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First-Order Optical Transformations

By making the relevant parameters of the various optical elements continuously adjustable, and controlling them systematically, photographs may be transformed, area by area, in quite arbitrary ways.

(Abstract on next page)

INTRODUCTION

THIS PAPER demonstrates the use of orthogonal matrices to simplify the analysis of optical systems which are designed to realize the general two-dimensional linear transform relation of image to object. In general, such systems include anamorphic lenses, and hence do not have cylindrical symmetry around the optical axis. The relations between system-transform coefficients and the relevant parameters of the component optical elements are found to be generally non-linear. As the solution of these relations can be quite tedious, it is deemed worthwhile to present an organized treatment of the subject.

As an example of application of the subject, NRI Systems has designed and manufactured instruments for data reduction of aerial photographs with substantial amounts of systematic (i.e., tilt, image motion and other types of) distortion. Stereo viewing of such photographs is sometimes required for operation of these instruments, and approximate optical correction of the distortions is hence necessary. Although the distortions are not linear, they may be treated as approximately so over the fields of view of the optical systems. In one such instrument a mini-computer first computes the distortions at the desired points of the photographs, then translates these into appropriate settings for the optical elements, and finally provides corresponding electrical signals to the motor-driven zoom lenses, continuously variable and rotatable anamorphic lenses and image rotators which are part of the optical system, and to X-Y stages which translate the photographs under the optical system. In this way

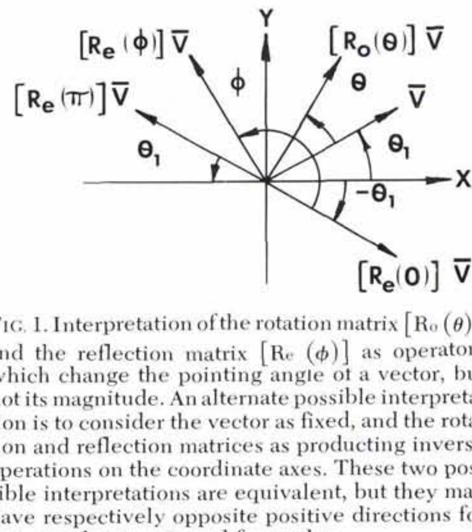

FIG. 1. Interpretation of the rotation matrix $[R_o(\theta)]$ and the reflection matrix $[R_e(\phi)]$ as operators which change the pointing angle of a vector, but not its magnitude. An alternate possible interpretation is to consider the vector as fixed, and the rotation and reflection matrices as producting inverse operations on the coordinate axes. These two possible interpretations are equivalent, but they may have respectively opposite positive directions for one-coordinate axis and for angles.

continuous automatic stereo viewing is provided with stage rates of translation up to about $(100/M)$ mm per second, where M is the particular setting of magnification (variable from 10× to 200×). At the time of writing this paper,* high performance systems utilizing these principles have been in successful operation for over two years.

Assuming that the field of view is sufficiently limited so linear treatment is a satisfactory approximation, one may take a pair of crossed vectors (in the object plane) as representative of objects to be projected. The corresponding image is then also a pair of crossed vectors (in the image plane). The op-

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tical transformation may hence be represented by the two-dimensional matrix equation:

$$\begin{bmatrix} \Delta x_1 & \Delta x_2 \\ \Delta y_1 & \Delta y_2 \end{bmatrix} = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} \Delta X_1 & \Delta X_2 \\ \Delta Y_1 & \Delta Y_2 \end{bmatrix} \quad (1)$$

where ΔX_i and ΔY_i are components of the object vectors, and Δx_i and Δy_i are components of the corresponding image vectors. Hence the objective is to express the overall transformation parameters A, B, C, D as functions of the setting parameters for the component optical elements in a form which may be solved for the latter.

Basic Principles

The technique used here is that of expressing general matrices as linear combinations of *orthogonal* matrices. An orthogonal matrix is one whose transpose is equal to its inverse, and there are two types—those whose determinant is equal to +1, and those whose determinant is equal to −1. In this paper these two types are called *rotation* and *reflection* matrices, respectively.

