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Block Triangulation with Independent Models

Perimeter control is sufficient for planimetric blocks.

INTRODUCTION

IN THE United States aerial triangulation has reached a high level of application due to sophisticated programs like COMBAT, MUSAT and one or two more. They all refer to the fully analytical solution (bundle adjustment), which is superior to any other method. Due to such advanced systems, aerial triangulation and its further development seems to have lost general interest.

However, the development of aerial triangulation is not yet really completed, a number of extensions still await their realization. And it is a fact that, in most countries of the world, the application of aerial triangulation lags behind. In particular, computer programs of similar sophistication as the ones mentioned are not available or accessible.

ABSTRACT: *Since 1968 a program package for strip and block adjustment by independent models has been developed at Stuttgart University. The wide scope and general applicability of the system and some technical features of the programs are described. The considerable practical experience, as gained from large scale applications mainly, is reviewed. It confirms the anticipated accuracy capability and economy of the system.*

At Stuttgart University a photogrammetric-mathematical team has attempted during the past three years to develop a program system for the adjustment of aerial triangulations with very general scope. The development has mainly been conducted by Dr. H. Ebner, H. Klein and (recently) H. Meixner while a second group (Dr. Kraus, K. Ballein, R. Bettin) concentrated on an early version of block-adjustment for cadastral application. All persons mentioned are members of, or associated with, the Photogrammetric Institute of Stuttgart University.

Here the system will be described and the results obtained so far by its application.

PHILOSOPHY AND SCOPE OF THE SYSTEM

We started our considerations by noting that in many countries analytical methods were applied to only a limited extent, if at all. In the Federal Republic of Germany, for instance, to this day only one stereocomparator (Landesvermessungsamt, Hannover) is used in actual production (others are available, of course, at scientific institutions). We therefore adopted the method of independent models because it is generally applicable. It was expected to yield very good accuracy. Also the new generation of precision instruments was becoming available. Those instruments (Planimat, A-10, PG-

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3, Stereometrograph, etc.) are designed and fully equipped for aerial triangulation by the method of independent models.

We were aware, of course, of the increasing importance of the analytical method, because of its inherent accuracy potential. We tried, therefore, to design our system in such a way that it would allow one to incorporate the block adjustment by bundles of rays, also known as fully analytical block-adjustment. For practical reasons, however, we concentrated first on the independent model method.

It was evident from the beginning that the task would not merely be of scientific interest. Because of the great initial effort required, the system and the actual programming would have to be pushed to complete practical applicability and to a high degree of optimization.

A few years ago the drawback of practically all existing programs for block-triangulation was that they lacked generality. They were specially adapted to certain computers (language, capacity), certain applications, special conditions of input, coding, control, etc. As a result the limitations were severe, from the point of view of systems capability, application and computers.

We planned, by contrast, a system, the essentials of which can be summarized as follows[1]:

- General applicability, no limitations concerning overlap, number of points per model, number and distribution of control points (other than the well known minimum requirements).
- Large and very large blocks.
- A problem-oriented language which is generally available (Fortran IV).
- Rigorous least squares adjustment, with all given data.
- High degree of automation, easy practical handling; as little input specifications as possible, preferably no practical restrictions of any kind.

It was clear from the beginning that great efforts would be required for the practical development of the programs because of the double function as a theoretically sound and powerful system and an optimized system for direct practical application. We hoped at the beginning that we would not have to abandon in the course of the development much of the initial wishful specifications. In fact it was not necessary, but we did underestimate considerably the efforts required, in spite of quite pessimistic initial estimates.

Some Features of the System

In the course of the development of the programs a number of decisions had to be taken, on which the overall capability of the system greatly depends. Such decisions were based partly on experiments, trial programs, or on theoretical and formal considerations. Here a few shall be mentioned.

METHOD FOR SOLVING EQUATIONS

A problem of central importance is the numerical solution of normal equations of up to 10^4 unknowns. We tried iterative solutions (method of conjugate gradients) but abandoned them after experimental and theoretical evidence favored a direct method. In our opinion the advantages of iterative solutions are outweighed by some of their properties which can have particularly awkward effects: the convergence depends strongly on the quality of the initial values and on the conditioning of the system. Thus the number and distribution of control points will affect the computing time.

In poorly conditioned systems very many iterations are needed and, in addition, it is very difficult to find reliable criteria for terminating further iterations. This means that it would be difficult to predict computing times and cost.

H. Klein thus developed the HYCHOL program for solving large systems of linear equations directly. We had not found a program which was fast and sophisticated enough for our purpose. The name HYCHOL refers to using submatrices as units together with a Cholesky-solution (*Hy*per-*Chol*esky). It can handle full coefficient matrices, but it is particularly suited for banded or banded/bordered matrices.

POINT NUMBER, ETC.

A number of other features concern approximate values, control points and point-numbering. The system does not require approximate values for the unknowns. It will always start from zero tilts referring to vertical photographs or levelled models respectively. Terrestrial control points can be weighted individually; in fact for each point even a 3×3 covariance matrix is admissible. Weighting terrestrial control is desired from a theoretical point of view. In addition, it helps considerably in locating gross errors.

Also the photogrammetric points (model points) can be weighted, but in groups rather than individually. We assign weights to the x-, the y-, and the z-coordinates of model points, and separately in the same way for the projection centers. Thus 6 different weights can be introduced, or rather two 3×3 covariance matrices, each of which may be occupied with nonzero elements.

