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Atmospheric Refraction

Convenient equations for aerial photogrammetry are derived in a simple and straight forward manner.

ABSTRACT: *Methods have been formulated for computing atmospheric refraction corrections to be applied to measured coordinates of photographic images used in analytic photogrammetry. The corrections may be calculated based on any combination of flying height, ground elevation, atmospheric pressure, atmospheric temperature, and measured image coordinates. The mathematical derivation of the equations is relatively simple and the parameters used in the equations are readily measured. A theoretically exact method is present first, followed by other methods which are approximate to simplify the calculations. Tests have shown that the approximations consistently yield results that agree with the theoretically exact values to within one micrometer.*

INTRODUCTION

IN RECENT YEARS several articles have been published on the subject of atmospheric refraction as it relates to the analytical aerial photogrammetric problem.[1, 2, 5, 6] The complexity of mathematical derivations in these articles varies. The input parameters that must be measured or otherwise determined in order to solve the equations also vary and, in some instances, the necessary parameters are not readily measured or available.

Here we develop an atmospheric refraction equation and we believe that this derivation is more straight forward and more easily understood by the average photogrammetrist than have previous developments. Four variations of the equation are presented utilizing readily measured data, and the equation may be selected that conforms to measured data for any particular application. The equations may be easily programed for analytical photogrammetric computations.

THEORETICAL DEVELOPMENT

Snell's Law from elementary physics states that for a light ray passing through a medium, the product of the index of refraction n and the sine of the incidence angle, *sine* θ, remains constant:

$$n \sin \theta = \text{a constant.} \tag{1}$$

Figure 1 illustrates the curved path of a light ray from a ground object point G to an exposure station C which is located a distance Z_c above mean sea level. The photographic nadir point is n and the ground nadir point is N. Object point G is imaged at g. Prior to applying photogrammetric principles of straight-line perspective in computing the position of the object G, this image point g must be corrected for atmospheric refraction to g', otherwise the object point will be incorrectly located at G'. Figure 2 is an enlarged view of the aerial photographic positive print.

The index of refraction of the atmosphere is a function of air density, and density is a function of height. The differential equation representing a light ray as it passes through the atmosphere can be developed with reference to Figures 1 and 2. In general the sine of the angle of incidence at any height Z above the ground may be expressed as (see Figure 1):

$$\sin \theta_z = dX/(dX^2 + dZ^2)^{\frac{1}{2}} = 1/[1 + (dZ/dX)^2]^{\frac{1}{2}}. \tag{2}$$

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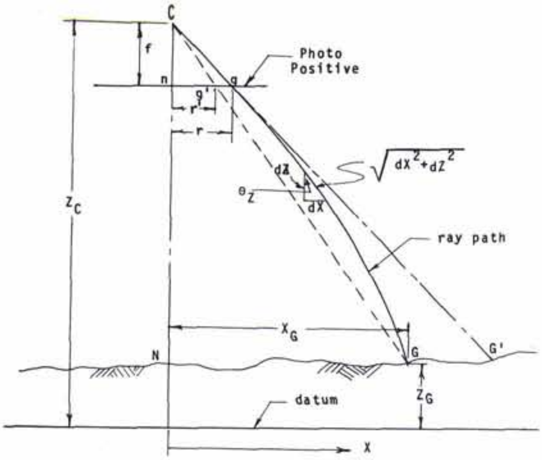

FIG. 1. Atmospheric refraction in aerial photography.

Equating Snell's Law for any height Z and for the height of the exposure station Z_C:

$$n_Z \sin \theta_Z = n_C \sin \theta_C = \text{a constant.} \tag{3}$$

Substituting Equation 3 into Equation 2, squaring and rearranging:

$$(dZ/dX)^2 = [n_Z^2/(n_C^2 \sin^2 \theta_C)] - 1. \tag{4}$$

By taking the square root of Equation 4 and solving for dX/dZ (note from Figure 2 that solving for dX/dZ is negative):

$$-dX/dZ = [n_Z^2/(n_C^2 \sin^2 \theta_C) - 1]^{-\frac{1}{2}} \tag{5}$$

If the index of refraction is known as a function of Z, the correct ground coordinate X_G of the object point G can be calculated from the expression:

$$X_G = -\int_{Z_G}^{Z_C} [n_Z^2/(n_C^2 \sin^2 \theta_C) - 1]^{-\frac{1}{2}} \, dZ. \tag{6}$$

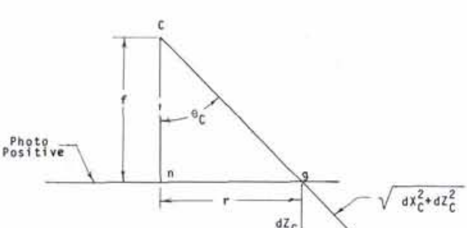

FIG. 2. Enlarged view of the aerial photographic image space.