Two dimensional (i.e., 2 by 2) orthogonal matrices have precisely one degree of freedom, i.e., their elements may be expressed as functions of only one parameter. If the latter is taken as an angle, then the two types are:

1. Rotation (two dimensional) matrix:

$$[R_o(\theta)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad (2)$$

2. Reflection (two dimensional matrix:

$$[R_e(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix} \quad (3)$$

By multiplying them out one may see that these two types of matrix satisfy the four matrix relations:

$$[R_o(\theta_2)][R_o(\theta_1)] = [R_o(\theta_2 + \theta_1)] \quad (4)$$
$$[R_o(\theta_2)][R_e(\phi_1)] = [R_e(\theta_2 + \phi_1)] \quad (5)$$
$$[R_e(\phi_2)][R_o(\theta_1)] = [R_e(\phi_2 - \theta_1)] \quad (6)$$
$$[R_e(\phi_2)][R_e(\phi_1)] = [R_o(\phi_2 - \phi_1)]. \quad (7)$$

Figure 1 illustrates geometric interpretation of the rotation, and reflection matrices.

In terms of these a general two-dimensional matrix may be expressed as:

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} = P\,[R_o(\theta)] + Q\,[R_e(\phi)] \quad (8)$$

with

$$P = \sqrt{\left[\left(\frac{A+D}{2}\right)^2 + \left(\frac{B-C}{2}\right)^2\right]} \quad (9)$$

$$Q = \sqrt{\left[\left(\frac{A-D}{2}\right)^2 + \left(\frac{B+C}{2}\right)^2\right]} \quad (10)$$

$$\theta = \tan^{-1}\left(\frac{B-C}{A+D}\right) \quad (11)$$

$$\phi = \tan^{-1}\left(\frac{B+C}{A-D}\right) \quad (12)$$

From Equations 4 through 12 it follows that

Abstract: *Optical systems, which include anamorphic lenses operating in tandem with other types of optical elements, are capable of general linear two-dimensional transformations of the input object plane. Over limited fields of view such linear transformations are often satisfactory approximations of more general types of transformations. By making the relevant parameters of the various optical elements continuously adjustable, and controlling them systematically as functions of photograph coordinates (by computer, for example) entire photographs may be transformed, area by area, in quite arbitrary ways. In particular, corrections for tilt, image motion, and other types of systematic distortion may be obtained optically, over each successive field of view. Implementation of this scheme is greatly facilitated by the use of orthogonal matrices. The "black box" point of view considers that the overall transformation may be represented as the matrix product of individual matrices representative of each of the component optical elements.*

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= P[R_o(\theta+\theta_1+\theta_2)] + Q[R_c(\phi+\theta_1-\theta_2)]. \quad (13)$$

Hence P and Q are invariant to rotations (whether *pre* or *post* applied). Thus A, B, C, and D on the one hand, and P, Q, θ, and ϕ on the other hand are two alternate equivalent sets of four parameters for the two-dimensional first-order transformation.

A rotatable anamorphic lens has two degrees of freedom which are here taken as the stretch ratio, a, and the angle from the X coordinate axis to the anamorph major axis, θ_1. In terms of these the appropriate matrix for the anamorphic lens is

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{bmatrix} \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$= \frac{a+1}{2\sqrt{a}} [R_o(0)] + \frac{a-1}{2\sqrt{a}} [R_c(2\theta_1)]. \quad (14)$$

Hence the overall transformation of an optical train consisting of a zoom lens followed by a rotatable anamorphic lens followed by an image rotator may be represented by

