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Interpolation of a Function of Many Variables, II

Two different methods pertain to control distribution for aerial triangulation and to the correction of coordinates on reseau photos.

ABSTRACT: *The paper presents two contrasted methods for interpolating a function of several variables with irregular distribution of controls. A previous (1965) method is updated to eliminate singularity problems and to minimize non-linearity characteristics in limiting cases. The second method is derived directly from G. H. Schut's planimetric adjustment and is essentially the determination of an osculating second-degree function with weights as functions of the distances. The derivation of the weights is critically examined. It seems that the normalizing distance in the weights depends on the average spacing of the controls and not on the largest dimension of the block.*

INTRODUCTION

THE PRESENT PAPER is in reply to criticisms of a previous paper by G. H. Schut (1970). The background very briefly is as follows. In 1957, when I was much involved in the automation of aerotriangulation computations, I made a first attempt to solve a problem of frequent occurrence in cartography, photogrammetry, astrographics, and geodesy, namely, the interpolation of a variable as a function of several arguments where the functional form is unknown and the control is irregularly distributed in the arguments. I initially proposed that this variable v be represented as the sum of components from each of the controls and thus could be expressed in the form

$$v = \sum_{i=1}^{r} K_i \varphi(r_i) \qquad (1)$$

where n is the number of control points, the K_i are constants characteristic of the controls, r_i is the *distance* from the point to the i-th control and φ is the attenuation function, always the same for all controls. Generally the distance r really is the Euclidean distance $(\Delta x^2 + \Delta y^2 + \cdots)^{1/2}$, but in non-geometrical applications it can be any appropriate function of the differences in the arguments. Clearly φ is designed to decrease monotonically from its maximum value at zero. Applications of Equation 1 to the controls themselves gives the set

$$\sum_{j=1}^{n} K_j \varphi(r_{ij}) = v_i \qquad (2)$$

in which r_{ij} is the distance between the i-th and j-th controls. Note that $r_{ii} = 0$. This set can generally be solved for the K_j and the interpolation then performed using Equation 1. In 1957 I set

$$\varphi(r) = 1 - r/a \qquad (3)$$

where a is greater than any of the r_{ij} and tested the method on blocks of height data with eight controls. In this form the method was communicated informally to interested German photogrammetrists at the 1958 Zeiss Photogrammetric Week at Munich (see B. Müller, 1963).

Before I published the method myself in 1965, I noted that the use of Equation 3 introduces discontinuities in the spatial derivatives of v at the controls themselves, so I substituted the form

$$\varphi(r) = 1 - r^2/a^2 \qquad (4)$$

in the paper as published.

Schut's Criticisms of the Distance Method

Schut (1970), in connection with the allied problem of adjusting the planimetry of a warped block to external controls, severely criticised the distance method. His criticisms are

i. the method has no theoretical basis;
ii. the method does not reduce to linear interpolation of the linear type $v = a + bx + cy + \cdots$ if the number of controls exceeds the number of arguments by unity;
iii. the method is precisely singular in the two-dimensional case where the four controls are at the corners of a square;
iv. the method tends to singularity with random control;
v. even with a new function φ the method is of little use.

Even though some of these criticisms are quite correct, they are overstated and I believe that they tend to mislead the reader as to the usefulness of the distance method.

I cannot answer the accusation that the method has no theoretical basis for the very simple reason that no interpolation method has such a basis. Either we know the functional form or we do not. In the latter case we must assign it on the basis of efficiency and convenience. The fact that second-degree forms, such as are generally used in the one-dimensional case, are more convenient than forms of higher order, and more efficient than linear forms, does not give them a theoretical basis.

The criticism concerning linearity has some merit but is overstated. In most applications v is the residual of a least-squares fit to the controls and very often the model is linear, so that the v's are then very definitely non-linear. If however there is a linear trend it is easily disposed of by a preliminary adjustment of the type $v = a + bx + cy + \cdots$. It should be noted, by the way, that Schut's planimetric adjustment (Schut, 1970) does *not* reduce to this form if there are three controls, although I myself would not criticize it on that account.