Another feature of practical importance refers to the point-numbering. Following the early version of the ANBLOCK program from ITC Delft, we also adopted the system that a point number refers to a terrain point. If a point is measured in several models its number will appear in several model listings. In this way complicated coding prescriptions are avoided, and the point numbering is natural and virtually free.

Of course, in the program search routines are needed which establish the ties between models and identify control points. Control points are such points which appear in the list of control points, called *0*-model. We use two *0*-models, one for planimetric and one for height control. The *0*-models may contain control points which have not been measured photogrammetrically.

For practical reasons the input specifications have been kept flexible. Various input formats can be accepted provided they are consistent.

PROJECTION CENTERS

It is a particular property of the method of independent models that the projection centers are treated as ordinary tie-points. For a three-dimensional treatment each model must contain two such projection centers unless sufficient overlap of models would allow one to drop them. With this approach the adjustment will leave residual errors at the projection centers. It is possible, however, through the weights assigned to them to constrain them, in particular to enforce the identity of the common projection centers of adjacent models.

How to measure projection centers cannot be discussed here. It is sufficient to state that simple and accurate methods are available such that the application of the method of independent models is practicable and economic.

ELIMINATION OF UNKNOWNS

The straightforward approach from the linearized observational equations to normal equations gives very large systems of equations containing two groups of unknowns: transformation parameters and unknown coordinates. Due to the special structure of the coefficient matrix it is easy to eliminate (by formulas) one group of unknowns out of the two. It is generally preferable to eliminate the unknown coordinates and to generate directly reduced normal equations for the transformation parameters only.

Having solved for them, the final coordinates are obtained by substitution back into the original (non-reduced) normal equations.

The coefficient matrix of the (partially reduced) normal equations for the transformation parameters is a banded-matrix. The band width depends on the ordering of the models. As the computing time required for solving the equations varies proportional to the square of the band width, it is essential to minimize band width. The program provides a highly automated procedure for that, requiring only the model numbers read in with which to start. From there on the models are automatically ordered according to their ties, irrespective of model numbering, thus keeping the computing time low, close to the theoretical minimum.

SIMILARITY TRANSFORMATION

The mathematical system is based on the concept of the similarity transformation of a model, the block adjustment performing it simultaneously for all models, taking tie points and control points into account appropriately. The mathematical formulation uses the adjustment approach for indirect observations. The observational equations for a point i measured in model j are:

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_{ij} = -\lambda j R j \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ij} - \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}_j + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i \qquad (1)$$

in which i is the point number, j is the model number, $[x\ y\ z]^{\mathrm{T}}_{ij}$ is the vector of model coordinates of point i measured in model j, $[X\ Y\ Z]^{\mathrm{T}}_i$ is the vector of terrain coordinates of point i (unknowns), $[V_X\ V_Y\ V_Z]^{\mathrm{T}}_{ij}$ is the vector of residual errors (corrections) to the transformed point i of model j, λ_j is the scale factor, R_j is a 3×3 orthogonal matrix (three independent unknowns) and $[X_0 Y_0 Z_0]^T_j$ is a shift vector. The last three terms are the orientation parameters of model j consisting of seven unknowns.

For R_j we chose a special form of the Rodriguesmatrix:

$$R_j = \frac{1}{k} \begin{bmatrix} 1 + \frac{1}{4}(a^2 - b^2 - c^2) & -c + \frac{1}{2}ab & b + \frac{1}{2}ac \\ c + \frac{1}{2}ab & 1 + \frac{1}{4}(-a^2 + b^2 - c^2) & -a + \frac{1}{2}bc \\ -b + \frac{1}{2}ac & a + \frac{1}{2}bc & 1 + \frac{1}{2}(-a^2 - b^2 + c^2) \end{bmatrix}_j \qquad (2)$$

$$k = 1 + \tfrac{1}{4}(a^2 + b^2 + c^2).$$

Weight coefficients for the observed model coordinates (scaled to terrain units) and for the terrain coordinates of the control points, respectively, can be introduced in the following form:

$$Q_{(ij)(ij)} = \begin{bmatrix} Q^{xx} & Q^{xy} & 0 \\ Q^{yx} & Q^{yy} & 0 \\ 0 & 0 & Q^{zz} \end{bmatrix}_{ij} ; \qquad Q_{(ij)(lk)} = 0. \qquad (3)$$

Different points are treated as uncorrelated.

The observational Equations 1 are non-linear in the unknown parameters. They need linearization, starting from approximate values. Regarding tilts (resp., the parameters a, b, c (2)) we always start from 0-values and obtain the linearized observational equations:

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_{ij} = \begin{bmatrix} 0 & -z & y & -x \\ z & 0 & -x & -y \\ -y & x & 0 & -z \end{bmatrix}_{ij} \begin{bmatrix} da \\ db \\ dc \\ d\lambda \end{bmatrix}_j - \begin{bmatrix} dX_0 \\ dY_0 \\ dZ_0 \end{bmatrix}_j + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i - \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ij} . \qquad (4)$$

The symbols have the same meaning as in Equation 1; *da, db, dc* refer to the increments of the parameters of Equation 2. The term (x,y,z) is understood to be the model coordinates with which to start. During the iteration process, they switch to be the model coordinates from the previous iteration.

