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New Math Model for Independent-Model Triangulation

Applying the collinearity principle, the system may be used as a substitute for fully analytical aerial triangulation.

Introduction

Triangulation by independent models is based on forming strips by joining independent models in planimetry,[4,9,10] or by spatial transformation,[13,18,19] or by performing the adjustment of relative orientation based on recorded y-parallaxes in the strip formation.[11] In the adjustment phase, standard polynomial procedures are used in these methods. The polynomial adjustment procedures suffer from several shortcomings.[1,2,15] The mathematical model in this paper is based on utilizing two collinearity equations. The first one involves the model point, perspective center of the stereoplotter and the image point. The second one involves the object point on the ground, the exposure station and the image point.

It is believed that the use of this model should overcome the shortcoming of the polynomial adjustment. Also it will avoid:

- The problem in the scale determination which arises in the seven-parameters An-block method.[27,28]
- The problem of assigning four parameters for planimetry and three others for height in the procedure described by v. d. Hout.[10]
- The problem of non-rigorous determination of the scale factor and the three shift parameters (which are determined not according to the least squares method) shown by Schut.[18]

At the same time, as shown in the paper, the suggested model provides complete statistical analyses for the observations and the parameters involved in the solution of this model.

Derivation of the Mathematical Model

The mathematical model is based on the collinearity equation involving the terrain point, the exposure station and the image point. The image point coordinates are obtained using another collinearity equation involving the model point, the perspective center of the stereoplotter and the image point. Thus the mathematical formulation for an image point i in photo k transformed from model j is †

$$\begin{bmatrix} V_X \\ V_Y \end{bmatrix}_{i,k} + \begin{bmatrix} X \\ Y \end{bmatrix}_{i,k} = f_k \begin{bmatrix} \bar{U}_M/\bar{W}_M \\ \bar{V}_M/\bar{W}_M \end{bmatrix}_{i,j} = f_k \begin{bmatrix} \bar{U}_G/\bar{W}_G \\ \bar{V}_G/\bar{W}_G \end{bmatrix}_{i,j} \quad (1)$$

where

$$\begin{bmatrix} \bar{U}_M \\ \bar{V}_M \\ \bar{W}_M \end{bmatrix}_{i,k} = A_R \begin{bmatrix} U_{M_i} - U_0 \\ V_{M_i} - V_0 \\ W_{M_i} - W_0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{U}_G \\ \bar{V}_G \\ \bar{W}_G \end{bmatrix}_{i,k} = A_{E_k} \begin{bmatrix} U_{G_i} - U_{E_k} \\ V_{G_i} - V_{E_k} \\ W_{G_i} - W_{E_k} \end{bmatrix}.$$

The Derived Observations

The derived observations are the image coordinates calculated using Formula 1 (for the left and right photos of a given stereomodel after performing a relative orientation) in the form:

$$Y_1 = f' \cdot \begin{bmatrix} \bar{U}_M'/\bar{W}_M' \\ \bar{V}_M'/\bar{W}_M' \end{bmatrix}_{i,j} \quad (1A)$$

$$Y_2 = f'' \cdot \begin{bmatrix} \bar{U}_M''/\bar{W}_M'' \\ \bar{V}_M''/\bar{W}_M'' \end{bmatrix}_{i,j}. \quad (1B)$$

For above calculation, $U'o$, $V'o$, $W'o$ and

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† For mathematical notations—see Appendix.

ABSTRACT: *A new mathematical model for independent-model aerial triangulation is based on the collinearity principle. The validity of the model was assured using various sets of test data. The stereomodels are first relatively oriented; then the model coordinates are observed, and transformed to the image plane. These transformed coordinates are used as observations together with exposure station unknowns and the pass-point coordinates as parameters in a simultaneous least-squares solution. This mathematical model was searched for by using different geometrical approaches and various statistical models. The approaches to the geometrical models are based on the collinearity equation, conditions between unknown parameters for the exposure stations, and conditions between the transformed image coordinates. A suitable geometrical model was chosen from a comparative test of these approaches, using the collinearity equation as a basic equation in the adjustment. This model gives the best agreement between the observations and the parameters in the solution, uses a half-photograph as a basic unit in a simultaneous least-squares solution, and assigns for each basic unit three rotation parameters. The choice of the statistical model for this geometrical model is obtained by using different weight coefficient matrices for the transformed image coordinates as observations to obtain the best agreement between these observations and the parameters in the solution. For the specific statistical model that satisfies this condition, it is found that the transformed image coordinates can be treated as uncorrelated observations with equal weights. Inasmuch as the mathematical model was used for the first time in a simultaneous adjustment, the results were checked by statistical and geometrical analyses. Furthermore, the mathematical model is used in a comparison study with the existing analytical systems, using various test data on different stereoplotters and on one stereocomparator. From the results of these tests, it is concluded that: (1) independent-model aerial triangulation may be used as a substitute for analytical aerial triangulation; (2) the formulation of the adjustment of aerial triangulation must include the collinearity equation; and (3) mathematical correlation between the image coordinates may be neglected.*

