PROF. EDWARD M. MIKHAIL, PH.D.
Purdue University
Lafayette, Indiana 47907
RAYMOND J. HELMERING, PH.D.
Defense Mapping Agency Aerospace Center
St. Louis, Missouri 63118

Recursive Methods in Photogrammetric Data Reduction

An algorithm is devised for applications where the size of the system of normal equations changes.

ABSTRACT: *Least squares adjustment of redundant data is the method chiefly used in computational problems of photogrammetry. Its use, however, has been mostly in a batch mode where all condition equations are collected first before a least-squares solution is derived. In several modern problems of photogrammetry, such as data editing and reinstatement and on-line data reduction, methods other than batch least squares are worthy of investigation. In this paper recursive processes are discussed and a set of algorithms are derived whereby observations may be added to or deleted from an existing set of observations. Consideration is given to two computational situations; if the change in the number of condition equations does not alter the number of mathematical parameters, and if the number of parameters, and hence the order of the normal equation matrix, changes.*

INTRODUCTION

IN ALMOST ALL of the problems encountered in computational photogrammetry, more observational information is normally available than required for an unique solution. Consequently, some method of estimation must be used to obtain the most probable values of the unknown parameters. For linear mathematical models, the method of least squares has been extensively used for this purpose. If the model is non-linear, as most photogrammetric problems are, some series expansion, such as Taylor's, is used to linearize the model. This allows for the use of least squares to estimate corrections to approximate values for the parameters rather than the parameters themselves. Depending on the values of the chosen approximations, the estimated corrections may or may not be those yielding the final answers upon adding them to the approximate values. If the approximations are coarse, the estimated corrections are added to them giving fresh approximations for a new, and normally closer, linear model. Least squares is then used again for estimating another set of corrections and the procedure is repeated until some criterion of convergence is satisfied. It must be emphasized that least-squares estimation only applies within each linearization.

The above discussion is related to general problems of computational photogrammetry where the mathematical conditions are formed at one time from an available set of input data. The data may be, for example, the set of photo coordinates to be used for solving conventional analytical problems such as multiple-photo triangulation. By contrast to these conventional methods of analytical photogrammetry, recent developments introduced newer and more dynamic concepts. Some of the most interesting are those involved with on-line methods in which a computer is interfaced with the comparator. The availability of computer-assisted comparators opens the door for more fundamental treatment and thorough evaluation of current data reduction and estimation techniques.

Data acquisition schemes for on-line comparator systems must be, by necessity, rather different from those currently used with conventional comparators. One of the obvious differences is the fact that data reduction (such as preprocessing) must overlap the data acquisition phase. This is because the computer may be used for processing data from measured image points while other points are being observed and edited. The sequential handling and processing of data necessitates different estimation methods than currently used in analytical systems.

Suppose that a set of observations yielded r linear condition equations* from which u parameters are to be estimated. A set of normal equations may then be formed and solved for the required estimates of the parameters. Suppose further that we have an on-line data acquisition system and that additional observations, such as the coordinates of another image point, yielding p additional condition equations, became available. The incorporation of these equations into the mathematical model and updating the estimates of the u parameters can be accomplished by different processing schemes which must be carefully considered. Another problem, which is akin to that of adding information, concerns the deletion of some condition equations due to measurement rejection in the editing phase of data reduction. The two problems, though not identical, are quite similar and can be handled with essentially the same mathematical formulation.

Returning to the example of p added condition equations, one of the methods which is used in off-line analytical systems is the so-called *batch* least squares. There are actually two variations of this method, depending on the size of the problem and the capacity and speed of the computer used in the reduction. In one procedure, only the input data, in the form of observed values, are stored and when the extra observations become available they are added to the stored data. The entire normal equation matrix is then formed, this time for all $(r + p)$ condition equations, and solved for the unknown parameters. This procedure is time consuming, but is normally used when problems of very large size are encountered and when the computer capacity cannot store the normal equation matrix.

In the second procedure, the normal equation matrix formed from the first r condition equations is stored into the computer. The contribution of the extra p condition equations† to the normal equations is computed and added to (or subtracted from) those already stored, and the updated set is then used to solve for the parameters. In this instance, it must be possible to store the entire set of normal equations which, of course, may require a good portion of the computer's storage capacity. Both types of batch least squares can handle additions as well as deletions, and the order of the normal equations can change due to these additions or deletions as a result of the increase in the number of parameters. In any event, the normal-equation matrix must be inverted anew after these changes take place.