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} = [R_o(\theta_2)]\,[R_o(\theta_1)] \begin{bmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{bmatrix}$$

$$[R_o(-\theta_1)]\; M\; [R_o(0)]$$

$$= \frac{M(a+1)}{2\sqrt{a}} [R_o(\theta_2)] + \frac{M(a-1)}{2\sqrt{a}}$$

$$[R_c(2\theta_1 + \theta_2)]. \quad (15)$$

In Equation 15 $[R_o(-\theta_1)]$ represents a rotation of coordinate axes—to the anamorph major and minor axes—and $[R_o(\theta_1)]$ represents rotating the coordinates back again. $[R_o(\theta_2)]$, however, represents rotation of vectors—hence has the opposite sign from the corresponding coordinate rotation. If the image rotator occurs optically ahead of the anamorph then the transformation is the same as Equation 15 except that θ_2 in the second term has its sign changed. If A, B, C, and D are regarded as known then Equations 8 through 12 may be used to solve Equation 15 for M, a, θ_1 and θ_2.

From Equations 9 and 10 it may be seen that the matrix determinant $(AD - BC)$ is equal to $(P^2 - Q^2)$. Because P and Q are, by definition, non-negative it follows that the necessary and sufficient conditions for M^2 and a in Equation 15 to be positive are that this determinant must be positive. Should a transformation be desired with values for A, B, C, and D such that $(AD - BD)$ is negative, then the expression equivalent to Equation 15 should include an image *reflector* in place of the image rotator which is shown. The solution would then yield positive values for M^2 and a—with the negative determinant.

In analyzing aerial photographs of the earth's surface it is common to choose coordinate systems in such a way that the determinant $(AD - BD)$ is generally positive, though the latter is sometimes taken as generally negative. In such applications the determinant is zero only for photograph perspective rays which are in the horizontal plane through the perspective center. As these do not usually occur, it is not likely that the determinant will change sign within any one photograph. Special provision must be made, however, for the situation when $A = D$ and $B = -C$, if the determinant is positive, or for $A = -D$ and $B = C$ if it is negative. Either of these special cases results in unity anamorphic ratio at an indeterminate angle.

As an application of these principles, we consider the net anamorphic effect of two crossed cylindrical lenses. Let each cylindrical lens have a stretch ratio N, and let the angle between the two be θ_1. Then the combined transformation of these two elements is

$$\begin{bmatrix} A_1 & C_1 \\ B_1 & D_1 \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{N} \end{bmatrix} = \begin{bmatrix} N\cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \left(\frac{1}{N}\right)\cos\theta_1 \end{bmatrix} \quad (16)$$

Hence

$$P = \sqrt{\left[\left(\frac{N^2+1}{2N}\right)^2 \cos^2\theta_1 + \sin^2\theta_1\right]^{1/2}} \quad (17)$$

$$Q = (N^2 - 1)\cos\theta_1/2N \quad (18)$$

and

$$P^2 - Q^2 = 1, \text{ as expected.}$$

Hence the effective anamorphic ratio is

$$a = \frac{P+Q}{P-Q} = (P+Q)^2$$

$$= \frac{N^4+1}{2N^2}\cos^2\theta_1 + \sin^2\theta_1$$

$$+ \frac{N^2-1}{N}\cos\theta_1 \sqrt{\left[\left(\frac{N^2+1}{2N}\right)^2\right.}$$

$$\left.\cos^2\theta_1 + \sin^2\theta_1\right] \quad (19)$$

This is a non-linear function of θ_1 which varies from N^2 to 1 as θ_1 varies from 0° to $\pm 90^\circ$. The analysis also shows that, in general, the two crossed anamorphs also produce a small net image rotation.

Generally Inclined Plane Mirror

The preceding discussion demonstrated the usefulness of rotation and reflection matrices in analyzing optical systems which include anamorphic lenses. For simplicity, the rotation and reflection matrices were treated abstractly, without discussing how they might be realized. Although it is well known that these matrices are realized with inclined plane mirrors, it nevertheless seems worthwhile to state briefly some details of how this is done. This discussion may also be considered as an application of the use of these matrices.