$U''o$, $V''o$, $W''o$, and the orthogonal matrices A'_R, A''_R should be used.

ADJUSTMENT PHASE

For any simultaneous adjustment using the image coordinates as observations, the mathematical model is

$$F(L^a, X_1^a, X_2^a) = 0. \qquad (2)$$

Linearizing Equation 2 leads to a linear observation equation having the form:

$$V = A_1 X_1 + A_2 X_2 + W. \qquad (3)$$

To find the values of X_1, X_2 and V of Equation 3 by the least-squares method, one needs the matrices (1) W, A_1, A_2 and (2) $\bar{Q}_Y$.

Calculation of W, A_1, A_2. Calculation of these matrices depends on the initial values for the parameters in the solution (X_1°, X_2°). Calculation of X_1° is accomplished using polynomial strip adjustment after strip formation is done using spatial transformation. For the calculation of X_2^o, space resection is performed either considering every whole photo or only half of it.

The Whole-Photo Case. The mathematical model in this case consists of Equations 2 and 3 in the form:

$$F(Y_1^a, Y_2^a, X_2^a) = 0 \qquad (4A)$$

where

$$V = W + A_2 X_2^0 \qquad (4B)$$

$$W = F(Y_1^b, Y_2^b, X_2^0). \qquad (4C)$$

In the space resection, the derived image coordinates Y_1 and Y_2 are used as observations. Because the resection in photogrammetry consists of the determination of the six orientation parameters (ω, φ, κ, U_E, V_E, W_E) from the positions of three or more image and object points, the values of X_1°, Y_1^b

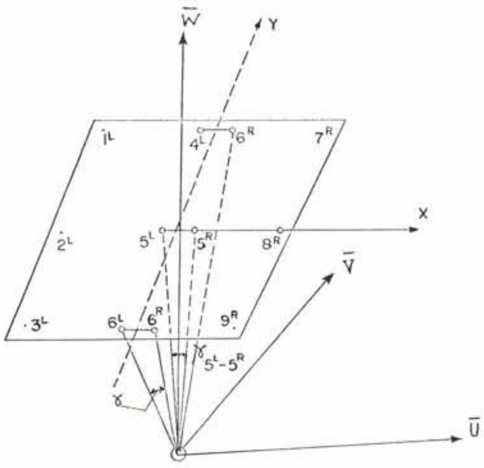

FIG. 1. Convergence condition.

and $Y_2{}^b$ are used for this purpose. The outcome of this phase is W, A_1 and A_2, needed for the final least-squares adjustment of Equation 3.

The Half-Photo Case. A half-photo as a unit in space resection is treated because it is needed for mathematical model D (which is indicated later). The resection is performed by using either $Y_1{}^b$ or $Y_2{}^b$ as observations for solving $X_2{}^o$, which constitutes six parameters for each half-photograph. The mathematical formulation is:

For the left photo of model j,

$$F(Y_1{}^a, X_2{}^a) = 0 \qquad (5A)$$

and

$$V_1 = A_2 X_2 + W_1 \qquad (5B)$$

where

$$W_1 = F(Y_1{}^b, X_2{}^o). \qquad (5C)$$

Similarly for the right photo of model j,

$$F(Y_2{}^a, X_2{}^a) = 0 \qquad (6A)$$

and

$$V_2 = A_2 X_2 + W_2 = 0 \qquad (6B)$$

where

$$W_2 = F(Y_2{}^b, X_2{}^o). \qquad (6C)$$

Calculation of Q_Y. Because Y_1 and Y_2 are calculated using Formulas 1 and 2, and are functions of the original variables (the model coordinates M and the relative orientation elements X_R), then these formulas have to be linearized with respect to M and X_R. Furthermore, to study the radial weighing proposed by Hallert,[8] and the effect of X_R on Q_Y, the following assumptions have been made.