The corrected image distance r' from the photographic nadir point to the corrected image point g' can then be calculated from (see Figure 1):

$$r' = f\, X_G/(Z_C - Z_G) \tag{7}$$

in which f is the focal length of the camera, Z_C is the height of the camera above mean sea level and Z_G is the height of the object point G above mean sea level.

In aerial photogrammetry, the atmospheric parameters that have a bearing on the magnitude of atmospheric refraction, and that are readily measured, are air temperature and air pressure. The index of refraction is related to these parameters through the Lorenz-Lorentz[4] equation as:

$$n^2 = (1 + 2K\rho)/(1 - K\rho) \tag{8}$$

in which K is a constant and ρ is the density of the air. Air density ρ_Z at any height Z is related to both temperature and pressure:

$$\rho_Z = P_Z/R\, T_Z \tag{9}$$

where T_Z and P_Z are air temperature and air pressure, respectively, at height Z and R is the gas constant.

From the foregoing it is seen that if temperature and pressure are known for all points in the column of air beneath the camera, Equation 6 can be numerically integrated to obtain X_G which, if substituted into Equation 7, yields the corrected image distance r'. In practice it is impossible to measure instantaneously the air temperature and air pressure for all points in the column of air below the camera. However, these measurements are quite routine if they are observed at the camera station or on the ground. As temperature and pressure cannot be instantaneously measured for all points in the column of air beneath the camera, a standard atmosphere model[7] was assumed and Equation 6 was numerically integrated using various combinations of ground temperature, ground pressure, ground elevation, and flying height above mean sea level. For the assumed atmospheric model, the equations that provide temperature T_Z and pressure P_Z at any height Z are:

$$T_Z = T_G - A(Z_C - Z_G) \tag{10}$$

$$P_Z = P_G\, (T_Z/T_G)^m \tag{11}$$

where T_G and P_G are ground air temperature and ground air pressure, respectively, A is a constant equal to 0.0065 degrees Kelvin per meter, and m is a constant equal to 5.256. Results from numerical integration of Equation 6 using one combination of data are

TABLE 1. ATMOSPHERIC REFRACTIONS FOR GROUND TEMP. 68°F
(Corrections in micrometers)

Flight Altitude (ft-MSL)	Radial Distance from Nadir in Centimeters 1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9	11.0
	Ground Elevation—0 feet						*Ground Pressure—960 mb.*			
10,000	0.4	0.7	1.1	1.5	2.0	2.5	3.1	3.8	4.5	5.4
20,000	0.6	1.2	1.9	2.6	3.4	4.3	5.3	6.5	7.8	9.2
30,000	0.8	1.6	2.4	3.4	4.4	5.5	6.8	8.3	9.9	11.8
	Ground Elevation—1,000 ft.						*Ground Pressure—926 mb.*			
10,000	0.3	0.6	1.0	1.4	1.8	2.3	2.8	3.4	4.0	4.8
20,000	0.6	1.2	1.8	2.5	3.2	4.1	5.0	6.1	7.3	8.7
30,000	0.7	1.5	2.3	3.2	4.2	5.3	6.5	7.9	9.5	11.3
	Ground Elevation—2,000 ft.						*Ground Pressure—894 mb.*			
10,000	0.3	0.6	0.9	1.2	1.6	2.0	2.4	3.0	3.6	4.2
20,000	0.5	1.1	1.7	2.3	3.0	3.8	4.7	5.7	6.9	8.2
30,000	0.7	1.5	2.2	3.1	4.0	5.1	6.3	7.6	9.1	10.8
	Ground Elevation—3,000 ft.						*Ground Pressure—862 mb.*			
10,000	0.2	0.5	0.8	1.0	1.4	1.7	2.1	2.6	3.1	3.7
20,000	0.5	1.0	1.6	2.2	2.8	3.6	4.4	5.4	6.4	7.6
30,000	0.7	1.4	2.1	2.9	3.8	4.8	6.0	7.3	8.7	10.3
	Ground Elevation—4,000 ft.						*Ground Pressure—831 mb.*			
10,000	0.2	0.4	0.6	0.9	1.2	1.5	1.8	2.2	2.6	3.1
20,000	0.5	1.0	1.5	2.0	2.7	3.3	4.1	5.0	6.0	7.1
30,000	0.7	1.3	2.0	2.8	3.7	4.6	5.7	6.9	8.3	9.9
	Ground Elevation—5,000 ft.						*Ground Pressure—801 mb.*			
10,000	0.2	0.3	0.5	0.7	1.0	1.2	1.5	1.8	2.2	2.6
20,000	0.4	0.9	1.4	1.9	2.5	3.1	3.8	4.7	5.6	6.6
30,000	0.6	1.3	1.9	2.7	3.5	4.4	5.4	6.6	7.9	9.4

