}_{s_i})\,\mathrm{dr}_{s_2}. \qquad (5)$$

DETERMINATION OF THE COEFFICIENT OF SYMMETRICAL LENS DISTORTION POLYNOMIAL AT ANY PRINCIPAL DISTANCE *C*

Let *dr* , *dr* be the two known lens distortion polynomials at the principal distance C_1 and C_2, respectively, where

$$\mathrm{dr}_{s_1} = \mathrm{k}_{11}\mathrm{r}^3 + \mathrm{k}_{12}\mathrm{r}^5 + \mathrm{k}_{13}\mathrm{r}^7 + \cdots. \qquad (6)$$

$$\mathrm{dr}_{s_2} = \mathrm{k}_{21}\mathrm{r}^3 + \mathrm{k}_{22}\mathrm{r}^5 + \mathrm{k}_{23}\mathrm{r}^7 + \cdots. \qquad (7)$$

where *r* is the radial distance from the point of symmetry. The terms *k* , *k* , *k* , . . . are the coefficients of the symmetrical lens distortion polynomial at the principal distance *C* .

For any chief ray making an angle a with the optical axis,

$$dr_{s1} = k_{11}C_1^3 \tan^3 a + k_{12}C_1^5 \tan^5 a + k_{13}C_1^7 \tan^7 a + \cdots \quad (8)$$

$$dr_{s2} = k_{21}C_2^3 \tan^3 a + k_{22}C_2^5 \tan^5 a + k_{23}C_2^7 \tan^7 a + \cdots. \quad (9)$$

Substituting by the values of dr and dr from Equations 8 and 9 into Equation 5, one can obtain the following:

$$dr_{s_i} = a_{s_i}(k_{11}C_1^3 \tan^3 a + k_{12}C_1^5 \tan^5 a + k_{13}C_1^7 \tan^7 a + \cdots) + (1 - a_{s_i}) \times (k_{21}C_2^3 \tan^3 a + k_{22}C_2^5 \tan^5 a + k_{23}C_2^7 \tan^7 a + \cdots) \quad (10)$$

$$dr_{s_i} = (k_{11}C_1^3 a_{s_i} + k_{21}C_2^3(1 - a_{s_i})) \tan^3 a \times (C_1^5 k_{12} a_{s_i} + C_2^5 k_{22}(1 - a_{s_i})) \tan^5 a + (C_1^7 k_{12} a_{s_i} + C_2^7 k_{23}(1 - a_{s_i})) \tan^7 a + \cdots \quad (11)$$

$$dr_{s_i} = \left[k_{11}\left(\frac{C_1}{C_i}\right)^3 a_{s_i} + k_{21}\left(\frac{C_2}{C_i}\right)^3 (1 - a_{s_i})\right] C_i^3 \tan^3 a + \left[k_{12}\left(\frac{C_1}{C_i}\right)^5 a_{s_i} + k_{22}\left(\frac{C_2}{C_i}\right)^5 (1 - a_{s_i})\right] C_i^5 \tan^5 a + \left[k_{13}\left(\frac{C_1}{C_i}\right)^7 a_{s_i} + k_{23}\left(\frac{C_2}{C_i}\right)^7 (1 - a_{s_i})\right] C_i^7 \tan^7 a. \quad (12)$$

The above equation can be reduced to the following form:

$$dr_s = k_{s1}r^3 + k_{s2}r^5 + k_{s3}r^7 + \cdots \quad (13)$$

where

$$k_{s1} = \left(\frac{C_1}{C_i}\right)^3 a_s k_{11} + \left(\frac{C_2}{C_i}\right)^3 (1 - a_s)k_{21}$$

$$k_{s2} = \left(\frac{C_1}{C_i}\right)^5 a_s k_{12} + \left(\frac{C_2}{C_i}\right)^5 (1 - a_s)k_{22} \quad (14)$$

$$k_{s3} = \left(\frac{C_1}{C_i}\right)^7 a_s k_{13} + \left(\frac{C_2}{C_i}\right)^7 (1 - a_s)k_{23}.$$

These are the basic formulas for the relations between the polynomial coefficients.

Brown's Extension of Magill's Formula

Brown gave the following equations expressing the relation between the coefficients of Equation 13 and those of Equations 6 and 7:

$$k_{s1} = a_s k_{11} + (1 - a_s)k_{21}$$
$$k_{s2} = a_s k_{12} + (1 - a_s)k_{22} \quad (15)$$
$$k_{s3} = a_s k_{13} + (1 - a_s)k_{23}$$

where

$$a_s = \frac{s_2 - s_i}{s_2 - s_1} \cdot \frac{s_1 - f}{s_i - f}. \quad (16)$$

The term a given by equations 16 and 4 have the same values. This can be proven by substituting all values of C, C_1, and C_2 by their corresponding values of s and f given by this relation:

$$\frac{1}{s} + \frac{1}{C} = \frac{1}{f}.$$

Here s_1, s_2, s are the various distances between the object and the lens corresponding to the principal distance C_1, C_2, C.

Comparison Between Brown's Formula and the Author's Formula

Table 1 gives the values of k, for symmetrical lens distortion polynomial ($dr = k\ r^3$) obtained by camera calibration in Reference 2 for different object distances s.

In Columns 5, 6 and 7 of Table 2 are the values of k calculated using Brown's formula (Equation 15), the author's formula (Equation 12) and the calibrated values, respectively.