(a) $\bar{Q}_Y = \sigma_0{}^2 \cdot Q_Y$ (7A)

where

$$\sigma_0{}^2 = \sigma_{01}{}^2 + \sigma_{02}{}^2. \qquad (7B)$$

(b) Linearizing, Formulas 1A and 1B are:

$$Y_1 = B_1 \cdot \Delta M + G_1 \cdot \Delta X_R \qquad (8A)$$

$$Y_2 = B_2 \cdot \Delta M + G_2 \cdot \Delta X_R. \qquad (8B)$$

(c) Applying error propagation to formulas 8A and 8B, one gets:

$$Q_{Y1} = B_1 \cdot Q_M \cdot B_1{}^t + K \cdot G_1 \cdot Q_R \cdot G_1{}^t \qquad (9A)$$

$$Q_{Y2} = B_2 \cdot Q_M \cdot B_2{}^t + K \cdot G_2 \cdot Q_R \cdot G_2{}^t \qquad (9B)$$

TABLE 1. DETERMINATION OF THE PERSPECTIVE CENTER AND RELATIVE ORIENTATION PARAMETERS

Step	*Determination of the Perspective Center*	*Relative Orientation Parameters*
Observation	Projected grid points in X, Y plan	Y-parallax in the model measured with ω
Number of Points	9	6
Statistical Model	Equal weight and no correlation	Equal weight and no correlation
Estimate Required for	Approximate values for U_o, V_o, W_o, ω_o, φ_o, κ_o	Approximate values for κ', κ'', φ', φ'', ω''
Geometrical Model	Linear form of the collinearity equation	Linear form of the coplanarity equation
Output	Initial values for the zero setting of the instrument and the perspective center coordinates	The model coordinates, the relative orientation parameters and their estimated variance-covariance matrix
References	11, 15	5, 15
Remarks	No iteration is necessary (15)	Iterative procedure, the change is done only for the vector of misclosure

TABLE 2. CASES OF Q_Y

Case	*Equation used for Q_Y and Q_{Y2}*	Q_{Y1Y2}	C_1	C_2	C_3	K
1	9A, 9B	9C	4.1	0.017	0.000567	σ_{PY}/σ_0
2	9A, 9B	—	4.1	0.017	0.000567	1
3	$Q_{Y1} = B_1\ Q_M\ B_1{}^t$ $Q_{Y2} = B_2\ Q_M\ B_2{}^t$	—	4.1	0.017	0.000567	1
4	Diagonal term of Case 3	—	4.1	0.017	0.000567	1
5	$Q_Y = Q_I$, 11A	—	4.1	0.017	0.000567	—
6	$Q_Y = Q_I$, 11A	—	6.1	0.017	0.000567	—
7	Q_Y = Unit Matrix	—	σ_0	0	0	—

$$Q_{Y1Y2} = K\ G_1\ Q_R\ G_2{}^t. \qquad (9C)$$

(d) Q_R can be obtained by utilizing the well known numerical procedure for relative orientation as shown in Table 1.
(e) To obtain Q_M, a space intersection can be used between the image and model coordinates:

$$Q_M = (B^t\ Q_I{}^{-1}\ B) \qquad (10)$$

where

$$Q_I = (1/\sigma_0{}^2)\cdot(S^2) \qquad (11A)$$

and

$$S = C_1 - C_2 r + C_3 r^2 \qquad (11B)$$

C_1, C_2 and C_3 are chosen according to Hallert[8] or similar assumptions; and they are introduced to take into account the radial weighing for the image coordinates. Their values are listed in Table 2.
(f) The previous assumptions have been made in various adjustment tests for deriving different patterns for Q_Y; and they are used as weight coefficient matrices in the least-squares solution for Equation 3. These assumptions are described briefly in Table 2. The main idea for these choices is to prove that:

1. It is not necessary to calculate Q_R and Q_M for the least-squares solution of the suggested mathematical model.
2. The transformed image coordinates (Y_1 and Y_2), derived from the model coordinates after performing relative orientation, are mathematically uncorrelated. This proof is shown later.