Ground control points are also treated as observations, they also get corrections v. In order not to disturb the simplicity of the approach of Equations 1 and 4 and not to violate the band structure of the (reduced) normal equations, we introduce the additional observational equations for the terrestrial control coordinates for a point i:

$$\begin{bmatrix} V_x{}^c \\ V_y{}^c \\ V_z{}^c \end{bmatrix}_i = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i - \begin{bmatrix} X^c \\ Y^c \\ Z^c \end{bmatrix}_i . \qquad (5)$$

Here $X^cY^cZ^c$ are the terrestrial control coordinates, to which can be associated weight coefficients of the same type as described in equation 3, dropping from there the model indices j, k.

One version of the program does not operate directly with the approach of Equation 4 but rather iterates between planimetric and height adjustment.

Instead of 7-parameter transformations now in sequence, 4-parameter and 3-parameter-transformations are used. The observational equations are:

Planimetry,

$$\begin{bmatrix} V_x \\ V_y \end{bmatrix}_{ij} = \begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}_{ij} \cdot \begin{bmatrix} a \\ b \end{bmatrix}_j - \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}_j + \begin{bmatrix} X \\ Y \end{bmatrix}_i \qquad (6a)$$

$$\begin{bmatrix} V_x{}^c \\ V_x{}^c \end{bmatrix}_i \quad \begin{bmatrix} X \\ Y \end{bmatrix}_i \quad \begin{bmatrix} X^c \\ Y^c \end{bmatrix}_i \qquad (6b)$$

This is the well-known Anblock approach, supplemented by treating the terrestrial control coordinates as observation Equation 6b. The special advantage of this adjustment is its linearity in the unknowns $(a,\ b,\ X_o,\ Y_o)$ and $(X,\ Y)_i$. For this part the projection centers are excluded from the list of points i; they are not used for the determination of the planimetric transformation parameters because the convergence of the plan-height iterations would be adversely affected.

Heights. For the vertical part of the adjustment the projection centers require special consideration because their planimetric coordinates are essential for the tilt determination of the models. We have, therefore, the following observational equations (linearized):

$$[V_z]_{ij} = [x \ -y]_{ij} \cdot \begin{bmatrix} dc \\ dd \end{bmatrix}_j - [dZ_0]_j + [Z]_i - [z]_{ij} \qquad (7a)$$

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_{ij}^{PC} = \begin{bmatrix} -z & 0 \\ 0 & z \\ x & -y \end{bmatrix}_{ij} \cdot \begin{bmatrix} dc \\ dd \end{bmatrix}_j - \begin{bmatrix} 0 \\ 0 \\ dZ_0 \end{bmatrix}_j + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i - \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ij} \qquad (7b)$$

$$[V_z{}^c]_i = [Z]_i - [Z^c]_i . \qquad (7c)$$

Here Equation (7a) refers to model tie points, Equation (7b) refers to projection centers (PC), and Equation (7c) to height-control points.

The determination of transformation parameters is thus iterated in groups of *4n* and

$3n$, respectively. After each determination the models are transformed rigorously using, in principle, the full spatial similarity transformation formula, substituting the increments of the parameters just determined. Thus the model coordinates $(x\ y\ z)_{ij}$ always refer to the latest stage of transformation. This allows one, in principle, to restart adjustment computations after any iteration. Note: The symbols a,b in Equation 6 and c,d in Equation 7 are different from the ones in Equations 2 and 4.

The system provides corrections for earth curvature and refraction, by correcting the model coordinates (projection centers included) accordingly.

The Program Package

The program package has been subdivided in a number of rather separate units which are briefly outlined.

THE HYCHOL PROGRAM

HYCHOL is a separate program for solving large systems of normal equations. It works with any coefficient matrix which is symmetric and positive definite, but it is particularly effective for banded matrices. HYCHOL can handle systems of linear equations of size 10^4 and larger. It is adaptable to core capacity through choosing the size of submatrices. Table 1 gives actual computing times obtained with a CDC 6600 computer for the solution of normal equations.

With HYCHOL the normal equations are subdivided into submatrices. During the solution only three submatrices of size $t \times t$ are in the core at a time. They are brought in successively from the external storage (disk); thus there is no direct limitation of the total size of equations to be solved. Through the choice of t the system can be adapted to available core capacity at the expense, of course, of computing time, which is of input-output time mainly.

For a given system of equations the computing time for solving it with HYCHOL can be predicted very accurately.[2]

THE STRIM PROGRAM

STRIM (*str*ip adjustment by *i*ndependent *m*odels). This program was developed first as a test case for a number of problems. It uses the direct seven-parameter approach of Equations 1 and 4, respectively, after having performed strip formation first with the same general approach.

The STRIM program will not be described here. It's use has been reduced to some extent by the block program that can handle single strips as well. It maintains, however, some independent applications.

THE PAT PROGRAMS

The block programs distinguish two main systems: PAT-B, block adjustment by *b*undles (fully analytical adjustment) and PAT-M, block adjustment by independent *m*odels. The PAT-M system is completed in two versions. PAT-B became operational in the spring 1972.