The singularity criticisms are correct and lead to some useful results both in connection with the distance method and Schut's own method of planimetric adjustment. In the two-dimensional case with the controls at the four corners of a square of side s the matrix in Equation 2 is singular if

$$\varphi(s) = \tfrac{1}{2} + \tfrac{1}{2}\varphi(s\sqrt{2}). \qquad (5)$$

It is readily verified that the matrix is singular for *all* values of a if φ is defined as in Equation 4. This result, coupled with Schut's findings that there is a tendency to singularity with random control, is a clear indication that Equation 4 is not appropriate for the purpose.

Schut's Planimetric Adjustment

Schut (1970) has published a method of external adjustment of the planimetry of an air-survey block which has considerable merit. It is reviewed briefly here for two reasons. First it leads directly to a general interpolation method; secondly it incorporates the extremely useful idea of *weights* which give the representing function an osculating character. Schut uses the well-known conformal relation

$$\Delta z = \alpha + \beta z + \gamma z^2 \qquad (6)$$

in which α, β and γ are complex constants whereas z is the complex vector $x + iy$. The constants in Equation 6 are determined afresh at each point at which Δx and Δy are required, using the weight w_i for each control i, where

$$w_i = (1 - \rho_i)^3(1 - \rho_i^2)^3\rho_i^{-1} \qquad (7)$$

and ρ_i is the normalized distance

$$\rho_i = r/a. \qquad (7.1)$$

As before, a is a distance greater than any of the r_{ij}. Two remarks are apposite here. First the form of Equation 7 is surprisingly complex. In relation to this it is not surprising that Schut does not appear to be entirely satisfied with his weight function. Secondly I note that Schut has followed my 1965 paper in normalizing his distances with the distance a. I now believe that I made incorrect choices for the function φ and was thereby forced into a completely wrong choice of the normalizing distance. If this is so, then Schut followed me into a rather important error. This is discussed later.

Interpolation with Second-Degree Functions

Schut's planimetric adjustment depends on the use of two strongly conditioned second-degree forms. As such it immediately suggests a general interpolation method with the free second-degree form

$$v = \sum_{i,k=0}^{m} a_{ik}x_i x_k \qquad (8)$$

with arguments $x_1, x_2, \ldots x_m$. The argument x_0 is defined to be always unity. There are $\frac{1}{2}(m + 1)(m + 2)$ coefficients a_{ik}, so for

$m > 2$ the computations rapidly become rather expensive. The control demands may also become excessive. The method is not likely to run into singularity problems but approximations to singularity cannot be ruled out. For example, in two dimensions, the above can be written as

$$v = a + bx + cy + dx^2 + 2exy + fy^2.$$

The typical row of the determinant is

$$1, x_i, y_i, x_i^2, 2x_iy_i, y_i^2.$$

The determinant vanishes if and only if there exists a single set A, B, C, D, E, F, such that

$$A + Bx_i + Cy_i + Dx_i^2 + 2Ex_iy_i + Fy_i^2$$

vanishes for **all** i, that is, for all controls. This happens when the controls all fall on some conic

$$A + Bx + Cy + Dx^2 + 2Exy + Fy^2 = 0.$$

This may be an ellipse, parabola, hyperbola or a pair of straight lines. For example, the solution fails if the controls fall on two parallel lines and nowhere else. Precise singularity is unlikely, but unstable solutions cannot be ruled out if the control configuration approximates to a critical case.

Even with the above drawbacks interpolation with osculating second-degree forms is quite an attractive proposition, because the method is usually very successful in bridging over awkward gaps in the control. In contrast to the above, Schut's planimetric adjustment is free of singularity problems.

The question of weights is deferred to a later section.