CHECKING THE MATHEMATICAL DERIVATIONS

In order to check the mathematical derivations, some instrumental work was conducted on two stereoplotters (A8 and Kern PG2) and on one stereocomparator (STK1). The test data of this experiment are given in Table 3. Test number 1 was performed first to check the mathematical derivations given in the first section, and it was conducted in the following steps.

(a) Determination of the coordinates of the perspective center and the relative orientation parameters as shown in Table 1. The outcome of this stage is:

1. *$U'o$, $V'o$, $W'o$ and $U''o$, $V''o$, $W''o$*
2. $\sigma_{02} = 4.23\ \mu m$
3. X_R, Q_R *and* σ_{PY}
4. U_M, V_M *and* W_M.

(b) Using Equations 1A and 1B to obtain Y_1 and Y_2.

(c) Determination of $X_1{}^o$ by polynomial adjustment (Carlin[25]) after strip formation.

(d) Calculation of different Q_Y as outlined in Table 2.

(e) A simultaneous least-squares adjustment for Equation 2 using geometrical Approaches *A,C,D,E* and *F* (as shown in the next section).

The other test data are used when the suggested mathematical model (Approach *D*) is compared with other analytical systems in practice (as shown in a later section).

TABLE 3. TEST DATA

Test	*Shediac Area New Brunswick*	*Swiss Test Block*
Type of Photography	Wide Angle	Super Wide Angle
Scale	1/16,000	1/63,000 − 1/78,000
Principal Distance	152.07 mm	88.24 mm
Format	230 × 230 mm	230 × 230 mm
Number of Strips	1	2
Number of Models	6	8
Instrument Used	Wild A-8	K-PG2, STK1

TABLE 4A. THE GEOMETRICAL APPROACH USED

Approach	X_2	*Geometry*		*Basic Unit of Adjustment*
A	6	$F(L^a, X_1^a, X_2^a) = 0$	(2A)	Photo
C	8	$F(L^a, X_1^a, X_2^a) = 0$	(2C)	Photo
D	9	$F(L^a, X_1^a, X_2^a) = 0$	(2D)	Half Photo
E	9	$F(L^a, X_1^a, X_2^a) = 0$ $F(X_{2R}^a) = 0$	(2E)	Half Photo
F	9	$F(L^a, X_1^a, X_2^a) = 0$ $F(L^a) = 0$	(2F)	Half Photo

CHOICE OF THE BEST PARAMETERS

Different approaches to various geometrical models were tested; they are listed in Table 4A. The following are remarks on these approaches.

★ Approaches *A*, *C* and *D* are based on Equation 2, the difference is the number of X_2 per photo.

★ Approach *E* used Equation 2 and the condition between unknown parameters X_2, which implied that the orientation elements for the left and right halves of photo *K* should be equal. This condition makes Approach *E* identical with Approach *A*.

★ Approach *F* also used Equation 2 together with the condition between observations (Y_1, Y_2), which implied that the space angle between any two identical pass points and the perspective center should equal zero, as shown in Figure 1. This condition occurs for a particular pass point in the triple lap area between two adjacent models. This condition is already implied in Approach *D* (and Equation 2) for the reason that in Approach *D* the collinearity condition is used for both rays specified in *F*. Hence, this condition is not an independent condition and it follows that geometrical models *F* and *D* should be identical.

★ The solutions of these geometrical approaches by a least-squares method are treated and developed by Maarek,[15] following the principles given in References 20, 17, 24, 3.

★ The orientation parameters X_2 represent the six exterior orientations per photo in case of Approach *A*, the six exterior orientations per photo and two shift components for the inner orientation per photo for Approach *C*, and the nine orientation elements for Approaches *D*, *E* and *F* (they are three translation parameters for each photo (X_{2T}) and three rotational parameters (X_{2R}) for each half-photo). (Discussion of the latter point is given next.)

THE NECESSITY OF INTRODUCING ADDITIONAL THREE ROTATIONAL ELEMENTS

The necessity of introducing three rotational parameters while keeping the translation elements the same is justified by the following items.

■ The fact that the transformed image coordinates $Y_1{}^b$ and $Y_2{}^b$ (which are calculated by Equations 1A and 1B) for a particular intermediate photo in a certain strip come from different relatively oriented models.

■ $Y_1{}^b$ and $Y_2{}^b$ are already rotated by the

TABLE 4B. APPROXIMATE PARAMETERS FOR X_2.