The PAT-M system will have three different versions according to the type of trans-

TABLE 1. COMPUTING TIMES (SYSTEM TIME) FOR SOLUTION OF NORMAL EQUATIONS WITH HYCHOL ON CDC 6600

Number of unknowns	900	3600	19901
(Half) bandwidth	60	480	150
Number of right hand sides	1	1	25
Size t of submatrices ($t \times t$)	30	120	50
System time (CP-time + a percentage of IO-time)	12 sec	915 sec	1236 sec

formations applied: PAT-M 7 operates directly with seven-parameter transformations according to Equations 1 and 4, together with Equation 5; PAT-M 43 operates with plan-height iterations determining the seven parameters of Equation 1 in groups of $4n$ and $3n$ according to Equations 6 and 7. M 43 is considerably faster than M 7.

PAT-M 4 is the planimetric block-adjustment only, according to Equation 6. This version has had separate application in the field of cadastral photogrammetry, and was the first version that we applied in practice. Its separate use will be greatly reduced in the future. It is a special case of PAT-M 43 and is fully included there. Its special feature is the linearity in the unknowns; therefore no iterations are needed, and computing times are short.

The PAT-M system is designed in such a way that other types of transformations also could easily be incorporated. For instance a version PAT-M 63 might be used which would iterate between planimetry and heights, the planimetric transformation being affine (six parameters). In principle the system could also be extended to accept the simultaneous adjustment of transformations of models with common unknown parameters for model deformations or other common degrees of freedom which might be of interest.

THE PAT-M PROGRAMS

All PAT-M programs are subdivided into four parts.

Part 1—read-in of the data, with initial checks, external storage.

Part 2—automatic ordering of data and formal checks. The tie and control points are searched for and ordered, also the interconnected models are grouped. As a result all models are ordered in their optimum sequence and again stored externally. This part is most important for the overall optimization of the program.

Part 3—formation of the reduced normal equations by submatrices and direct solution of the system by HYCHOL; transformation of the models. In versions M 7 and M 43 this part is iterated until certain convergence criteria are met. M 43 requires usually about three iterations (counting one plan-height sequence as one iteration) although no preliminary strip formation or any other preliminary transformation of the models or computation of approximate values are applied.

Part 4—computation of the adjusted final coordinates by taking the weighted mean of tie points appearing twice or more in different (transformed) models (identical with back substitution of the parameters into the normal equations). This program includes a printout of the transformed model coordinates and their residuals and the final list of coordinates, together with statistical data, standard errors, etc.

The partitioning of the program in four parts has several advantages. It allows one, firstly, to restart the parts separately, which is of importance after partial clearing of errors, reducing the overall computing times for several runs considerably. Secondly, the total program (of about 40 K words) would unnecessarily occupy core storage. By the partitioning, each part of the program does not occupy more than 12 K words of core storage at a time.

STORAGE CAPACITIES

Although the programs are not tied to certain computers and can be applied rather generally, certain minimum capacity requirements must be observed. It is very difficult to specify lower limits exactly. The following statements must, therefore, be considered as general indications only.

The STRIM program for strip adjustment asks for core-storage capacity of preferably 32 K words. No external storage is required.

For PAT-M 43 the minimum core storage capacity should be 64 K words. In addition external disk storage is needed. For large blocks, however, or for blocks with very many points per model, the capacity should be about double. For such applications the computer should also be fast in order to avoid computing times lasting many hours.

Thus, for instance computers comparable to IBM 360/50 and below would, in principle, be capable of handling the PAT-M program. They are, however, too slow and cannot be used for the size of blocks or the quantities of data with which block adjustments become really interesting and economical.

Powerful computers like the CDC 6600, with which we are operating at Stuttgart University, are suited to adjust effectively and economically blocks of up to several thousand models, which are required for certain most interesting applications.

THEORETICAL AND EXPERIMENTAL STUDIES

Along with the development of the system a number of practical and theoretical studies were performed in order to support the validity of the approach and to estimate the accuracy capability of the system.

PLANIMETRIC ACCURACY

Most investigations refer to theoretical questions of accuracy of adjusted strips and blocks. The former theoretical case studies of the planimetric accuracy of adjusted blocks by F. Ackermann[3] were extended by H. Ebner in two ways. He determined first[1] the theoretical accuracy of large and very large blocks of independent models, up to blocks of 20,000 models. Table 2 shows the results of the very favorable logarithmic law. It can safely be concluded that *blocks with planimetric perimeter control can be used up to virtually any size.* Vice versa, by increasing the scale of photographs, covering a certain area, any accuracy can be reached down to the centimeter level. *The average accuracy of the block remains in the order of magnitude of the accuracy of a fully controlled single model.* This demonstrates the great accuracy capability of blocks.

The second extension relates to cadastral applications of block adjustment, which are distinguished by strong ties between models. In cadastral application a model may contain up to several hundred points, a considerable percentage of which (see Table 5) can be tie points. The theoretical study (H. Ebner[6]), referring to random errors only, showed the favorable effect of strong ties on the resulting accuracy of blocks (Table 3). A block with strong ties has better average planimetric accuracy than fully controlled single models. This remains valid even for blocks controlled by very few (<20) planimetric control points only, the distribution of which can be rather arbitrary.