Batch least squares may not be an efficient process of on-line systems, as both computer capacity and processing times are more critical than with off-line analytical systems. Consequently, other methods of successive estimation must be investigated. Mikhail,[9] Gambino and Stilwell,[5] Schmid and Schmid,[11] and others have discussed a process by which not the normal equation matrix, but rather its inverse is modified due to addition or deletion of data. These published accounts, however, represented a first step and entailed one situation in which the size of the normal equations, and consequently the number of parameters, does not change.

In this paper we shall discuss several aspects of recursive methods of data reduction. First is presented the straight-forward situation where the size of the normal equations remains unaltered. Next there is the case where the addition of information increases the number of parameters, whereas deletion of information effects the opposite. Finally, problems of estimation in non-linear models will be discussed, particularly in regard to the exactness of the recursive methods.

Fixed Number of Parameters

This is the situation that has been discussed in the literature and derived mostly in relation to least-squares estimation. Actually, the recursive formula may be given strictly as a matrix algebra operation, Equation 1. For example, given the expression,

$$\underset{u,u}{M} = \underset{u,u}{N} \pm \underset{u,p}{U}\,\underset{p,p}{W}\,\underset{p,u}{V} \tag{1}$$

the inverse of M may be evaluated from

$$M^{-1} = N^{-1} \mp N^{-1}U[W^{-1} \pm VN^{-1}U]^{-1}VN^{-1} \tag{2}$$

provided, of course, the inverses shown in Equation 2 do exist. Equation 2 may now be

* In this paper, all mathematical equations involving observations (with or without parameters) are termed *condition equations*, while those involving only parameters are called *constraint equations*.

† The extra conditions should not in this case contain any of the original observations, nor should the new observations be correlated with the original ones.

applied to *normal-equation augmentation* as follows. Let a set of condition equations be,

$$\underset{r,n}{\dot{\boldsymbol{A}}}\ \underset{n,1}{\dot{\boldsymbol{V}}} + \underset{r,u}{\dot{\boldsymbol{B}}}\ \underset{u,1}{\Delta} = \underset{r,1}{\dot{\boldsymbol{F}}^0} \tag{3}$$

and the set to be added or subtracted be,

$$\underset{s,t}{\ddot{\boldsymbol{A}}}\ \underset{t,1}{\ddot{\boldsymbol{V}}} + \underset{s,u}{\ddot{\boldsymbol{B}}}\ \underset{u,1}{\Delta} = \underset{s,1}{\ddot{\boldsymbol{F}}^0} \tag{4}$$

noting that the parameter vector is identical in both sets.* The final coefficient matrix of normal equations can readily be written in terms of both sets of condition Equations 3 and 4 as,

$$\boldsymbol{B}^t(\boldsymbol{AQA}^t)^{-1}\boldsymbol{B} = \dot{\boldsymbol{B}}^t(\dot{\boldsymbol{A}}\dot{\boldsymbol{Q}}\dot{\boldsymbol{A}}^t)^{-1}\dot{\boldsymbol{B}} \pm \ddot{\boldsymbol{B}}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\ddot{\boldsymbol{B}} \tag{5}$$

where Q, $\dot{Q}$, and $\ddot{Q}$ are the cofactor matrices which are the inverses of the weight matrices P, $\dot{P}$, and $\ddot{P}$, respectively. The modified inverse of the normal equation matrix may now be obtained directly by applying Equations 2 to 5. Thus,

$$[\boldsymbol{B}^t(\boldsymbol{AQA}^t)^{-1}\boldsymbol{B}]^{-1} = \boldsymbol{N}^{-1} \mp \boldsymbol{N}^{-1}\ddot{\boldsymbol{B}}^t[(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t) \pm \ddot{\boldsymbol{B}}\boldsymbol{N}^{-1}\ddot{\boldsymbol{B}}^t]^{-1}\ddot{\boldsymbol{B}}\boldsymbol{N}^{-1} \tag{6}$$

in which

$$\boldsymbol{N}^{-1} = [\dot{\boldsymbol{B}}^t(\dot{\boldsymbol{A}}\dot{\boldsymbol{Q}}\dot{\boldsymbol{A}}^t)^{-1}\dot{\boldsymbol{B}}]^{-1} \tag{7}$$

and the upper *signs in Equation 6 relate to* adding Equation 4 to 3, whereas the lower signs are for subtracting. In a direct manner, the relationship, similar to Equation 5, for the normal-equations constant term is given by,

$$\boldsymbol{B}^t(\boldsymbol{AQA}^t)^{-1}\boldsymbol{F}^0 = \dot{\boldsymbol{B}}^t(\dot{\boldsymbol{A}}\dot{\boldsymbol{Q}}\dot{\boldsymbol{A}}^t)^{-1}\dot{\boldsymbol{F}}^0 \pm \ddot{\boldsymbol{B}}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\ddot{\boldsymbol{F}}^0. \tag{8}$$

