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Modulated Normal Distribution

Modulated normal densities must be taken into consideration if one wishes to test the distribution of photogrammetric observations.

Abstract: *A statistical analysis of photogrammetric samples by means of matching them with normal and modulated normal densities was conducted. The tested samples are composed of residuals of grid measurement observations and residuals of image coordinates. The grid measurement contained 19 independent sets of observations, whereas the image coordinates were used in a simultaneous least-squares adjustment of independent model aerial triangulation. The instrument used for the grid measurement and the independent model triangulation was the A8 stereoplotter. The result of the statistical analysis of the samples shows that they conform to modulated normal density curves.*

Introduction

The common statistical tests generally applied, in photogrammetry and geodesy, to the results of a least-squares adjustment (LSA), are based on the assumption that the residuals are normally distributed. Also, in the classical approach the method of rejection commonly used assumes a normal distribution of residuals (relying on the central limit theory) and, accordingly, only the second moment is tested. This may lead to a wrong conclusion, especially if the observations have certain weight coefficient matrices. In addition, a wrong conclusion may be drawn if the diagram of the distribution of the residuals and their higher order moments is not considered. Therefore, a statistical analysis of photogrammetric samples, particularly their matching with normal or modulated normal distributions, was conducted. This work is reported in this article and is organized into four sections as follows:

A brief presentation of the modulated normal densities.
An analysis of the grid measurement observations.
An analysis of the residuals of the image coordinates.
Conclusion.

The Modulated Normal Densities (MN)

The theory of random errors, which is based on the concept of MN distributions, differs from the classical theory of elementary errors (first proposed by Gauss and developed by Hagen[11] on the concept of the total number of elementary errors k in a given measurement). According to the classical theory, the most probable case is that there are $k/2$ positive and $k/2$ negative errors. The theory of the MN distribution introduces the assumption that in each measurement some of the elementary errors are equal to zero. This new principle was first introduced by Romanowski and Green[9,10] who applied it to geodetic and astronomical observations.[6,8] The introduction of the MN distribution replaces the normal curve, defined by two parameters (mean and variance), by density defined by three parameters (mean, variance, and non-normality parameter a). The MN density has a higher peak than the Gaussian curve for the same variance. Two modulated curves have been considered in this article. The first one is the radico normal, RN, with a equal to one half and the second one is the lineo normal, LN, with a equal to one. It is worthwhile to note that with a infinitely large, the curve becomes normal, but with a equal to three the curve is practically normal.

The ratio between the peak of the MN curve and the peak of the corresponding (i.e., with the same variance) normal curve is 1.16 and 1.09 for the RN and LN curves, respectively. Tables for the MN distributions are presented in References [7,10]. It should be noted that, in the statistical analysis of th

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closure errors in the primary triangulation of Central Europe, an attempt has been made[8] to fit the distribution of these closure errors to the LN, RN, and Gauss normal GN curves. This study showed that MN curves are superior to the GN curve.

Analysis of Grid Measurement Observations (GMO)

The GMO were composed of 10 sets of monocular observations of 9 points each and another 9 sets of monocular observations of 25 points each. The residuals of the GMO were obtained from the output of an LSA solution using space resection and applying the well known collinearity equation in its linear form. This equation in matrix notation can be written as follows:

$$-V + AX + L = O$$

where V is the vector of the residuals of the GMO, X represents the six unknown parameters, namely the three coordinates of the perspective center and the three initial values for the rotational elements, and L is the vector of the misclosure.

Because these sets of the GMO were taken at different dates, and to ensure the homogeneity of the samples, the test of the homogeneity of equal variances[4] was applied as shown in Appendix 1. From this test it was concluded that the hypothesis of equal variances cannot be rejected. The individual variances, S_i, in the two instances of 9 and 25 points were tested against their pooled variances by the χ^2-test, and passed.

After the homogeneity was tested, the estimated standard deviation S_o and the estimated mean M_o were calculated for V (two times enlargement) and they were found to be

$$S_o = 7.92\ \mu m$$
$$M_o = -0.333\ \mu m$$

Next, the minimized quadratic form of the residuals $V^T V$* was passed by using a χ^2-test for each individual adjustment for the 19 sets of GMO considered.