Photo	x^o_{2T} *(in ft.)*. U^o_E	V^o_E	W^o_E	x^o_{2R} *Grads* ω^o_E	φ^o_E	κ^o_E
1	1489597.8	900103.7	7974.80	0.8821	−.1616	199.8237
2R	1494356.1	900090.3	7987.20	0.8223	−.1784	199.8276
2L	1494351.7	900089.9	7987.40	−0.2525	−0.0322	199.9378
3R	1499488.7	900084.7	7989.13	−0.2476	−0.0139	199.9423
3L	1499487.5	900088.4	7987.80	−0.0352	0.1281	200.2045
4R	1504391.8	900102.5	7977.98	−0.0369	0.1278	200.2051
4L	1504390.9	900105.71	7974.96	0.4222	−0.1506	199.8838
5R	1509025.3	900097.8	7986.68	0.4160	−0.1618	199.8834
5L	1509030.1	900101.2	7986.80	0.3859	−0.2128	199.9089
6R	1513684.6	900057.3	8001.50	0.4169	−0.2130	199.3976
6L	1513685.0	900057.0	7999.10	0.5607	−0.0219	199.2506
7	1518305.7	900003.9	8000.00	0.5453	−0.0177	199.2515

angular parameters of the relative orientation X_R.

■ The positions of these $Y_1{}^b$ and $Y_2{}^b$ are not related to a fixed coordinate system as in the case of a normal analytical triangulation measured on a monocomparator.

■ A study of the values of $X_2{}^o$, after space resection using the half-photo (Table 4B), shows that the differences between the translation elements for the right and the left photo do not exceed 2.5 feet (deviation from the mean). This is in the order of 30 μm at image scale, and in the order of the standard deviations of the perspective center of the stereoplotter. Furthermore, the vector of misclosure W in Equation 3 is always calculated by returning to the original Equation 2. Thus it is only necessary to introduce X_{2T} as three parameters per photo.

■ On the other hand, from the same Table 4B, the range of change between the sets of $X^o{}_{2R}$ for the right and the left halves of the photos has reached the value of one grad. Thus, it is essential to introduce three rotational parameters for each half-photo. It is very essential to introduce this change because it does not correspond with the change (which does not exceed 0.0005 grad) of the rotational elements of the perspective center of the projectors of the stereoplotter.

The Choice of the Suggested Model

Analyses of the different geometrical approaches using different Q_Y (Table 2), were conducted to clarify two points: (1) the necessity of introducing three rotational parameters for each half of photo, and the best geometrical approach, and (2) the choice of weight coefficient matrix for the suggested model.

Table 4C. Results of Using the Geometrical Approaches

Approach	*Statistical Case*	$\sigma_0{}^2 (mm^2)$	*df*
A	7	0.932	137
A	2	0.790	137
E	2	0.797	137
C	2	0.798	123
D	2	0.000349	122
F	7	0.000150	122

With respect to the first point, the results obtained are listed in Table 4C. From these results, it is obvious that the unbiased geometrical model corresponds to Approach *D* because its estimated variance of unit weight $\hat{\sigma}_0{}^2$ is the smallest.

To study the second point, the different weight coefficients given in Table 2 are used, and the results obtained are shown in Table 5. Comparing the estimated variance $\hat{\sigma}_0{}^2$ of unit weight of each case with each other, and checking $\hat{\sigma}_0{}^2$ of each case with the tabulated value of the Fisher test, it is found that the unbiased $\hat{\sigma}_0{}^2$ corresponds with the statistical model Case 7 (Table 5). This leads to the conclusion that the transformed image coordinates behave as uncorrelated observations and with equal weights.

Statistical Validity of the Results

Statistical tests have been applied for testing the behaviour of V and it is found that the following characteristics are valid.

▲ With respect to the estimated mean of $V(U)$; it is found that U is an unbiased estimate by checking it against the theoretical mean U_t (U_t equals zero). The statistical test used in this case is the normal distribution test.

▲ With respect to the shape and the density of V*, V was plotted in histogram and it was found that it belongs to normal distribution with modulated structure Romanowski.[22,23] The statistical test applied in this case was χ^2-test.

▲ The number of rejected observations were found as listed in Table 6.