To give the reader some estimate for the computational efficiency of recursive versus batch processing, a system of 90 linear equations in 60 parameters was generated. As a start, a 60 × 60 normal-equation matrix was formed from all 90 equations and inverted and the time consumed was noted to be approximately eight seconds. Next, several instances of deleting a progressively larger number of equations from the original set were computed. The computation was performed both by our conventional batch least-squares process as well as by the recursive method given by Equation 6. The results are summarized in Figure 1. It can be seen that there is a cut-off point, where the two curves intersect, beyond which recursive methods offer no advantages. It must be emphasized, however, that these results hold for the size of normal equations chosen, for the computer used (Univac 1108), and the inversion algorithm of Gauss elimination. Deviation from these conditions may produce somewhat different results, but with essentially the same characteristics as those in Figure 1. As a matter of fact, if the equations were to be formed within the solution, the recursive procedure is expected to offer further time savings.

* The observations associated with $\ddot{\boldsymbol{V}}$ must not be correlated with the observations associated with $\dot{\boldsymbol{V}}$.

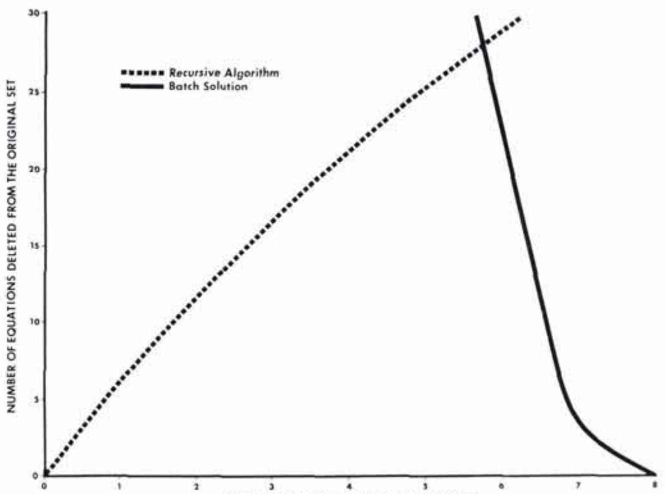

FIG. 1. Plot of the change in computation time versus the number of equations deleted from the solution.

Variable Number of Parameters

In many photogrammetric problems, the most common of which is triangulation, the increase or decrease in condition equations also increases or decreases the number of parameters in the model. For example, if the collinearity equations are used as the basic mathematical condition, every time another image for a pass point is measured and added to the model, three new parameters are added. These parameters are the object-space coordinates of the pass point. A similar situation occurs if a pass point is completely deleted, thus reducing the parameters by three.

Starting with the general case, let the total set of equations be,

$$\underset{r,n}{\boldsymbol{A}}\;\underset{n,1}{\boldsymbol{V}} + \underset{r,(u+w)}{\boldsymbol{B}} \begin{bmatrix} \underset{u,1}{\Delta} \\ \\ \underset{w,1}{\delta} \end{bmatrix} = \underset{r,1}{\boldsymbol{F}^0} \qquad (9)$$

which may be partitioned to the form,*

$$\begin{bmatrix} \dot{\boldsymbol{A}} & \boldsymbol{0} \\ \boldsymbol{0} & \ddot{\boldsymbol{A}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{V}} \\ \ddot{\boldsymbol{V}} \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{B}} & \boldsymbol{0} \\ \ddot{\boldsymbol{B}} & \boldsymbol{b} \end{bmatrix} \begin{bmatrix} \Delta \\ \delta \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{F}}^0 \\ \ddot{\boldsymbol{F}}^0 \end{bmatrix} \qquad (10)$$