The Improved Distance Method

The drawbacks of the second-degree-function method of interpolation, namely rather high computing costs and excessive control demands, make it worthwhile to remodel the distance method of Equations 1 and 2 to meet as well as possible the criticisms of Schut. The trouble evidently lies in the forms assigned to φ. What are the required characteristics of this function? Evidently, it must be a maximum at zero and die away with r increasing. Also it must be flat-topped at zero to avoid singularities at the controls. It must also be positive for all values of r, because any other behavior would be quite inappropriate. Evidently φ cannot be much different from the Gaussian function

$$\varphi(r) = \exp(-c^2r^2) \qquad (9)$$

and this is now adopted. The constant c determines the rate of attenuation and must be such that the interpolation is dominated by the values at the immediately adjacent controls. The next ring or shell of controls beyond then should contribute a small fraction of v and the next ring beyond these very little indeed. The avoidance of singularity in the matrix of Equation 2 leads to much the same ideas. As the elements on the principal diagonal are all unity, we can be sure that singularity will not arise if all the off-diagonal elements are all positive and all rather small. Thus two lines of reasoning point to a fairly small value of $\varphi(h)$, where h is the *average distance between adjacent controls*. This last concept is rather vague, but gives no special difficulty in practice.

It would be possible to stipulate a value for c which makes $\varphi(h)$ an arbitrary small value, but a more interesting approach is to try to kill two birds with one stone. To meet Schut's objection to the non-linearity, as nearly as the use of Equation 9 allows, suppose that c is chosen so that in the case of two points only the non-linear interpolation at a point midway between them is precisely equal to the linearly interpolated value. Write $\theta = \exp(-b^2c^2)$ where b is the separation of the two controls, then from Equation 1,

$$v_1 = K_1 + K_2\theta,$$
$$v_2 = K_1\theta + K_2.$$

Solving for K_1 and K_2,

$$K_1 = (v_1 - \theta v_2)/(1 - \theta^2)$$
$$K_2 = (v_2 - \theta v_1)/(1 - \theta^2).$$

At the midpoint $r_1 = r_2 = b/2$, the non-linear interpolation gives

$$\begin{aligned} v_m &= K_1\varphi(b/2) + K_2\varphi(b/2) \\ &= (K_1 + K_2)\varphi(b/2) \\ &= (K_1 + K_2)\theta^4. \end{aligned}$$

Equating this to the linearly interpolated value $\frac{1}{2}(v_1 + v_2)$ we get the quartic

$$1 - 12\theta + 6\theta^2 + 4\theta^3 + \theta^4 = 0.$$

Apart from the non-significant root $\theta = +1$, the only real root of this is $\theta = +0.08738$. This is equivalent to

$$\varphi(r) = \exp(-2.4375r^2/b^2). \qquad (10)$$

It is convenient to approximate the constant to 2.5. Now we cannot make b equal to *all* the distances between the adjacent controls, but we can equate it to the *average* value h.

Thus the improved distance method makes use of

$$\varphi(r) = \exp\,(-2.5r^2/h^2). \qquad (11)$$

Note that this does not eliminate the non-linearity. It merely ensures in a well-conditioned network of controls that the non-linear interpolation never differs too much from the linear interpolation. Is $\varphi(h)$ acceptably small? From the last we have $\varphi(h) = 0.082$ and $\varphi(2h) = 0.00005$. These values seem to be in line with the requirements detailed above, so that Equation 11 also avoids singularity problems. Thus the improved distance method is embodied in Equations 1, 2 and 11. A preliminary linear adjustment is used if the values at the controls show a definite trend. The distance method is affected by a difficulty not noted by Schut, namely that the method is not very satisfactory if the distribution of controls is markedly anisotropic, that is, with different intervals in one fixed direction as compared to another. This can be eliminated by a preliminary affine transformation of the coordinates or, what comes to the same thing, a generalization of the definition of distance from the Euclidean $\sqrt{\Sigma \Delta x^2}$ to the Riemannian $\sqrt{\Sigma g_{ik}\,\Delta x_i\,\Delta x_k}$ with constant g_{ik}.