The problem now is to trace an appropriate theoretical normal curve that fits in the

* V^T is the transpose of V.

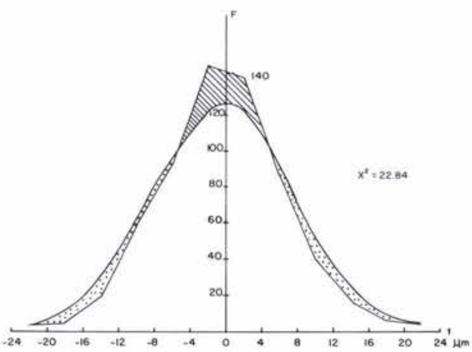

FIG. 1. Fitting of a normal curve into V using S_o.

TABLE 1. STATISTICAL ANALYSIS OF V USING S_o, M_o

Class mean of V $t(\mu m)$	Number of Values F	$Ft.$	$Ft.^2$	Class limit b μm	$\frac{b - M_o}{S_o}$	Theoretical Values (T) Normal	Lineo Normal	Radico Normal
24	1	36		24	3.0702			
22	4	88	1936	20	2.5644	2.52	3.09	3.38
18	7	126	2268	16	2.0608	8.61	9.13	9.34
14	18	252	3528	12	1.5561	25.60	23.09	22.52
10	40	400	4000	8	1.0514	54.72	49.73	47.34
6	88	528	3168	4	0.5467	91.5	88.83	86.18
2	142	284	568	0	0.0420	120.01	128.95	131.74
−2	149	−298	596	−4	−0.4627	122.59	132.88	138.17
−6	94	−564	3384	−8	− .9674	97.49	96.11	93.85
−10	59	−590	5900	−12	−1.4721	60.57	55.34	52.91
−14	20	−280	3920	−16	−1.9768	33.71	26.57	25.76
−18	5	−90	1620	−20	−2.4815	11.63	10.48	10.95
−22	3	−66	1452	−24	−2.9862	3.39	3.75	4.56
Sum	629		32340					

$M_o = (Ft.)/n = -0.333$; $S_o = \sqrt{(32{,}340/515)} = 7.92$; $S_M = \sqrt{(32{,}340/629)} = 7.19\ \mu m$.
M_o represents the estimated mean of the sample.
n represents the number of observations.

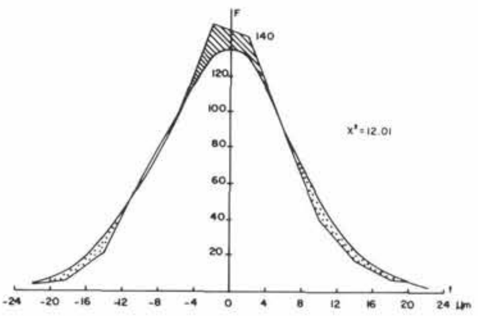

FIG. 2. Lineo-normal curve for V.

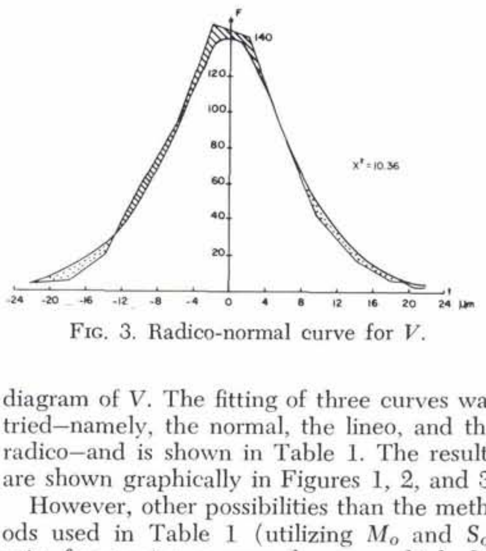

FIG. 3. Radico-normal curve for V.

diagram of V. The fitting of three curves was tried—namely, the normal, the lineo, and the radico—and is shown in Table 1. The results are shown graphically in Figures 1, 2, and 3.