The δ is a subvector of parameters by which the total number of parameters will change due to either addition or deletion. The lower Equation of 10 may be rewritten as,

$$\underset{s,t}{\ddot{\boldsymbol{A}}}\;\underset{t,1}{\ddot{\boldsymbol{V}}} + \underset{s,(u+w)}{\ddot{\bar{\boldsymbol{B}}}} \underset{u+w,1}{\begin{bmatrix} \Delta \\ \\ \delta \end{bmatrix}} = \underset{s,1}{\ddot{\boldsymbol{F}}^0} \qquad (11)$$

where

$$\ddot{\bar{\boldsymbol{B}}} = [\ddot{\boldsymbol{B}} \quad \boldsymbol{b}].$$

It is important to note that Equation 6 cannot be applied to Equations 9 and 11 directly. For example, if N^{-1} is the inverse of the normal equations for Equation 9, and M^{-1} is the inverse of the net normal equations after Equations 11 are subtracted from Equation 9, the following expression is singular:

$$[(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t) - \ddot{\bar{\boldsymbol{B}}}^t\boldsymbol{N}^{-1}\ddot{\bar{\boldsymbol{B}}}].$$

The preceding assertion gives an indication that some other means must be sought to obtain a recursive method for such cases. The total normal equation matrix for the system in Equation 9 is,

$$\boldsymbol{M} = \boldsymbol{B}^t(\boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}^t)^{-1}\boldsymbol{B}.$$

* Again, the sets of observation with $\ddot{V}$ and $\dot{V}$ are assumed uncorrelated.

Or, in view of Equation 10:

$$\boldsymbol{M} = \begin{bmatrix} \dot{\boldsymbol{B}}^t(\dot{\boldsymbol{A}}\dot{\boldsymbol{Q}}\dot{\boldsymbol{A}}^t)^{-1}\dot{\boldsymbol{B}} + \ddot{\boldsymbol{B}}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\ddot{\boldsymbol{B}} & \ddot{\boldsymbol{B}}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\boldsymbol{b} \\ \boldsymbol{b}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\ddot{\boldsymbol{B}} & \boldsymbol{b}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\boldsymbol{b} \end{bmatrix} \qquad (12)$$

or, more compactly,

$$\boldsymbol{M} = \begin{bmatrix} \dot{\boldsymbol{M}} & \bar{\boldsymbol{M}} \\ \bar{\boldsymbol{M}}^t & \ddot{\boldsymbol{M}} \end{bmatrix} \qquad (13)$$

with obvious correspondence in terms for the submatrices shown. The inverse of Equation 13 may be symbolically written as,

$$\boldsymbol{M}^{-1} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{G} \\ \boldsymbol{G}^t & \boldsymbol{H} \end{bmatrix} \qquad (14)$$

and from the fact that $MM^{-1} = I$, one can readily write,

$$\boldsymbol{E} = (\dot{\boldsymbol{M}} - \bar{\boldsymbol{M}}\ddot{\boldsymbol{M}}^{-1}\bar{\boldsymbol{M}}^t)^{-1} \qquad (15a)$$

$$\boldsymbol{G} = -\boldsymbol{E}\bar{\boldsymbol{M}}\ddot{\boldsymbol{M}}^{-1} \qquad (15b)$$

$$\boldsymbol{H} = \ddot{\boldsymbol{M}}^{-1} - \ddot{\boldsymbol{M}}^{-1}\bar{\boldsymbol{M}}^t\boldsymbol{G}. \qquad (15c)$$

The matrix E is of particular interest because it can be evaluated using Equation 2 as,

$$\boldsymbol{E} = \dot{\boldsymbol{M}}^{-1} + \dot{\boldsymbol{M}}^{-1}\bar{\boldsymbol{M}}[\ddot{\boldsymbol{M}} - \bar{\boldsymbol{M}}^t\dot{\boldsymbol{M}}^{-1}\bar{\boldsymbol{M}}]^{-1}\bar{\boldsymbol{M}}^t\dot{\boldsymbol{M}}^{-1}. \qquad (16)$$

From Equations 12 and 13 it is obvious that

$$\dot{\boldsymbol{M}} = \dot{\boldsymbol{B}}^t(\dot{\boldsymbol{A}}\dot{\boldsymbol{Q}}\dot{\boldsymbol{A}}^t)^{-1}\dot{\boldsymbol{B}} + \ddot{\boldsymbol{B}}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\ddot{\boldsymbol{B}}. \qquad (17)$$

or

$$\dot{\boldsymbol{M}} = \dot{\boldsymbol{N}} + \ddot{\boldsymbol{B}}^t(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t)^{-1}\ddot{\boldsymbol{B}}.$$