The Weight Problem

The discussion of the last section throws an interesting light on the problem of normalizing the distances, either for an attenuation function φ, or for the calculation of the weights in Schut's planimetric adjustment and the interpolation method discussed previously. It will be remembered that the use of the distance a in the 1957 and 1965 distance method was entailed by a rather poor choice of the function φ. The normalizing distance had to be larger than the largest dimension of the block to prevent negative values of φ. There was really no other reason. Similarly I assume that Schut wished his weights to drop to zero at this maximum dimension and he followed me mistakenly, I now believe, in the use of a for this purpose. It is quite evident from the behavior of his rather complex w in Equation 7 that he wished his w to approximate to zero long before r approached the value a.

In the foregoing section I think it becomes quite clear that the correct normalizing distance is the average spacing of the controls, that is, h is the average separation of adjacent controls. This idea is supported by appeal to that case in which there is no doubt. In the interpolation of a function of a single variable the intervals between the controls enter very strongly into the interpolation formulas. The total range of the argument, which is the equivalent of a, is irrelevant and does not come into the discussion. As we are dealing with processes which are inevitably arbitrary to some extent, the statement that the use of a is incorrect is too strong. However, I think it can be said that the use of a is not an optimum procedure and that normalization with h gives better results.

If the distances are scaled with h it follows at once that Equation 7 must be abandoned as it does not behave correctly if ρ exceeds unity. Within the range $0 \leqslant \rho < +1$ Schut's w is a J-shape function dropping rapidly from infinity at zero. Its behavior suggests a weight function of the type

$$w = \rho^{-m} = h^m/r^m. \qquad (12)$$

Experiments with synthetic two-dimensional blocks in which the interpolations could be demonstrated visually as contours soon led to the conclusion that the most plausible interpolations resulted from the very simple expression

$$w = \rho^{-2} = h^2/r^2. \qquad (13)$$

The vagueness of the definition of h as the average separation of adjacent controls may bother some. Apparently this is no great handicap in practice. This constant h is evidently related to the control density, but I have not taken the time to work out its definition from that point of view.

Figures 1, 2, and 3 show the same block interpolated by three methods. Figure 1 shows the results of visual contouring, that is, sketching of contours so as to be consistent with the values at the controls. In practice this is never very different from linear interpolation. Figure 2 shows as contours the same block interpolated by the distance method with $h = 1.16$ in units of the sides of the coordinate squares. Notice the tendency to isolations at the edges of the block. Figure 3 shows the results of interpolation with an osculating second-degree function with $h = 1.16$ as before and $w = \rho^{-2}$. Clearly this third method is superior in its results.

It may be inquired whether the new weights are so very different from the old. This is best seen by expressing them in terms of the same parameters. If ρ is taken to mean r/h only and $m = a/h$, then Schut's weight is

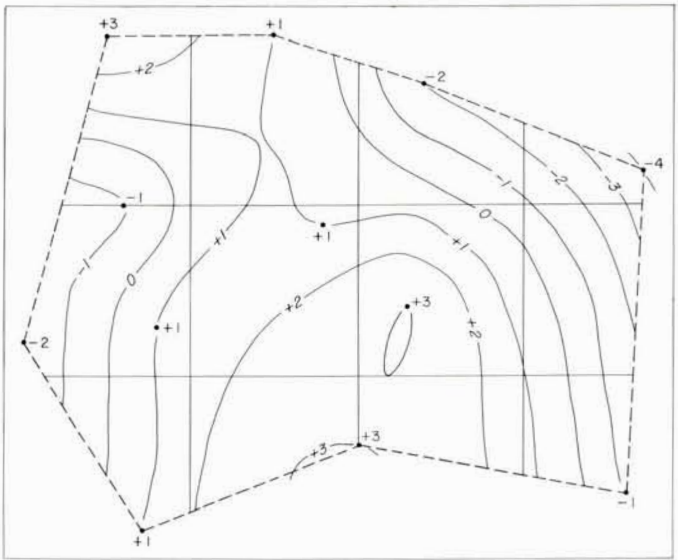

FIG. 1. Contours interpolated by eye for a typical block with 12 controls in the two-dimensional case.

$$w_s = \frac{1}{\rho}\left[\frac{(m-\rho)(m^2-\rho)}{(m-1)(m^2-1)}\right]^3. \qquad (14)$$

This has a rough resemblance to the new weight ρ^{-2} if m is small (say $m \leqslant 2$), but the similarity decreases very rapidly as m increases.