VERTICAL ACCURACY

Very recently the investigations were extended to the theoretical height accuracy of

TABLE 2. THEORETICAL PLANIMETRIC ACCURACY ($\sigma = \sigma_X = \sigma_Y$) OF LARGE SQUARE-SHAPED BLOCKS WITH PERIMETER CONTROL

Number of models	200	800	1800	5000	9800	20000
σ_{max}/σ_o	1.19	1.30	1.36	1.43	1.48	1.52

TABLE 3. THEORETICAL PLANIMETRIC ACCURACY (STANDARD ERRORS $\sigma_X = \sigma_Y$) OF BLOCKS WITH STRONG TIES BETWEEN MODELS (EXAMPLE: 32 MODELS, 16 CONTROL POINTS ALONG PERIMETER, ABOUT 6,500 POINTS, OF WHICH 777 ARE TIE POINTS)

Standard Errors $\sigma = \sigma_X = \sigma_Y$	Strong ties (60 tie-pts. per model)	∞ ties	Normal Case, 4 Tie Points per Model	Single Models, No Ties Used, 4 Control Pts. per Model
Max. value	$1.20\,\sigma_o$	$1.04\,\sigma_o$	$1.48\,\sigma_o$	$1.22\,\sigma_o$
Mean value	$1.06\,\sigma_o$	$0.99\,\sigma_o$	$1.30\,\sigma_o$	$1.14\,\sigma_o$

blocks. In particular, the density of control was studied which is required for obtaining uniform accuracy (see also Gyer[4] and similar results from Kunij, Talts and Kilpelä). The classical concept of bridging distance has been abandoned in favor of a grid pattern of height control points, the spacing of which determines the resultant accuracy. 60 percent side overlap increases the plan accuracy by about 40 percent, the height accuracy by about 70 percent. The investigations confirmed furthermore that the fully analytical methods yield significantly better accuracies, both for plan and height, than the independent-model method, as judged from random errors.

A number of other investigations were conducted which can only briefly be mentioned here. They refer to the theoretical accuracy of various methods for strip triangulation and adjustment, to instrumental investigations on the measurement of independent models, and to the empirical weight determination (variance estimation) of model coordinates. Extensive empirical accuracy studies (OEEPE *Oberschwaben* test) will be referred to in the next section. Other studies concerned the computing times required for different methods of adjustment. The M-43 version of our block programs turned out to be considerably faster than the M-7 version or the bundle adjustment.

The combined results of the theoretical and experimental studies confirm and support the development of computer programs capable of rigorous adjustment of even very large blocks, or blocks with large numbers of tie points. They also encourage very much the consistent application of block triangulation at any level of photo scales.

A recent, most valuable study of K. Kubik[5] showing the effects of systematic errors in blocks does not seriously violate that conclusion. The study shows that in well-controlled blocks the systematic errors become visible in the residuals. They are compensated rather than propagating. It is only with poor control that some types of systematic errors have greatly deteriorating effects. Besides, our empirical results (see next section) do not indicate the presence of large systematic errors in blocks.

Experience From Practical Application of Strip and Block Adjustment by Independent Models

During the past two years considerable experience was gained from the practical application of strip and block adjustments by the method of independent models. The computer programs were used in preliminary and final versions. The majority of the data originated from practical routine production; in only a few instances did they refer to research projects. Up to now all applications (with the exception of two) have had only 20 percent lateral overlap. In the following comments some practical results are reported, classified according to various points of view.

THE STRIM PROGRAM

The results from strip adjustment by the STRIM program are not displayed here because the applications are too diverse. We have adjusted numerous strips from practical production. The scales range from 1:4,000 to 1:50,000. The computing time (system

TABLE 4. ABSOLUTE ACCURACY OF ADJUSTED STRIPS, FROM CHECK POINTS, AVERAGE FROM 8 STRIPS WITH 25 MODELS EACH (OEEPE *Oberschwaben* TEST BLOCK, FRANKFURT, WIDE-ANGLE 1:28,000 PHOTOGRAPHY, $h = 4300\ m$) STANDARD ERRORS IN CM

Bridging Distance	STRIM				*Polynomial Second Degree*			*Polynomial Third Degree*		
	σ_0	μ_X	μ_Y	μ_Z	μ_X	μ_Y	μ_Z	μ_X	μ_Y	μ_Z
2 b	24	26	27	40	47	60	65	39	39	50
4 b	24	28	29	42	49	61	65	40	39	53
6 b	24	30	29	46	52	65	72	41	39	57
8 b	24	32	33	54	53	67	78	43	41	62
12,5 b	24	48	44	88	62	78	86	—	—	—

time) with CDC 6600 is now down to 0.4 sec/model. The input/output is, however, usually more expensive than the actual adjustment. Due to gross errors, two to five runs are needed (average, three). The total treatment requires still 0.3 to 1.0 man hour/model. We found that the separation of first strip formation from adjustment with control was useful for error-finding, which is more difficult with strips than with blocks. Compared with strip formation the final strip adjustment with control usually has increased values of σ_o (standard error of unit weight) by about 25 percent.

THE OBERSCHWABEN TEST

The absolute accuracy of strip adjustment with independent models can be compared with polynomial adjustments. With the available OEEPE *Oberschwaben* test material (photo scale 1:28,000, signalized control and check points, signalized pairs of tie points, measurements by stereocomparator, models formed analytically) the absolute accuracy of strips was checked with about 80 check points per strip. For comparison polynomial adjustments of second and third degree also were computed. Table 4 shows some results, which display the inherent accuracy of strip adjustment if point identification and point transfer do not constitute significant sources of error. The effect of bridging distance on the accuracy turns out to be less marked than hitherto believed.