The inverse of $\dot{M}$ can be directly obtained by applying Equation 2 to Equation 17:

$$\dot{\boldsymbol{M}}^{-1} = \dot{\boldsymbol{N}}^{-1} - \dot{\boldsymbol{N}}^{-1}\ddot{\boldsymbol{B}}^t[(\ddot{\boldsymbol{A}}\ddot{\boldsymbol{Q}}\ddot{\boldsymbol{A}}^t) + \ddot{\boldsymbol{B}}\dot{\boldsymbol{N}}^{-1}\ddot{\boldsymbol{B}}^t]^{-1}\ddot{\boldsymbol{B}}\dot{\boldsymbol{N}}^{-1}. \qquad (18)$$

We now have all the elements for handling both cases of addition as well as subtraction of condition equations.

Adding Information

Referring to Equation 10, the top line would constitute the original set, and the second line the set to be added, which includes the new unknown parameter vector δ. For the original set, we would have the inverse of the normal equation matrix,

$$\dot{\boldsymbol{N}}^{-1} = [\dot{\boldsymbol{B}}^t(\dot{\boldsymbol{A}}\dot{\boldsymbol{Q}}\dot{\boldsymbol{A}}^t)^{-1}\dot{\boldsymbol{B}}]^{-1} \qquad (19)$$

and the constant term vector,

$$\dot{\boldsymbol{T}} = \dot{\boldsymbol{B}}^t(\dot{\boldsymbol{A}}\dot{\boldsymbol{Q}}\dot{\boldsymbol{A}}^t)^{-1}\dot{\boldsymbol{F}}^0. \qquad (20)$$

In order to get the new inverse M^{-1} of the augmented normal equation matrix, we first compute $\dot{M}^{-1}$ from Equation 18 noting that all matrices involved are already known. Next, the matrices $\bar{M}$ and $\ddot{M}$ may be computed using the information in Equations 12 and 13. From these and $\dot{M}^{-1}$, Equation 16 can be used to compute E, and then Equation 15 leads to computing G and H. Having these three matrices, M^{-1} may be readily constructed using Equation 14.

The new augmented constant column T would be obtained as

$$T = B^t(AQA^t)^{-1}F^0 \tag{21}$$

or, using the partitioning in Equation 10,

$$T = \begin{bmatrix} \dot{B}^t(\dot{A}\dot{Q}\dot{A}^t)^{-1}\dot{F}^0 + \ddot{B}^t(\ddot{A}\ddot{Q}\ddot{A}^t)^{-1}\ddot{F}^0 \\ b^t(\ddot{A}\ddot{Q}\ddot{A}^t)^{-1}\ddot{F}^0 \end{bmatrix} \tag{22}$$

Realizing that

$$\dot{T} = \dot{B}^t(\dot{A}\dot{Q}\dot{A}^t)^{-1}\dot{F}^0$$

equals constant-term vector from the original set of equations, Equation 22 therefore becomes

$$T = \begin{bmatrix} \bar{T} \\ t \end{bmatrix} = \begin{bmatrix} \dot{T} + \ddot{B}^t(\ddot{A}\ddot{Q}\ddot{A}^t)^{-1}\ddot{F}^0 \\ b^t(\ddot{A}\ddot{Q}\ddot{A}^t)\ddot{F}^0 \end{bmatrix}. \tag{23}$$

Deleting Information

In this case, Equation 9, and its partitioned form, Equation 10, would represent the original layer set of condition equations, from which the lower line of Equation 10 is to be deleted. Thus we would start with having the total inverse M^{-1} and seek to obtain the net inverse $\dot{N}^{-1}$. Consequently, all we need to consider is the submatrix E of M^{-1} and disregard all the rest. Using Equation 15a we evaluate $\dot{M}^{-1}$ as follows:

$$E^{-1} = \dot{M} - \bar{M}\ddot{M}^{-1}\bar{M}^t$$

or,

$$\dot{M} = E^{-1} + \bar{M}\ddot{M}^{-1}\bar{M}^t$$

and, applying Equation 2,

$$\dot{M}^{-1} = E - E\bar{M}[\ddot{M} + \bar{M}^tE\bar{M}]^{-1}\bar{M}^tE \tag{24}$$

where $\bar{M}$ and $\ddot{M}$ are computed from the condition Equations 11 which are to be deleted, utilizing the information in Equation 12 and 13.