However, other possibilities than the methods used in Table 1 (utilizing M_o and S_o) exist for tracing a normal curve which fits into the diagram of V. These methods are:

1. Using the same M_o but with (a) the use of S_M instead of S_o where S_M is given by $S_M = [V^TV/(n-1)]^{\frac{1}{2}}$ (n is the number of observations); (b) the use of the normalized V, obtained by dividing the elements of V by their corresponding standard deviations. These standard deviations are obtained from the covariance matrices of the residuals Q_V.

2. The use of the optimum mean M_P and the optimum standard deviation S_P where the criterion of optimality is the minimization of the χ^2-function given by $\chi^2 = (T_j - F_j)^2 / T_j$, $j = 1 \ldots m$, where T is the class count of the Gaussian curve, F is the class count of V shown in Table 1.

The methods outlined in a and b are given in Tables 2 and 3 and Figures 4 and 5. The values of M_P and S_P of Method 2 are given in Table 4.

TABLE 2. STATISTICAL ANALYSIS OF V. USING S_M, M_O

Standard Class Limit $\frac{b - M_o}{S_M}$	P	$T = nP$	$[F-np]$	$\frac{[F-np]^2}{np}$
3.8372	1.00	1.51	2.49	4.106
2.8274	0.9976	5.73	1.27	0.281
2.2711	0.9884	19.90	1.90	0.181
1.7148	0.9568	50.53	11.53	2.631
1.1585	0.8766	94.62	6.62	0.463
0.6022	0.7264	131.16	10.84	0.896
0.0458	0.5182	134.06	14.94	1.665
−0.5090	0.3054	101.93	7.93	0.617
−1.0653	0.1436	57.54	2.54	0.112
−1.6216	0.0524	23.75	3.75	0.592
−2.1779	0.0147	7.18	2.18	0.662
−2.7342	0.0033	2.07	0.93	0.418
−3.2905	0.000			
Sum		629.98		

S_M, M_O are given in Table 1.

TABLE 3. STATISTICAL ANALYSIS OF V USING Q_V, M_O

F	Standardized Class Limit	P	$T = nP$	$[F - T]$	$\frac{[F - T]^2}{T}$
8	+3	0.9986	3.19	3.71	6.95
8	+2.5	0.9938	10.44	2.44	0.56
25	+2.0	0.9772	27.67	2.67	0.25
54	+1.5	0.9332	57.81	3.81	0.25
84	+1.0	0.8413	94.22	10.22	1.11
121	+0.5	0.6915	120.45	0.55	0.00
136	0.0	0.5000	120.45	15.55	2.00
98	−0.5	0.3085	94.22	3.78	0.15
63	−1.0	0.1587	57.81	5.29	0.48
20	−1.5	0.0668	27.67	7.67	2.12
9	−2.0	0.0228	10.44	1.44	0.19
4	−2.5	0.0062	3.19	0.71	0.15
	−3.0	0.0014			

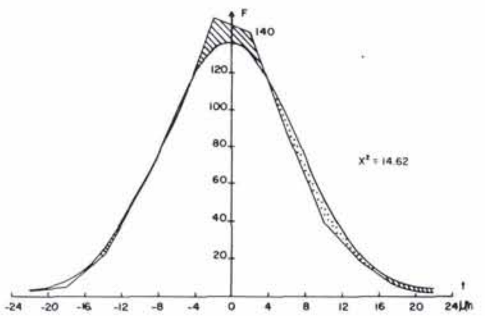

FIG. 4. Normal curve for V using S_M.

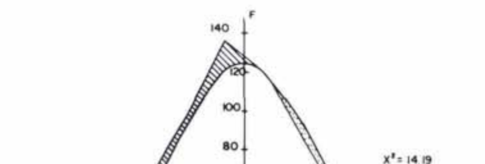

FIG. 5. Normal curve for V using Q_V.

The $\chi^2 r,\alpha$-values† obtained from the above outlined developments together with the tabulated value of this test are given in Table 4.