Table 5. Choice of the Weight Coefficient Matrix

Statistical Case	ΣV $mm \times 10^{-2}$	ΣV^3 $mm^2 \times 10^{-3}$	$\sigma_0{}^2$ μm^2	*Remarks*
1	8.89	0.139	349.62	
2	8.60	0.130	349.62	
3	8.52	0.127	349.62	
4	8.52	0.127	349.62	
5	−1.65	−0.234	314.15	
6	8.21	0.222	290.64	
7	−0.079	−0.687	150.36	n = 304
7'	−0.095	−0.224	82.23	n = 298

* See also "Modulated Normal Distribution and Photogrammetric Measurements," by the author, Photogrammetric Engineering, 39:8, 1973.

TABLE 6. NUMBER OF REJECTED OBSERVATIONS

Test	*No. of Observations*	*Rejected Observations*
Grid Measurement	630	2
Test 1	304	6
Test 2	406	8

These rejected observations were relatively few, which agrees with the ideas of Rosenfield.[21] Also these residuals have biased character and gross magnitude and they do not belong to the normal distribution; furthermore their magnitude is larger than $4\ \hat{\sigma}_0$.

▲ With respect to the estimated $\hat{\sigma}_0^2$, this is usually carried out by the use of χ^2-test as follows:

$$(1-\alpha) = P\left[\frac{\hat{\sigma}_0^2 \cdot df}{\chi_{df,\alpha/2}{}^2} < \sigma_0^2 < \frac{\hat{\sigma}_0^2 \cdot df}{\chi_{df,(1-\alpha/2)}{}^2}\right] = 0.95. \quad (12A)$$

The necessary statistical quantities for Equation 12A are given in Table 7.

$\hat{\sigma}_0^2$ is obtained from Table 5, σ_0^2 is calculated using Equation 7B. Substituting in Equation 12A, one gets:

$$63.83 < 66.89 < 110.88 \quad (12B)$$

This means that $\hat{\sigma}_0^2$ is an unbiased estimated variance of unit weight.

▲ Finally, it is of great interest on concluding the statistical analyses for the suggested model, to check the power of the χ^2-test on accepting $\hat{\sigma}_0^2$ to represent the sample variance; in other words, one likes to know the probability of error Type II (B) when the χ^2-test is used to test the hypothesis concerning the variance $\hat{\sigma}_0^2$. Such a test is shown in Hamilton[7] (page 83). Using this test, it is found the $B = 60$ percent.

COMPARISON OF THE SUGGESTED MATHEMATICAL MODEL WITH OTHER TRIANGULATION SYSTEMS

In the following the obtained results from the two test data given in Table 3 were compared with the other triangulation systems in practice. Such a comparison is difficult due to the lack of information about an equivalent value to $\hat{\sigma}_0^2$ in the published aerial triangulation adjustments by other organizations. The limited size of the test materials used, the distribution of the ground control, bridging distance, type of photography and scale of photography question the validity of such comparison. Furthermore, in photogrammetric practice, RMSE (Anderson[6]) constitutes the normal criterion for comparison, for which either a test area is required[12] or simulated blocks[6] must be used.

TABLE 7. STATISTICAL QUANTITIES FOR EQUATION 12A

$\hat{\sigma}_0^2$	df	σ_0^2	$\chi^2_{116,0.025}$	$\chi^2_{116,0.975}$	α
82.23	116	66.89	87.87	150.13	0.05

TABLE 8A. COMPARISON OF $\bar{S}_U$, $\bar{S}_V$, $\bar{S}_W$ WITH RMSE FROM ANDERSON[6], VALUES IN METERS (c REPRESENTS SCALE FACTOR)

Estimated Standard Deviation for X_1	S_U	S_V	S_W	*Remarks*
Calculated	0.212	0.179	0.352	
S×c	0.870	0.740	1.450	c = 4.11
RMSE	$\bar{S}_U$	$\bar{S}_V$	$\bar{S}_W$	(6)
Minimum	0.570	0.520	0.690	
Maximum	1.400	1.700	11.900	