Next, we rearrange Equation 17 and write

$$\dot{N} = \dot{M} - \ddot{B}^t(\ddot{A}\ddot{Q}\ddot{A}^t)^{-1}\ddot{B}$$

to which we apply Equation 2 and get,

$$\dot{N}^{-1} = \dot{M}^{-1} + \dot{M}^{-1}\ddot{B}^t[\ddot{A}\ddot{Q}\ddot{A}^t - \ddot{B}\dot{M}^{-1}\ddot{B}^t]^{-1}\ddot{B}\dot{M}^{-1} \tag{25}$$

which is the inverse we are seeking.

The constant term vector $\dot{T}$ may be readily obtained from T by simple inspection of Equation 23, thus:

$$\dot{T} = T - \ddot{B}^t(\ddot{A}\ddot{Q}\ddot{A}^t)^{-1}\ddot{F}^0. \tag{26}$$

The Recursive Algorithm and Non-Linear Models

In all the preceding derivations regarding recursive formulation we assumed a linear mathematical model. In photogrammetric applications, however, the estimation problems encountered are usually non-linear. This situation of originally non-linear models needs to be discussed more in detail, particularly with regard to the application of recursive methods. In order to facilitate the explanation of the various possibilities that could occur, we shall begin our discussion with the batch method and realizing that the problem of adding information is quite similar to that of deleting information, we shall consider the former without loss of generality.

Suppose that we have a set of non-linear equations,

$$F(L, X) = \mathbf{0} \tag{27}$$

in which L is the observation vector and X is the vector of parameters to be estimated. For the actual observations L^0, and the set of approximations X^0, Equation 27 may be linearized to the form,

$$\underset{r,n}{A}\,\underset{n,1}{V} + \underset{r,u}{B}\,\underset{u,1}{\Delta} = \underset{r,1}{F^0} \tag{28}$$

where the matrices A, B, and F^0 are evaluated at L^0 and X^0. A system of u normal equations can be formed from Equation 28 and solved to obtain the first set of corrections Δ_1 which, if added to the approximations, yields the first updated parameter vector,

$$X_1 = X^0 + \Delta_1. \tag{29}$$

This process may be repeated through m linearizations to give,

$$X = X^0 + \sum_{k=1}^{m} \Delta_k. \tag{30}$$

Suppose that in addition to the condition equations of Equation 27 we have an added set which we will assume contains no more than the original parameter vector X and which is denoted by

$$F_a(L, X) = \mathbf{0}. \tag{31}$$

The combination of Equations 27 and 31 gives a larger set of condition equations from which the final estimate X_f of the u parameters will obviously be different from that given by Equation 30. The most direct process, but not necessarily the most efficient, is to linearize Equation 31 at an approximation vector $X_f{}^0$ equal to X obtained from Equation 30 to give,

$$\underset{p,n}{A_a}\ \underset{n,1}{V} + \underset{p,u}{B_a}\ \underset{u,1}{\Delta_t} = \underset{p,1}{F^0}. \tag{32}$$

Next, Equation 28 is re-evaluated at the same value $X_f{}^0$ and combined with Equation 32 to form the total set of $(r + p)$ condition equations. The corresponding normal equations are formed and solved to obtain a correction, Δ_t, and the solution repeated, if necessary, until convergence is achieved. It should be noted that in this process the normal equation matrix of size u will have to be inverted each time a correction vector is computed.

Another possibility occurs if Equations 28 are stored and kept until all the additional Equations 31 become available. In this instance, Equation 31 is linearized at the approximation vector X^0 used for Equation 27 and the set of linear equations is combined with Equation 28. Normal equations are formed and solved and relinearizations applied until the solution converges to a preset tolerance.

A number of possibilities arise if the recursive algorithm is used for non-linear estimation problems. Some of these possibilities entail exact solutions whereas others involve certain approximations. To begin with, the sequential algorithm is exact if used inside each linearization. For example, suppose that the $\Delta_{t\ 1}$ is obtained from the first cycle of the solution from Equations 27 and 31 together, linearized at the same vector of approximation, X^0. If then the inverse of the normal equations and the constant term vector arising from the set of Equation 27 linearized at X^0, are modified by the sequential algorithm to include the set of Equation 31, also linearized at X^0, the product of the resulting inverse and constant vector will give $\Delta_{t\ 1}$ exactly.