From Table 4 it is obvious that by using S_o and M_o, the hypothesis that the observed sample is taken from a normal distribution must be rejected. However, the hypothesis that the observation is taken from a modulated normal distribution is acceptable. Furthermore, applying the Fisher test ensured the campatibility of S_M normal, Q_V normal, and S_P normal by comparing their different χ^2-values shows in Table 4. It should be noted that the hypothesis that V is taken from a normal density population is acceptable only if S_M normal, Q_V normal, and S_P normal are used.

This concludes that the use of lineo normal, radico normal, S_M normal, Q_V normal, and S_P normal distributions strongly substantiated the hypothesis that the population from which 630 elements have been drawn is of a modulated structure.

† r represents the degrees of freedom and α the level of significance.

ANALYSIS OF THE RESIDUALS OF THE IMAGE COORDINATES

The model coordinates were observed in independent model triangulation, after performing a relative orientation on the A8 stereoplotter. The image coordinates were obtained from these model coordinates using the collinearity equation between the models, the perspective center of the plotter, and the image point. These image coordinates are refined for the known systematic errors. The adjustment procedure was divided into two stages. In the first stage, the pass points were given approximate values for their coordinates from a result of a polynomial adjustment. These preliminary coordinates, together with the refined image coordinates and the known ground control points, were entered in a space resection program.

This resection program gives approximate values for the rotation elements of each half photo and for the coordinates of the exposure stations. The second stage is a simultaneous least-squares adjustment using the collinearity equation between the image point, the exposure station, and the ground point. In this second stage, the image coordinates were used as observations to obtain the incremental correction for the unknown parameters which are X_1 and X_2. The approximate values of X_1 and X_2 were available from the first stage. Thus, the mathematical model used in the

TABLE 4. χ^2-VALUES FOR THE DIFFERENT DISTRIBUTION USED FOR V. $r = 9$, $\alpha = 0.05$

	Modulated		Normal				
Distribution	*Lineo*	*Radico*	S_o	S_M	Q_V	S_P	*Theoretical*
Mean (μm)	−0.333	−0.333	−0.333	−0.333		−0.270	
Standard Deviation (μm)	7.92	7.92	7.92	7.92		7.28	
χ^2, α = 0.05 9	12.01	10.36	22.84	14.62	14.19	14.40	16.92

r represents degrees of freedom and α the level of significance.

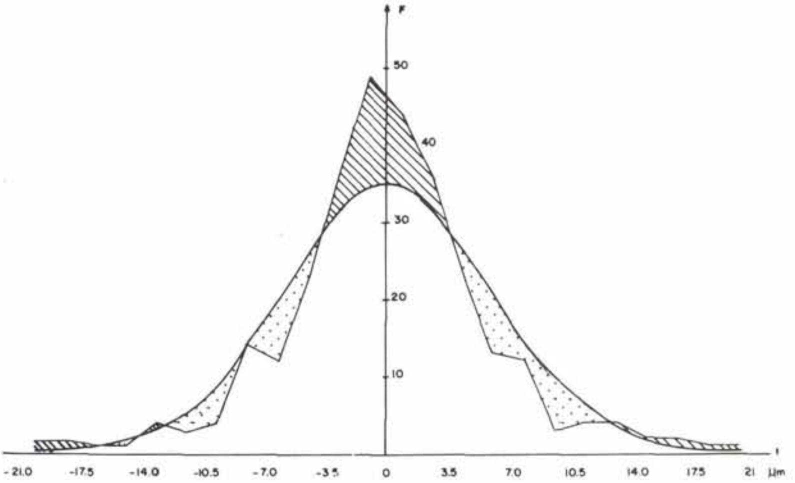

FIG. 6. Normal curve for U.

second stage can be expressed in an observation equation in the following form:

$$-U + A_1X_1 + A_2X_2 + W = 0$$

where U is the vector of the residuals of the image coordinates, X_1 is the incremental correction for the approximate values of the unknown coordinates of the pass points, X_2 is the incremental correction for the approximate values of the orientation unknowns, A_1 is the matrix of partial derivatives of the image coordinates with respect to the pass points coordinates, A_2 is the matrix of partial derivatives of the image coordinates with respect to the orientation elements, and W is the vector of misclosure.

Finally, it may be of value to point out that both V and U were subjected to study to ensure that (1) systematic errors were not present (assured by checking their first and third moments using the normal and students tests) and (2) systematic errors behave as random variables free from gross errors.