The *black box* view of a plane mirror is, of course, that there are *input* and *output* planes normal, respectively, to the incident and reflected sections of the optical axis, and that only vectors in these planes are considered as objects and corresponding images. It is then convenient to use two different coordinate systems for object space and image space. The latter are taken so their respective +Z axes are parallel to the two sections of the optical axis, and so that one points toward, and the other points away from, the mirror. (In other words, the two +Z axes are like an axial incident and reflected ray.) In this way the object and image planes are both represented by *X-Y* coordinate planes, which are both normal to the optical axis, and the transformation is analogous to those which were discussed previously.

The reflective transformation of a plane mirror may be represented by the vector equation

$$V_i = \sum_{k=1}^{3} \left[C_{ik} - 2\,\lambda_i\lambda_k\right] V_k \qquad (20)$$

where V_k are the object-space coordinate system components of an arbitrary object vector, V_i are the image-space coordinate system components of the corresponding image vector, C_{ik} are the components of the 3×3 rotation matrix which turns the object-space coordinate system into the image space coordinate system, and λ_i and λ_k are the image- and object-system components of a unit vector normal to the mirror. It may be shown that the matrix that is bracketed in Equation 20 satisfies the requirements for being a 3×3 reflection matrix. It will be represented by X_{ik}.

If the object-space and the image-space coordinate systems are oriented as was described earlier (i.e., with their respective Z axes parallel to the object space and image space sections of the optical axis), then the reflection matrix in Equation 20 may be put in the form:

$$X_{ik} = \frac{1}{1-C_{33}} \begin{bmatrix} (C_{11} - C_{22}) & (C_{12} + C_{21}) & 0 \\ (C_{21} + C_{12}) & (C_{22} + C_{11}) & 0 \\ 0 & 0 & (1 - C_{33}) \end{bmatrix}. \qquad (21)$$

Evidently this is equivalent to:

$$X_{ik} = \begin{bmatrix} [R_e(\phi)] & 0 \\ 0 & 1 \end{bmatrix} \qquad (22)$$

where $[R_e(\phi)]$ is the *X-Y* (2×2) sub-matrix of Equation 21 and

$$\phi = \tan^{-1} \frac{C_{21} + C_{12}}{C_{11} - C_{22}} \qquad (23)$$

Hence the Z-axis portion may be ignored if the object and image vectors are constrained to be normal to the two sections of the optical axis.

Thus, within the framework of the main subject of this paper, the transformation produced by a generally inclined plane mirror may be represented by a (2×2) reflection matrix whose angle is given by Equation 23. From this it follows (see Equations 4 through 7) that one may combine an odd number of mirrors to realize a desired reflection matrix, or an even number of mirrors to realize a desired rotation matrix. (Note that, in this respect, Dove prisms and Pechan prisms are image reflectors—rather than image rotators.)

In applying Equation 23 it is sometimes convenient to use the familiar process of treating a coordinate rotation matrix as the matrix product of rotation matrices representing plane rotations about successive positions of particular coordinate axes.

As an example of Equation 23, let the two segments of the optical axis be stated in terms of their respective aximuth and elevation angles—with respect to a common base coordinate system whose +Z axis is vertically upward. Azimuth is taken as a four-quadrant angle, positive right handed about the base +Z axis, from the base +*X* axis to the horizontal projection of the rotated +Z axis. Elevation is taken as a two-quadrant angle, positive *left* handed about the rotated +*X* axis, from the base horizontal plane to the rotated +Z axis. As stated before, the two rotated +Z axes are parallel to the two segments of the optical axis, and one (either one) points toward the mirror, whereas the other one points away from it. The mirror inclination is, of course, that which is particularly required to reflect one segment of the optical axis into the other.