Comparison of the values of S_U, S_V and S_W by the results given by Anderson[6] are shown in Table 8A where S_U, S_V and S_W are calculated using the equations,

$$S_U = \sqrt{(\Sigma \hat{S}_{U_i}/n_u)}$$
$$S_V = \sqrt{(\Sigma \hat{S}_{V_i}/n_v)}$$
$$S_W = \sqrt{(\Sigma \hat{S}_{W_i}/n_w)}$$
$$S_P = \sqrt{(S_U{}^2 + S_v{}^2)} \quad (17)$$

where $\hat{S}_U$, $\hat{S}_V$, $\hat{S}_W$ are obtained using the diagonal elements of the variance-covariance matrix of the adjusted pass point coordinates (E_{X_1}) after the least-squares adjustment of equation 2; n_U, N_V, n_W are the number of the coordinates of the pass points in X, Y, Z directions respectively. Another comparison is done between the estimated variance of unit

TABLE 8B. COMPARISON OF $\hat{\sigma}_O$, WITH $\sigma_{XY}(\sigma_X, \sigma_Y$ FROM JAKSIC[12], P. 400, 401)

	Computation		
	Unrefined Coordinates	*Refined Coordinates*	*Remarks*
σ_X	± 8.7	± 7.6	
σ_Y	± 7.4	± 6.2	(12)
σ_{XY}	±11.4	± 9.8	
$\hat{\sigma}_O$	±12.3	±12.3	n = 304
	± 9.1	± 9.1	n = 298

TABLE 9. RMSE OF SIMULTANEOUS, MATHEMATICAL MODEL D AND POLYNOMIAL ADJUSTMENTS (USING STEREOCOMPARATOR DATA; $n_c = 18$ POINTS

	Simultaneous AMAREO (25)	*Mathematical Model D*	*Polynomials (29)*	*Units*
$\bar{S}_U$	2.63	3.05	3.72	meters
$\bar{S}_V$	2.87	3.26	3.81	meters
$\bar{S}_W$	2.93	3.76	4.18	meters
number of iterations	2	1	15	

TABLE 10. RMSE OF MATHEMATICAL MODEL D AND POLYNOMIAL ADJUSTMENT (USING KERN PG2 DATA), $n_c = 18$ POINTS

	Polynomials	*Mathematical Model D*		*Units*
$\bar{S}_U$	4.04	3.50	3.40	meters
$\bar{S}_V$	4.00	3.82	3.23	meters
$\bar{S}_P$	5.68	5.26	4.70	meters
$\bar{S}_W$	3.68	4.03	3.16	meters
number of iterations	12	1	2	

weight $\hat{\sigma}_0^2$ and the similar value given by Jaksic[12] shown in Table 8B.

The suggested mathematical model was used by Okuwa[16] (in the case of the second test) to prove its validity; the results obtained are given in Tables 9, 10, 11. Table 9 shows the results using the suggested model, polynomial adjustment by Schut[29] and simultaneous adjustment described by U.S.C.G. and modified by Carlin.[25] The basis of the comparison in Table 9 is RMSE, using 18 check points. Table 10 shows a comparison between the suggested model and polynomial adjustment.[29] The instrument used in the comparison given in Table 9 was stereocomparator STK1 whereas for Table 10 it was Kern PG2.

As they were available, a comparison was made between the RMSE in planimetry and height for Kern PG2 (Table 10) and the values of S_P and S_W, calculated from Equation 17, shown in Table 11. This comparison shows that it is practically accurate to consider S_P and S_W as representative for RMSE in planimetry and height respectively, despite the fact that the statistical basis for such comparison is not so rigorous.

CONCLUSIONS

Considering the results achieved if the test data are subjected to the suggested mathematical model, the following conclusions can be drawn.

C It is obvious from Tables 4A, 4B, and 4C that the best parameters for the solution came from mathematical model *D*, which assigned to each half-photo three rotations different from the other half.

C The derived observations, computed from model coordinates and relative orientation elements in a least-squares solution, behave as observations of equal weight and free from mathematical correlations. This becomes clear from Table 5 and from different choices of weight coefficient cases.

C The variances of the adjusted parameters of the coordinates of the pass points were checked and they were found to be satisfactory for the use in large scale mapping. This is obvious from the values of S_U, S_V and S_W, which are given in Table 8A.

C Comparison of the values of S_U, S_V and S_W by the results shown in (Anderson,[6] Jaksic[12]); and the comparison of the suggested system with other systems in practice given in Tables 8A, 8B, 9 and 10 prove that independent-model aerial triangulation can be used as a substitute for analytical systems, providing that the suggested mathematical model is used in the adjustment.