CONCLUSION

Considering the results, it is possible to conclude that:

- V follows a certain probability curve which can be either a modulated normal with *lineo or radico structure or a normal curve* with a standard deviation S_M or S_P; or a normal curve using Q_V;
- S_M normal, S_P normal, and Q_V normal are competitive, which is clear from their obtained χ^2 values;

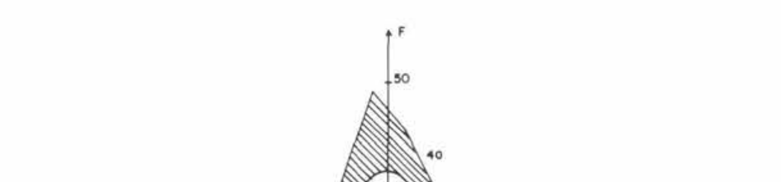

FIG. 7. Lineo-normal curve for U.

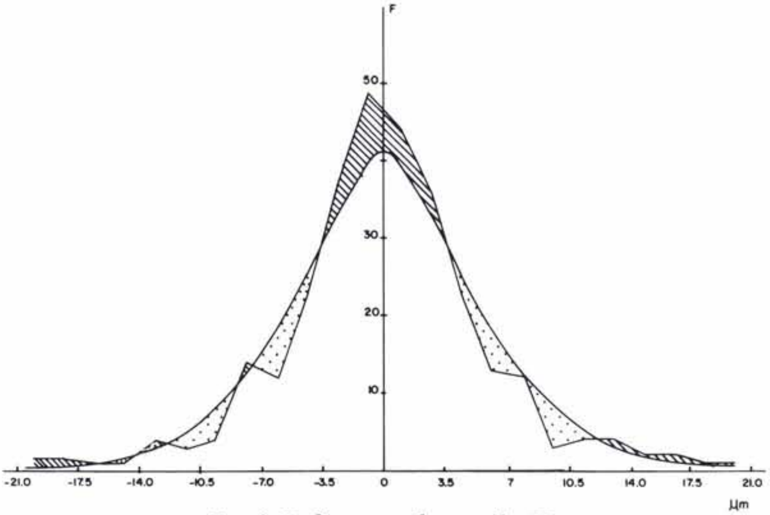

FIG. 8. Radico-normal curve for *U*.

- *U* follows a certain probability curve which is a modulated normal with either lineo or radico structure; and
- Modulated normal distributions have to be taken into consideration if one wishes to test the distribution of photogrammetric observations.

Subjecting *U* (Table 5) to the outlined analysis of modulated distribution curves, mentioned heretofore, we obtained the normal and the modulated normal curves for *U*, which are shown in Figures 6, 7, and 8. These figures correspond to the data of Table 5. This table has been constructed in the same way as Table 1. Moreover, the values of χ^2 for the fitting of S_o normal and the modulated normal curves into the diagram of *U*, are shown in Table 6.

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TABLE 5. STATISTICAL ANALYSIS OF *U*

Class mean t	*F*	*Ft.*	*Ft.*^{2}	*Class Limit*	$\frac{b-M_o}{S_o}$	*Normal*	*Lineo*	*Radico*
>21.0	2			21.00	3.5928			
19.25	2	38.50	741.12	17.50	2.9560		0.63	.75
15.75	4	63.00	922.25	14.0	2.3658	2.18	2.31	2.46
13.13	4	52.52	689.58	11.25	2.0692	2.39	3.11	3.13
11.38	4	45.52	518.02	10.50	1.7360	6.49	4.08	4.60
9.63	3	28.89	278.21	8.75	1.4780	8.52	8.66	8.27
7.88	12	94.56	745.13	7.00	1.1824	14.60	13.22	12.63
6.13	13	79.56	488.49	5.25	.8868	20.56	19.20	18.39
4.38	24	105.12	460.42	3.50	0.5912	26.61	25.98	25.32
2.63	36	94.68	249.00	1.75	0.2956	31.64	32.67	32.83
0.88	44	38.72	34.07	0.00	0.0000	34.71	37.37	39.24
0.88	49	−43.12	37.94	−1.75	0.2956	34.71	37.37	39.24
−2.63	37	−97.31	255.92	−3.50	0.5912	31.64	32.67	32.83
−4.58	24	−105.12	460.42	−5.25	0.8868	26.61	25.98	25.32
−6.13	12	−73.55	450.92	−7.00	1.1824	20.56	19.20	18.39
−7.88	14	−110.32	869.32	−8.75	1.4780	14.60	13.22	12.63
−9.63	4	−38.52	370.94	−10.50	1.7360	8.52	8.66	8.27
−11.38	3	−34.14	388.51	−11.25	2.0692	6.49	4.08	4.60
−13.13	4	−52.52	689.58	−14.00	3.5928	3.39	3.11	3.13
−15.75	2	−31.50	496.13	−17.50	2.9560	2.18	2.31	2.46
−19.25	3	−57.75	1111.68	−21.00	3.5928		0.63	0.75
<21								
Sum	298	−1.9	10436			298	297.21	296.24