shown in Table 1. Corrections for atmospheric refraction given in the table are the differences between image distance r and the corrected image distance r' in micrometers.

APPROXIMATIONS

Because Equation 6 cannot in general be integrated in closed form, the following approximations of Equation 8 were made to simplify the solution:

$$n_Z^2 = 1 + 3K\rho_Z$$
$$n_C^2 = 1 + 3K\rho_C$$
$$n_C = 1 + (3/2)K\rho_C \qquad (12)$$

where n is the index of refraction, ρ is air density and K is a constant equal to 1.5159×10^{-4}. Subscripts Z refer to any flight elevation Z and subscripts C refer to the height of the camera. Equations 12 are reasonable approximations because K is approximately 1.5×10^{-4} and ρ is on the order of unity. Substituting Equations 12 into Equation 6, dropping terms that contain K raised to powers of 2 or higher, and reducing, the following simplified expression is obtained for X_G:

$$X_G = -\tan\theta_C \int_{Z_G}^{Z_C} \left[1 + \frac{3K\rho_C}{2\cos^2\theta_C} - \frac{3K\rho_Z}{2\cos^2\theta_Z}\right] dZ. \qquad (13)$$

Substituting Equations 9, 10, and 11 into Equation 13, integrating the first two terms and reducing, the following expression containing only a simple integral is obtained:

$$X_G = -\tan\theta_C \left[(Z_G - Z_C) + \frac{3K\rho_C}{2\cos^2\theta_C}(Z_G - Z_C) - \frac{3K}{2\cos^2\theta_C}\int_{Z_G}^{Z_C} \rho\, dZ\right] \qquad (14)$$

Evaluating the simple integral and reducing:

$$X_G = \tan\theta_C \left\{(Z_C - Z_G) + \frac{3K\rho_C}{2\cos^2\theta_C}(Z_C - Z_G) - \frac{3K\rho_G}{2m\,A\,R\cos^2\theta_C}\left[1 - \left(\frac{T_G - A(Z_C - Z_G)}{T_G}\right)^m\right]\right\}. \qquad (15)$$

From Figure 2,

$$\tan \theta_c = r/f \qquad (16)$$

where r is the distance from the photographic nadir point to the image point g. Let $\Delta r = r - r'$ where Δr is a correction to be applied to a measured photo distance. Solving Equation 7 for X_G and in turn substituting into Equation 15, and also making substitutions of $r/f = \tan \theta_c$ and $\Delta r = r - r'$, the following basic equation is obtained for the amospheric correction Δr:

$$\Delta r = \frac{3rK}{2 \cos^2 \theta_c} \left\{ \rho_c - \frac{P_G/(Z_c - Z_G)}{mAR} \left[1 - \left(\frac{T_G - A(Z_c - Z_G)}{T_G} \right)^m \right] \right\}. \qquad (17)$$

Variations of the Basic Refraction Equation

The following expression my be obtained from Equation 17 which uses only variables of ground air temperature and ground air pressure in addition to the usual image coordinates, flying height, focal length and ground elevation:

$$\Delta r = \frac{3rKP_G(f^2 + r^2)}{2Rf^2} \left[\frac{T_G - A(Z_c - Z_G)}{T_G} \right]^m \times \left\{ \frac{1}{T_G - A(Z_c - Z_G)} - \frac{1}{mA(Z_c - Z_G)} \left[\left(\frac{T_G}{T_G - A(Z_c - Z_G)} \right)^m - 1 \right] \right\}. \qquad (18)$$

Another variation of the basic equation utilizes air pressure and air temperature at the exposure station, as follows:

$$\Delta r = \frac{3rKP_c(f^2 + r^2)}{2Rf^2} \left\{ \frac{1}{T_c} - \frac{1}{mA(Z_c - Z_G)} \left[\left(\frac{T_c + A(Z_c - Z_G)}{T_c} \right)^m - 1 \right] \right\}. \qquad (19)$$