With these conventions the two rotation matrices which carry the base coordinate system into the object and image systems are

$$[R_1] = [R_x(\tfrac{\pi}{2}-\theta_2)]\,[R_z(\theta_1 + \tfrac{\pi}{2})] \tag{24}$$

$$[R_2] = [R_x(\tfrac{\pi}{2}-\theta_4)]\,[R_z(\theta_3 + \tfrac{\pi}{2})] \tag{25}$$

where $\theta_1, \theta_2 \ldots$ are respectively the two pairs of azimuth and elevation angles. Hence the rotation matrix that carries the object system into the image is

$$\begin{aligned}[R_o] &= [R_2]\,[R_1]^{-1} \\ &= [R_x\ (\tfrac{\pi}{2}-\theta_4)]\,[R_z\ (\theta_3-\theta_1)]\,[R_x\ (\theta_2-\tfrac{\pi}{2})] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & S_4 & C_4 \\ 0 & -C_4 & S_4 \end{bmatrix} \begin{bmatrix} \cos(\theta_3-\theta_1) & \sin(\theta_3-\theta_1) & 0 \\ -\sin(\theta_3-\theta_1) & \cos(\theta_3-\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & S_2 & -C_2 \\ 0 & C_2 & S_2 \end{bmatrix}\end{aligned} \tag{26}$$

Multiplying these out and applying Equation 23 gives

$$\phi = \tan^{-1} \frac{(S_2-S_3)\sin(\theta_3-\theta_1)}{-C_2C_4 + (1-S_2S_4)\cos(\theta_3-\theta_1)} \tag{27}$$

with $S_i = \sin\theta_i$ and $C_i = \cos\theta_i$ This is in a form which may be readily applied to each of two or more plane mirrors in tandem. It is derived for coordinates taken with the X axis always in the horizontal plane, and with the $+Y$ axis always either in or above the horizontal plane. The image-space coordinate system for any particular mirror is, of course, the same as the object space coordinate system for the next mirror in the sequence (if there is one).

As an example, Equation 27 has been used in determining the inclinations of two mirrors and a film projection plane so as to efficiently fold an optical path into a prescribed volume, with a prescribed orientation of the doubly-reflected image.

Summary

The *black box* view of an optical system is that the optical system acts like a 2×2 matrix operator, which transforms the input object into a corresponding output image. Factoring this matrix operator into component submatrices is equivalent to analyzing the overall optical system as a group of individual *elements*, operating in tandem, with this same *black box* view taken for each individual element—and with the output plane for each element, except the last one, also being the input plane for the next element in the sequence. Hence the overall transform is constructed as the matrix product of the matrices representing individual elements, for which relevant parameters are easily specified.

Inasmuch as the purpose of this analysis is to predict the overall performance of the system as a whole, it is not necessary that the analytical subdivision correspond to a real optical subdivision. In other words, the so called *object* and *image* planes for individual elements are not necessarily planes at which the system produces intermediate optical real images. The question of where (or whether) the system requires intermediate real images is part of the design problem—which is not the subject of this paper. Nevertheless it is assumed that the system *as a whole* produces an actual optical image (final output) of an actual optical object (initial input). The assertion is that the analysis correctly describes the relation between the input (object) and output (image) for the system as a whole, and not that it implies anything about possible intermediate optical real images.

In developing this method of analysis, it turns out that the relations between the transform matrices and the relevant optical parameters of individual elements are generally non-linear, and hence tedious to work with if not treated in a systematic way. Calculation is found to be greatly facilitated if the general matrices are resolved into linear combinations of *orthogonal* matrices. Hence discussion is included of properties and applications of the two types of orthogonal matrix, which for convenience are referred to as *rotation* and *reflection* matrices.

In particular, applications of rotation and reflection matrices to anamorphic lenses, and also to plane mirrors, are discussed in some detail. As additional examples of application of orthogonal matrices, it is observed at this point that reflection matrices may be used to represent the difference between projection of *positive* and *negative* film, and also the difference between projecting either film type *emulsion side up* versus *emulsion side down*. Finally, reflection matrices also correspond to the difference between *front screen* and *rear screen* viewing. Hence the overall transform for the complete system may include matrices for the input and/or output planes themselves, as well as for the various individual optical elements of which the system is composed, and is the matrix product of all these individual matrices.