Four different versions of polynomials were applied, each for second and third degree (xyz interrelated; x y z independent; xy z; xy z conformal in xy). The results of the four versions differ surprisingly little ($<$ 10 percent; one exception 13 percent). Therefore Table 4 contains only the situation of interrelated polynomial adjustments for xyz.

Apart from very good overall accuracy, the test results show that it is worthwhile to apply rigorous methods of adjustment. The results can be considered representative. They are confirmed by similar results from the *Oberschwaben* superwide-angle test.

PLANIMETRIC BLOCK ADJUSTMENT

The planimetric version of the block program was completed first. Therefore most of our practical block adjustments refer to planimetric blocks in which the fields of application are mainly photogrammetric cadastral surveys. For such large-scale applications the boundary points to be measured are usually signalized in the terrain, with signals of size 10 cm $\times$ 10 cm up to 25 cm $\times$ 25 cm. The control points are also signalized in the same way. Table 5 summarizes some statistical data on a number of such planimetric block adjustments. Practical block adjustments usually do not provide one an opportunity to check the absolute accuracy by check points. Therefore σ_o (standard error of unit weight) is the only precision estimate that is regularly obtained.

Table 5 shows that most of our practical applications of block adjustment refer to large scales, signalized points and strong ties. A model having 300 points of which 150 are tie points is not exceptional.

With the exception of two blocks from abroad, the values of σ_o range from 6 μm to 13 μm, in most instances not exceeding 10 μm, referred to the negative scale. This confirms that the inherent accuracy of photogrammetric point determination is effective also in everyday routine application with conventional analogue instruments.

Because of gross errors, the block adjustments have to be repeated usually about three times. The two examples of Table 5 with 11 and 16 runs are not representative, due to special circumstances. The rate of gross errors at tie points is about 1 to 2 percent. Table 5 shows that we had relatively little trouble with ground control except for one or two instances. It should be noted that some of the examples of Table 5 were the first photogrammetric cadastral projects for the particular organization, implying initial difficulties, especially with signalization.

TABLE 5. STATISTICAL DATA ON PLANIMETRIC BLOCK-ADJUSTMENTS BY INDEPENDENT MODELS

			Number of				*Number of*				
								Gross Errors			
Type of Project	*Photo Scale*	*Instr.*	*Models*	*Control Points*	*Unknown Points*	*Tie Points*	*Runs*	*Control*	*Ties*	σ_0	
Reallotment	6000	C 8	32	42	4800	892	–	–	–	4,4 cm	
	6000	C 8	58	72	6000	2284	4	0	20	6,0	
	4300	Planimat	54	65	3178	1548	4	0	23	4,5	
	4300	C 8	30	47	2709	1075	4	0	32	4,8	
	6000	C 8	29	49	2674	1050	3	0	51	5,6	(1)
	4300	Planimat	42	62	4586	1768	5	1	41	4,8	
	4000	Planimat	46	19	2925	1750	11	2	47	4,1	
	10000	C 8	33	19	3791	406	3	–	–	8,2	
Cadaster	7500	A 7	170	32	1065	950	5	0	29	5,7 cm	
		C 8	14	18	656	480	2	0	1	3,9	(1)
	5000	C 8	12	17	2121	296	4	0	5	4,5	
	4200	C 8	6	34	493	107	16	9	50	5,1	
	4000	C 8	17	37	2502	540	2	0	7	2,9	
	3600	Planimat	9	61	419	226				3,8	
	3600	PSK	9	62	392	213				2,0	
	6000	PSK	3	60	384	39				3,8	
	1800	PSK	6	21	125	55				1,7	
	3400	C 8	50	17	244	177				5,5	
	7500	A 7	4	12	173	160				5,5	
Vine-yards reall.	6000	C 8	3	8	932	168	1	0	0	6,0 cm	
	6000	C 8	5	11	3811	1490	4	0	32	7,0	
	6000	C 8	3	9	1845	576	3	1	18	7,8	
	6500	C 8	3	13	1702	530	2	0	5	3,7	
	28000	PSK	200	40	1400	900				20 cm	
	14000	A 8	129	36	442	366				28	(2)
	84000	Stecometer	243	54	1363	825	2	0	5	2,46 m	(2)

(1) Double overlap.
(2) Tie points marked artificially.

TABLE 6. ABSOLUTE ACCURACY ($\sigma_{check} = \sigma_x = \sigma_y$) OF PLANIMETRIC BLOCKS OF INDEPENDENT MODELS

Photo-scale 1:	Instr.	Number of Models	Number of Control Points	Number of Check Points	σ_o	σ_{check}	Control
6,000	C 8	32	42	48	4,4 cm	8,0 cm	Scattered
7,500	A 7	170	32	14	5,7 cm	8,0 cm	Perimeter
3,600	PSK	9	62	392	2,0 cm	3,9 cm	Scattered
1,800	PSK	6	21	125	1,7 cm	3,3 cm	Scattered
10,000	C 8	33	13	6	8,1 cm	12,0 cm	Perimeter
28,000	PSK	200	40	500	20 cm	35 cm	Perimeter
28,000	PSK	32	16	80	19 cm	28 cm	Perimeter

ABSOLUTE ACCURACY

A number of projects had enough ground control to use some of them (for separate test adjustments) as check points. With check points the absolute accuracy of the adjusted blocks can be estimated. Planimetric ground control along the block perimeter (the interior of the block having no control) is of particular interest. Table 6 gives results for the absolute planimetric accuracy for such points as estimated from check points. Except for the *Oberschwaben* test material (1:28,000 photo scale) the examples refer to practical routine projects. Control, tie and check points were signalized in all instances.