A different variation which could be convenient under certain conditions uses camera air temperature and ground air pressure as follows:

$$\Delta r = \frac{3rK(f^2 + r^2)P_G}{2Rf^2} \left[\frac{T_c}{T_c + A(Z_c - Z_G)} \right]^m \times \left\{ \frac{1}{T_c} - \frac{1}{mA(Z_c - Z_G)} \left[\left(\frac{T_c + A(Z_c - Z_G)}{T_c} \right)^m - 1 \right] \right\}. \qquad (20)$$

A final variation of the equation uses ground air temperature and camera air pressure as follows:

$$\Delta r = \frac{3rK(f^2 + r^2)P_c}{2Rf^2} \left\{ \frac{1}{T_G - A(Z_c - Z_G)} - \frac{1}{mA(Z_c - Z_G)} \times \left[\left(\frac{T_G}{T_G - A(Z_c - Z_G)} \right)^m - 1 \right] \right\}. \qquad (21)$$

The symbols and their units for Equations 18 through 21 are:

- Δr–Refraction correction in millimeters.
- r–Measured distance from photo nadir to photo image in millimeters.
- K–A constant equal to 1.5159×10^{-4}.
- f–Camera focal length in millimeters.
- P_G–Ground air pressure in millibars.
- Z_G–Ground elevation above mean sea level in meters.
- T_G–Ground air temperature in degrees Kelvin.
- P_c–Camera air pressure in millibars.
- Z_c–Camera elevation above MSL in meters.
- T_c–Camera air temperature in degrees Kelvin.
- A–A constant equal to 0.0065.
- R–A constant equal to 2.8704.
- m–A constant equal to 5.256.

Equation 19 is perhaps the most convenient of the four Equations 18 through 21 because it requires no ground support at the time of photography. Rather, it uses only values obtained in the aircraft or from measurements made later in the laboratory. Each of the equations are rather formidable to solve by hand methods, but they are easily programed for computer solution and can be added as a preliminary calculation in analytical photogrammetry programs.

For a given strip or block of photographs taken on the same day, all atmospheric and camera position parameters may be assumed constant so that each of Equations 18 through 21 may be reduced to the following simplified form:

$$\Delta r = \left[r(f^2 + r^2)/f^2 \right] B. \qquad (22)$$

Thus for any given block or strip of aerotriangulation, the constant B need be calculated only once, and the refraction correction is readily evaluated for each point depending on its photographic posion (r-value). Explicitly, the values of B for the four Equations 18 to 21 are indicated next.

Ground air temperature and ground air pressure:

$$B = (0.7922 \times 10^{-4}) P_G \left[\frac{T_G - A(Z_C - Z_G)}{T_G}\right]^m \times \left\{\frac{1}{T_G - A(Z_C - Z_G)} - \frac{1}{mA(Z_C - Z_G)} \left[\left(\frac{T_G}{T_G - A(Z_C - Z_G)}\right)^m - 1\right]\right\}. \quad (23)$$

Air pressure and temperature at camera station:

$$B = (0.7922 \times 10^{-4}) P_C \left\{\frac{1}{T_C} - \frac{1}{mA(Z_C - Z_G)} \left[\left(\frac{T_C + A(Z_C - Z_G)}{T_C}\right)^m - 1\right]\right\}. \quad (24)$$

Camera air temperature and ground air pressure:

$$B = (0.7922 \times 10^{-4}) P_G \left[\frac{T_C}{T_C + A(Z_C - Z_G)}\right]^m \times \left\{\frac{1}{T_C} - \frac{1}{mA(Z_C - Z_G)} \left[\left(\frac{T_C + A(Z_C - Z_G)}{T_C}\right)^m - 1\right]\right\}. \quad (25)$$

Ground air temperature and camera air pressure:

$$B = (0.7922 \times 10^{-4}) P_C \times \left\{\frac{1}{T_G - A(Z_C - Z_G)} - \frac{1}{mA(Z_C - Z_G)} \left[\left(\frac{T_G}{T_G - A(Z_C - Z_G)}\right)^m - 1\right]\right\}. \quad (26)$$

Equations 12 are approximations that were made in arriving at simplified Equations 18 through 26. These approximations have been verified as reasonable by solving for refraction corrections for various combinations of conditions. In every application tested the results agreed within 1 micrometer or less of the results obtained using the same conditions and numerically integrating exact Equation 6.

Summary

A basic atmospheric refraction equation has been presented for which the mathematical derivation is simple and straight forward. Variations of the basic equation utilize readily measured or generally available data. The equations may be easily programed for analytical computation on an electronic computer.

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