SUMMARY AND CONCLUSIONS

The paper presents two very different methods of interpolating a function of several variables with irregular control. The first, depending on the use of Equations 1, 2 and 11, is generally the less expensive and demanding in relation to control. It

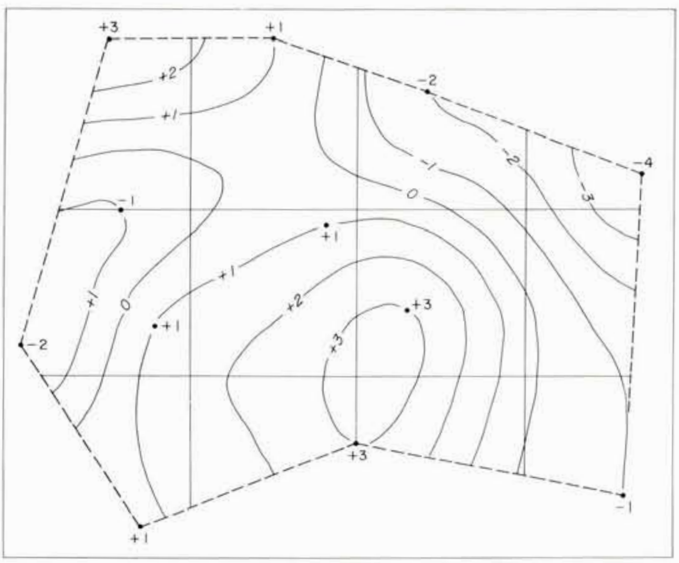

FIG. 2. Contour representation of interpolation using $v = \Sigma K\varphi(r)$ for the case illustrated in Figure 1.

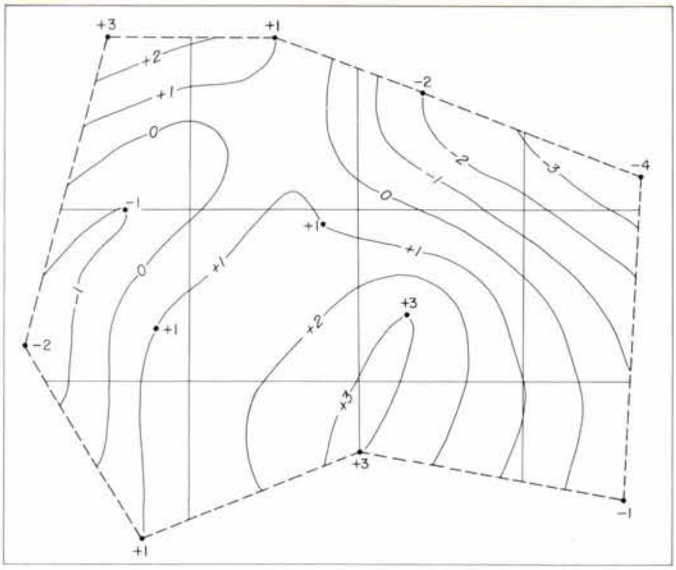

FIG. 3. Contour representation of interpolation using an osculating second-degree function for the case illustrated in Figure 1.

seems to be adequate for the correction of coordinates on reseau-bearing photographs and for the smoothing out of the residuals of least-squares adjustments. It involves one inversion (or solution of equations) of the same order as the number of controls. It has the possible advantage of expressing the variable as a function of the arguments after this single inversion.

The second method is derived very directly from Schut's planimetric adjustment. It is generally more expensive and more demanding as to control than the first method but, provided the control demands can be met, it appears to give better interpolations.

Schut's weight concept, an extremely useful idea in this context, has been critically examined and an improved version developed for handling control configurations of widely different sizes and densities.

ACKNOWLEDGEMENTS

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