The ratio σ_{check}/σ_o ranges between 1.4 and 1.8 except for the very large photo scales with which the limited accuracy of the terrestrial ground survey becomes noticeable. The results confirm in general the good absolute planimetric accuracy which is expected from perimeter-controlled blocks. The *Oberschwaben* test material will be used for more detailed investigations.

SINGLE-MODEL SOLUTION

One block allowed us to compare single-model restitution with block adjustment because it had originally been measured and computed by single models. The statistical data are: photo scale 1:6,000, measured on a C-8, 32 models, 91 control points, 4,800 points of which 860 tie points. The same measurements (planimetry only) were processed first by separate transformations (based on control points only) and then by block adjustment (using control and tie points). The results compare as shown in Table 7. The internal fit of the models is very much improved by the block adjustment, whereas the residuals at the control points increase somewhat, which is due to the strong ties between the models in this instance.

COMPARATOR VS. PLOTTER

We have no comparison, as yet, for the accuracy obtained from fully analytical triangulation versus independent models. We have one case, however, where we can compare independent models obtained from comparator measurements (Zeiss-PSK) with independent models obtained directly from a precision stereoplotter (Zeiss-Plan-

TABLE 7. SINGLE MODEL VS. BLOCK

	σ_o	Mean Coordinate Residuals at Control Points	Mean Coordinate Residuals at Tie Points
Single Models	7.0 cm = 11.6 μm	5.6 cm = 9.3 μm	6.9 cm = 11.5 μm
Block Adj.	4.8 cm = 8.0 μm	7.3 cm = 12.2 μm	2.9 cm = 4.8 μm

imat). In both cases the same photographs were used. The example refers to: photoscale 1:3,600, 9 models, 390 points, 61 control points. The values for σ_o and σ_{check} as obtained by the planimetric block adjustments with independent models are:

PSK $\sigma_o = 2.0\ cm \simeq 5{,}6\ \mu m;\ \sigma_{check} = 3.9\ cm \simeq 10.8\ \mu m$

Planimat $\sigma_o = 3.8\ cm \simeq 10{,}6\ \mu m;\ \sigma_{check} = 4.6\ cm \simeq 12.8\ \mu m$

Thus, there is evidence that comparator measurements will yield considerably better results even if processed by independent models.

LARGE-SCALE EXAMPLE

Table 5 contains one example referring to the very large photo scale of 1:1,800. Here the camera had a focal distance of $f = 60$ cm. The results confirm that the accuracy rules for block-adjustment remain valid for very large scales provided the signalization is appropriate. That result is confirmed by a recent investigation of students (Förstner/Gönnenwein) from a test area flown at negative scales of 1:1,500 and 1:1,000 with 2 cameras of focal distances 15 cm and 30 cm, respectively. From this test the absolute accuracies of planimetric coordinates obtained by well-controlled strip-adjustment by independent models (models from comparator measurements) are shown in Table 8.

THREE-DIMENSIONAL ADJUSTMENT

The version PAT-M 43 of the three-dimensional block adjustment became available only very recently. The program has, so far, been applied to two practical blocks (apart from test runs). The statistical data of the two blocks are shown in Table 9.

Table 10 shows the rate of convergence of the plan-height iterations of the adjustments, by listing the maximum coordinate differences between successive iterations. Counting a plan-height sequence as one step, the maximum alterations to the previous iteration is down to 6 mm (1 mm) in the terrain with three steps, starting from 1,758 m (10,229 m). The factor between steps is here about 100. According to tests with poorer initial conditions it is safe to assume at least a factor 10. In the two applications of Table 10 one could have safely stopped after two steps. The rapid convergence of the plan-height iterations is remarkable in view of the fact that no preliminary transformations are applied whatsoever to the machine coordinates of the independent models.

TABLE 8. LARGE-SCALE PHOTOS

Focal Length f	*Photo-scale, 1:*	*Check Points*	μ_X	μ_Y
15 cm	1,500	117	1,2 cm	1,3 cm
	1,000	89	1,5 cm	1,3 cm
30 cm	1,500	111	1,1 cm	1,4 cm
	1,000	78	1,2 cm	1,3 cm

TABLE 9. THREE-DIMENSIONAL ADJUSTMENT

Block	*Photo Scale 1:*	*Instr.*	*Number of*					*Computing (system) Time with CDC 6600*
			Models	*Control Plan*	*/Height*	*Points Measured*	*Iterations*	
A	3,400	C8	50	17	23	532	3	1,3 sec/model
B	14,000	A8	129	36	26	1354	3	1,3 sec/model