TABLE 11. COMPARISON OF S_P AND S_W WITH $\bar{S}_P$, $\bar{S}_W$ (VALUES OF $\bar{S}_P$, $\bar{S}_W$ FROM TABLE 10)

Number of Iterations	S_U	S_V	S_P	$\bar{S}_P/S_P$	S_W	$\bar{S}_W/S_W$
1	3.22	4.05	5.26	1.00	5.25	0.74
2	2.87	3.56	4.62	1.02	3.66	0.86

C The applied statistical analyses prove that the suggested mathematical model is an unbiased mathematical model and applies a rigorous least-squares solution.

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Appendix—Mathematical Notations

INDICES

- i Image point.
- j Model number.
- k Photo number.

SUBSCRIPTS

- M Model coordinates.
- G Ground coordinates.
- T Translation elements.
- R Rotational elements.

SUPERSCRIPTS

- t Transpose.
- a Adjusted quantities.
- b Observed quantities.
- o Approximate quantities.
- $'$ Left photo.
- $''$ Right photo.
- -1 Inversion.

LETTERS

X, Y	Image coordinates in X, Y directions.
V_X, V_Y	Residuals of the image coordinates.
U_M, V_M, W_M	Model coordinates in X, Y, Z directions, respectively.
f	Principal distance.
Uo, Vo, Wo	The perspective center coordinates in the stereoplotter.
U_E, V_E, W_E	The exposure station coordinates for photo k.
U_G, V_G, W_G	The ground coordinates in X, Y, Z directions, respectively.
A_R	Orthogonal matrix, relates the image coordinates with model coordinates.
A_E	Orthogonal matrix, relates the image coordinates with ground coordinates.
μm	micrometers.
mm	millimeters.
Y_1, Y_2	The vectors of the image coordinates of left and right photos of a certain stereomodel.
A_1	The matrix of partial derivatives of the image coordinates with respect to pass-point coordinates.
A_2	The matrix of partial derivatives of the image coordinates with respect to orientation elements.
X_1	The incremental correction for the pass point coordinates.
X_2	The incremental correction for the orientation elements.
X_R	The relative orientation elements $(\omega'', \varphi'', \varphi'', \kappa'', \kappa'')$.
σ_0^2	*A priori* variance of unit weight for the image coordinates.
σ_{01}^2	*A priori* variance of unit weight related to the wideangle photography. It is assumed to be 49 μm^2 (8).
σ_{02}^2	*A priori* variance of unit weight related to the measuring instrument. σ_{02} is obtained by grid measurement (Table 1).
S	Standard error of image coordinates.
σ_{PY}^2	The estimated variance of unit weight for the y-parallax measurements.
$\hat{\sigma}_0^2$	The estimated variance of unit weight for the observations obtained from a least-squares adjustment.
$\overline{Q}_Y$	Variance-covariance matrix for the image coordinates.
RMSE	Root mean squares error of the discrepancies on the check points.
$\overline{S}_U, \overline{S}_V, \overline{S}_W$	RMSE in X, Y, Z directions.
S_U, S_V, S_W	The unbiased estimated standard deviations for the pass point coordinates in X, Y, Z directions.
r	The radial distance of the image point.
n	number of observations.
n_C	number of check points.
df	degree of freedom.

EQUATIONS USED IN THE PAPER

$$(Vx, Xy)^t = V_1^t$$
$$(Vx, Vy)^t = V_2^t$$
$$(x, y)^t = Y_1^t$$
$$(x, y)^t = Y_2^t$$
$$(U_M, V_M, W_M)^t = M^t$$
$$(U_G, V_G, W_G)^t = X_1^t$$
$$(U_E, V_E, W_E)_K^t = X_2^t$$
$$(\kappa, \omega, \varphi)_K^t = X^t_{2R}$$
$$X_2 = \frac{X_{2T}}{X_{2R}}$$
$$V = \frac{V_1}{V_2}$$
$$W = F(L^b, X_1^o, X_2^o)$$
$$x_1^a = X_1 + X_1^o$$
$$x_2^a = X_2 + X_2^o$$
$$Qy = \sigma_o^2 . Qy$$
$$P = Qy^{-1}$$
$$B = \frac{B_1}{B_2}$$
$$B_1 = \partial Y1/\partial M$$
$$B_2 = \partial Y2/\partial M$$
$$G_1 = \partial Y1/\partial XR$$
$$G_2 = \partial Y2/\partial XR$$
$$\hat{\sigma}_o^2 = V^t PV/df$$