$M_o = -0.006$ µm; $S_o = \sqrt{(10{,}436/298)} = 5.92$ µm

TABLE 6. χ^2 VALUES FOR THE DIFFERENT DISTRIBUTIONS USED FOR U
$S_o = 5.92$, $M_o = -0.006$, $\alpha = 0.05$

Distribution	*Normal*	*Lineo*	*Radico*	*Theoretical*	*r*
	26.36	20.15	17.05	22.36	13
χ^2 Values	32.55	22.06	17.49	25.00	15
		26.80	20.49	27.59	17

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APPENDIX 1. TEST OF HOMOGENEITY OF EQUAL VARIANCES

This test, which is commonly used, has been proposed by Barlett.[4] To apply the test one proceeds as follows:

(1) The pooled variance estimate S^2 is given by

$$S^2 = \sum_{i=1}^{n} r_i S_i^2 / \sum_{i=1}^{n} r_i \qquad \text{(A-1)}$$

where n is the number of the samples, and S_i and r_i are, respectively, the variances and the degree of freedom of the samples;

(2) The statistical argument θ is given by

$$\theta = \left(\sum_{i=1}^{i=n} r_i\right) \log S^2 - \sum_{i=1}^{i=n} r_i \log S_i; \qquad \text{(A-2)}$$

(3) The distribution of θ is rather complicated, but if $r_i \geqslant 5$, θ is satisfactorily approximated by χ^2_{n-1}. In fact, under the null hypothesis,

$$H_0\colon S_1^2 = S_2^2 = \cdots = S_n^2, \qquad \text{(A-3)}$$

the inequality

$$P\,(\theta > \chi^2_{n-1}) \qquad \text{(A-4)}$$

is strictly true.

Hence, for the application if $r \geqslant 5$, *a* χ^2-test is all that is required.

Examples of the application of the Barlett test to the grid measurement samples are as follows. For 9 points $n = 10$, $r_i = 12$, the unit of S_i is μm and the values of S_i for samples 1 . . . 10 are, respectively, 8, 7, 10, 9, 10, 9, 11, 8, 7, and 9. The results are

$S^2 = 81.1\ \mu m^2$, $\theta = 2.84$, $\chi_{9,\,0.05} = 16.92$.

Because $\theta < \chi^2$ the hypothesis of equal variances cannot be rejected.

For 25 points, $n = 9$, $r_i = 44$, the unit of S_i is μm and the values of S_i for samples 1 . . . 9, respectively, are 9, 8, 9, 7, 8, 8, 7, 8, and 9. The results are

$S^2 = 66.3\ \mu m^2$, $\theta = 2.84$, $\chi_{8,\,0.05} = 15.151$.

As $\theta < \chi^2_{8,\,0.05}$ the hypothesis of equal variance cannot be rejected.

ABBREVIATIONS USED IN THE TABLES AND GRAPHS

The area under the density curve is P, assuming this area is equal to 1. The count of the area is $-\infty$ to $+\infty$. The number of observations is n. T or nP is the class count of the frequency according to the distribution. F is the number of observations. Brackets ([ ]) are used to indicate the absolute value.

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