TABLE 10. RATE OF CONVERGENCE OF ITERATIONS OF THREE-DIMENSIONAL BLOCK ADJUSTMENTS WITH PAT-M 43

Adjustment Iteration	*Plan* 1	*Height* 2	*Plan* 3	*Height* 4	*Plan* 5	*Height* 6
A Δx_{max}	1,501 m	8 m	0.78 m	0.18 m	0.005 m	0.001 m
Δy_{max}	1,758 m	23 m	0.93 m	0.19 m	0.006 m	0.001 m
Δz_{max}	318 m	147 m	0.47 m	0.21 m	0.005 m	0.001 m
B Δx_{max}	10,229 m	30 m	0.92 m	0.52 m	0.000 m	0.000 m
Δy_{max}	7,303 m	85 m	2.68 m	0.35 m	0.000 m	0.000 m
Δz_{max}	389 m	527 m	0.78 m	0.65 m	0.000 m	0.000 m

RESIDUAL ERRORS

The practical block-adjustments indicated from the beginning that the residual errors at the control points tend to be considerably larger than at the tie points. The (root) mean value of the residual errors has a strict ratio $\mu_v/\sigma_o = \sqrt{(r/n)}$ (r is redundancy, n is the number of observations) against the value of σ_o. In ordinary blocks (few tie points) the ratio is about 1:2, hence the average magnitude of residuals is only about 0.5 σ_o. For blocks with strong ties (unknown parameters becoming negligible in r) the ratio tends towards $\sqrt{(1 - 1/\alpha)}$, where α is the average number of measurements of a tie point in a block. From our cadastral blocks we have $\alpha \approx 2.4$, thus $\mu_v/\sigma_o \approx 0.76$. The practical cadastre blocks give an average ratio of $\mu_v \approx 0.7\ \sigma_o$. The residual errors at control points, however, are distinctly different in most cases. The average empirical ratio is $\mu_v(contr.) \approx 1.7\ \sigma_o$. The rather large residual errors are here due mainly to strong ties compared with very few control points (see Table 5) in combination with systematic errors. They also reflect some systematic errors left, and possibly tensions in the geodetic system. Some blocks, however, did not show such effects.

If needed, we apply a program *least squares interpolation* (K. Kraus[6]) for a post-treatment of the adjusted blocks. It is taken from the theory of stochastical processes (linear Wiener prediction with Wiener filtering) and corrects for systematic errors, reducing the residual errors at control points accordingly.

With the block program PAT-M the terrestrial coordinates of control points can be given weights, thus also attributing corrections to them. The main advantage relates to easier error finding, but it is also suited for taking poor ground control into account appropriately. The weights of the terrestrial values can also be determined empirically during the block adjustment by varying them until the sums of the squares of the photogrammetric residual errors of control points and tie points balance. First tests have shown that the method converges giving reasonable standard errors for the terrestrial control coordinates.

It was hoped that the computing times required would be short enough to be economical in spite of the high performance of the programs, concerning both generality and rigor of the adjustment and automated administration of the data. The computing times depend to some extent on the size of a block, and on the number of points involved. Most of our applications refer to blocks with unusually large numbers of points per model. As empirical values, for moderate numbers of points per model, we can indicate the computing time (system time) with CDC 6600 to be:

Strip-adjustment (STRIM)	~ 0.4 sec/model/run
Planimetric block-adjustment (PAT-M 4)	~ 0.5 sec/model/run
Three-dim. block-adjustment (PAT-M 43)	~ 1–2 sec/model/run.

Those figures should remain valid within rather wide limits. For the three-dimensional adjustment our experience is still limited. The adjustments of the two blocks quoted in

Table 9 both took for three iterations 1.3 sec/model/run. Due to gross errors, normally three to four runs are necessary (see Table 5).

RESULTS, CONCLUSIONS

During the past two years considerable experience has been gained from the practical application of the programs STRIM and PAT-M for strip and block adjustment at Stuttgart University. Although most of the applications up to now refer to cadastral surveys, a number of conclusions to be drawn are general.

The basic philosophy of the system has been strongly confirmed. In particular, the generality of the approach, the absence of severe limitations, and the high degree of optimization of the programs has proven most advantageous, too many practical blocks being non-standard in one way or another. The computing times have been cut back to be truly economical, even for very large blocks. With fast computers the costs of data processing are moderate or negligible, considering in particular that no pre-programs have to be applied.

Regarding accuracy, the original high expectations have been surpassed in two ways. Firstly, the independent-model method with measurements from analogue precision plotters, has consistently reached the 10 μm-level, provided point identification is accordingly precise. This accuracy capability of the independent-model method is significantly better than was expected a few years ago. Thus the suitability of precision plotters for aerial triangulation is fully proven, a result which is of paramount practical importance. It is particularly interesting to the practitioner that the accurate results have been obtained from practical routine work by straight-forward data-processing, without any special treatment or corrections applied. It is also noticeable that the theory of perimeter control being sufficient for planimetric block adjustment has been confirmed and is regularly relied upon in planning practical blocks.

The second point where the expectations have been surpassed is the high accuracy from comparator measurements, although no special corrections were applied. The evidence from our material is not yet strong, but it looks as if independent models as computed from comparator measurements are significantly more accurate than from analogue precision plotters. Possibly the 5 μm-level for σ_o can be reached with signalized points. Of course, the fully analytic method is expected to give still more accurate results.

Concluding this paper, it can only be emphasized how powerful a tool numerical photogrammetry has become through the sophisticated and general adjustment programs for block triangulation. The practical experience with our programs for strip and block adjustment confirms the high accuracy and economical potential of photogrammetric point determination, and recommends highly its consistent application at all scales.

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