For the non-linear situation, the set of Equations 28 is solved by the linearization process as shown in Equation 30. The new set of Equations 32 are then added by the sequential algorithm after being linearized with the best estimates of the parameters from the original set of equations, that is, X from Equation 30. In this situation, the resulting estimated parameter vector Δ_t will not necessarily be precisely the same as that $(\Delta_{t\ 1})$ estimated from the linear least-squares estimation problem with all equations considered together. Obviously, the degree of difference between Δ_t from the sequential and $\Delta_{t\ 1}$ from the batch will depend on the quality of the first vector of approximation, X^0.

To obtain an appreciation of this method of sequential reduction for the non-linear models, we have solved different examples of a relative orientation problem of a pair of aerial photographs. Table 1 summarizes the results obtained. First, six points were used in a regular batch least squares and the solution iterated until convergence. The answers (that is, the values of the five exterior orientation parameters) from this solution are given in the first line. Adding three more points, the second line gives the results from a batch solution where all nine points are used directly, whereas the fifth line gives the corresponding results when the three points are added sequentially. In the third and sixth

TABLE 1. COMPARISON OF BATCH AND SEQUENTIAL SOLUTIONS OF THE NON-LINEAR MODEL OF RELATIVE ORIENTATION

Type of Solution		*Estimated Parameters*				
Batch Points	*Sequential Points*	y	z	ω	φ	κ
6	0	.179 mm	151.766 mm	0°1'42.851	0°36' 0.070	0°6'17.920
9	0	.168	151.770	1'56.564	35'59.249	6'19.278
15	0	.158	151.768	2'07.057	36'33.944	6'12.551
40	0	.152	151.773	2'12.462	36'38.931	6'12.551
6	3	.168	151.770	1'56.573	36'59.238	6'19.277
6	9	.158	151.768	2'07.097	36'33.939	6'12.207
6	34	.152	151.773	2'12.462	36'38.931	6'12.551

lines the results of a total of 15 points are given, whereas those in the fourth and seventh lines are for 40 points. One can note that if an extensive case of redundancy exists, such as for a set of 40 points, the two solutions are identical. For other examples, there is some difference between the two methods of solution reflecting the already mentioned fact that the sequential algorithm is not exact for non-linear applications. However, such differences, at least for the example given, are so small that they can be neglected. Of course, we must emphasize that this is only one example which shows tendencies, but more experimentation is needed before one can formulate a firm concept.

Conclusions

An attempt has been made to deal with sequential data-reduction problems as they arise in computational photogrammetry. Continuing earlier efforts, an algorithm suitable for applications where the size of the system of normal equations changes has been formulated and tested. Also, test results are given to compare times used for sequential solutions compared to batch solutions indicating a cutoff point. Finally, a discussion of the sequential process and non-linear mathematical models points out the complexity of the problem. A possible scheme is given which, although not exact, yielded results which were very close to the exact solution, at least for the limited test given. The authors believe that for many of the computational problems of photogrammetry this scheme may work satisfactorily. However, it is recommended that more testing and experimentation must be performed before a firm opinion is formulated.

Appendix

In the text, attention was given solely to the problem of parameter estimation. Actually, least squares adjustment usually includes a second operation, namely that of *a posteriori* precision estimation (traditionally called error propagation). The cofactor matrix of the estimated parameters, Q_Δ, is perhaps, the most desired. For the case of fixed number of parameters, Q_Δ is equal to the inverse given by Equation 6. If one is dealing with a case of variable number parameters, then Q_Δ is equal to M^{-1} from Equation 14 for adding conditions, and to N^{-1} from Equation 25 if deleting conditions. All that needs to be remembered is that when sequential procedures are applied, one must use the *updated* inverse as the *a posteriori* cofactor matrix of the estimated parameters.

Once Q_Δ is evaluated, other cofactor matrices may also be obtained. For example, it can be shown that the cofactor matrix of the residuals is

$$Q_v = QA^t W_e AQ - QA^t W_e BQ_\Delta B^t W_e AQ$$

where $W_e = (AQA^t)^{-1}$ and Q is the *a priori* cofactor matrix of the net observations. Also, if the estimated observations are those equal to the original observations plus the residuals, then the cofactor matrix of the estimated observations is

$$\hat{Q} = Q - Q_v$$

All cofactor matrices may be converted to covariance matrices using the reference variance, $\hat{\sigma}^2$, which is equal to $V^t W V$ divided by the degrees of freedom.

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