In Columns 1 and 2 of Table 2 are the known object distances s_1 and s_2 from which the values of k and k were computed.

Table 1. The Values of k for Symmetrical Lens Distortion Polynomial (dk = k r³) Obtained by Camera Calibration in Reference 2 for Different Object Distances s.

Object Distance (ft.)	*Principal Distance (inches)*	10^6 k *(mm)*$^{-2}$	10^6 o *(mm)*$^{-2}$
3	6.222	−0.628	0.0023
4	5.950	−0.719	0.0017
6	5.692	−0.825	0.0023
∞	5.30	−1.024	0.0028

° Obtained by plumb line method as in Reference 2.

TABLE 2. COMPARISON BETWEEN THE COEFFICIENT OF SYMMETRICAL LENS DISTORTION POLYNOMIAL k CALCULATED BY BROWN'S FORMULA AND THOSE OBTAINED BY CALIBRATION IN REFERENCE 2

Known dr_1 *and* dr_2 *at* S_1 *(ft.)*	S_2 *(ft.)*	*Unknown* dr *at (ft.)*	a	*Calculated* k *in* 10^{-6} $(mm)^{-2}$ *Brown's Formula*	*Author's Formula*	*Calibrated*
3	6	4	0.479	−0.731	−0.720	−0.719
		∝	−0.853	−0.996	−1.028	−1.024
3	4	6	−0.920	−0.802	−0.823	−0.825
		∝	−2.558	−0.952	−1.028	−1.024
3	∝	4	0.719	−0.739	−0.719	−0.719
		6	0.460	−0.842	−0.824	−0.825

Column 3 of Table 2 gives the focus distances s at which we want to know k . Column 4 of Table 2 gives the values of a . Tables 3 A, B, and C give the radial lens distortion values calculated by using Brown's formula (Equation 15) and the author's formula (Equation 14) and those obtained by calibrations as in Table 1.

DISCUSSION

It can be seen from Tables 2 and 3 that the author's formula given by Equation 14 gives much closer results to the calibrated values than those obtained by using Equation 15.

The maximum difference between the estimated distortion by author's formula (Equation 14) and the calibrated values is 1.7 micrometers (Table 2-A). The corresponding difference by using Equation 15 is 12.2 μm. The maximum difference between the estimated distortion by Equation 15 and that by calibration is 31 μm (Table 3-B). The corresponding difference using Equation 14 is 0.4 μm. It can also be seen from Tables 2 and 3 that, without any exceptions, all the

TABLE 3-A. COMPARISON OF CALIBRATED DISTORTION dr_4 $(s = 4\ ft.)$ AND dr $(s = \propto ft.)$ WITH RESULTS COMPUTED FROM dr_3 $(s = 3\ ft.)$ AND dr_6 $(s = 6\ ft.)$ USING BROWN'S AND THE AUTHOR'S FORMULAS

r in cm	dr_4 *computed by*		dr_4	dr *computed by*		dr
	Brown's Formula (μm)	*Author's Formula (μm)*	*Calibrated (μm)*	*Brown's Formula (μm)*	*Author's Formula (μm)*	*Calibrated (μm)*
15	2.5	2.4	2.4	3.4	3.5	3.5
30	19.7	19.4	19.4	26.9	27.8	27.7
45	66.6	65.6	65.5	90.8	93.7	93.3
60	157.9	155.5	155.3	215.1	222.1	221.2
75	308.4	303.7	303.3	420.2	433.7	432.0

TABLE 3-B. COMPARISON OF CALIBRATED DISTORTION dr_6 $(s = 6\ ft.)$ AND dr $(s = \quad ft.)$ WITH RESULTS COMPUTED FROM dr_3 $(s = 3\ ft)$ AND dr_4 $(s = 4\ ft.)$ USING BROWN'S AND THE AUTHOR'S FORMULAS

r in cm	dr_6 *computed by*		dr_6	dr *computed by*		dr
	Brown's Formula (μm)	*Author's Formula (μm)*	*Calibrated (μm)*	*Brown's Formula (μm)*	*Author's Formula (μm)*	*Calibrated (μm)*
15	2.7	2.8	2.8	3.2	3.5	3.5
30	21.7	22.2	23.3	25.7	27.6	27.7
45	73.1	75.0	75.2	86.8	93.2	93.3
60	173.2	177.8	178.2	205.6	221.0	221.2
75	338.3	347.2	348.1	401.6	431.6	432.0

TABLE 3-C. COMPARISON OF CALIBRATED DISTORTION dr_4 ($s = 4$ ft.) AND dr_6 ($s = 6$ ft.) WITH RESULTS COMPUTED FROM dr_3 ($s = 3$ ft.) AND dr ($s =$ ft.) USING BROWN'S AND THE AUTHOR'S FORMULA

r in cm	dr_4 *computed by*		dr_4	dr_6 *computed by*		dr_6
	Brown's Formula (μm)	*Author's Formula (μm)*	*Calibrated (μm)*	*Brown's Formula (μm)*	*Author's Formula (μm)*	*Calibrated (μm)*
15	2.5	2.4	2.4	2.8	2.8	2.8
30	20.0	19.4	19.4	22.7	22.2	22.3
45	67.3	65.5	65.5	76.7	75.1	75.2
60	159.6	155.3	155.3	181.9	178.0	178.2
75	311.8	303.3	303.3	355.2	347.6	348.1

values calculated by Equation 14 (author's formula) gives much closer results to the calibrated values than those obtained by Equation 15.

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