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Expected Accuracy of Convergent Photos

It is recommended that the normal-case of photogrammetry be applied if possible, otherwise the angle of convergence must be kept as small as possible and as far as possible from the critical angle.

ABSTRACT: *In photogrammetric literature one finds many contradictory statements about the accuracy of convergent photography. For example, some investigators have shown that the accuracy of stereometric model decreases by having convergent photographs, whereas others state that the accuracy increases. This article derives a new formula which states the accuracy of convergent photography. Moreover, the applications of this formula on different experimental results by other investigators are also given.*

INTRODUCTION

THE EXPECTED accuracies of normal photography, convergent photography, and a comparison between them have been studied for several years, and until now, according to Konecny,[5] the correct conclusions have generally not yet been drawn. In photogrammetric literature one finds a statement by Hallert[2] that states, "The comparison between convergent and vertical pictures concerning the geometrical accuracy has shown that convergent pictures in general are superior."

Another statement by Malhotra and Karara[6] states, "Precision decreases with an increase of convergence from ∓0° to ∓30°. Precision then improves with convergence from ∓30° to ∓50°."

Another statement by Kenefick[4] states, "Although Karara[3] does not recommend photographing with more than 60° convergence, we maintain that 60° should be considered as being a minimum convergence angle and 90° or more should be employed whenever possible."

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There are more contradictory statements. Evidently no definite answers exist as to whether the accuracy increases or decreases by using convergent photographs.

The accuracy of any photogrammetric system can be determined by the accuracy of the object-space coordinates obtained from the system. The accuracy of the object-space coordinates obtained from convergent photographs (refer to Figure 3) is a function of those parameters B, D, C, ϕ, mx and my, where mx, my, are the standard errors of the image coordinates x and y, respectively. C is the camera principal distance, B is the base, ϕ is the angle of convergence, and D is the height.

These parameters can be categorized into two groups:

> *Configuration parameters*, D, B, and ϕ, which change by changing the configuration of the two photos (outer orientation parameters of the two photos).
>
> *Camera parameters*, mx, my and C, which change by changing the camera properties (inner orientation parameters).

This paper gives new formulas which define the accuracy of the photogrammetric system as a function of the *configuration parameters* D, B, and ϕ of the two photos that form the model.

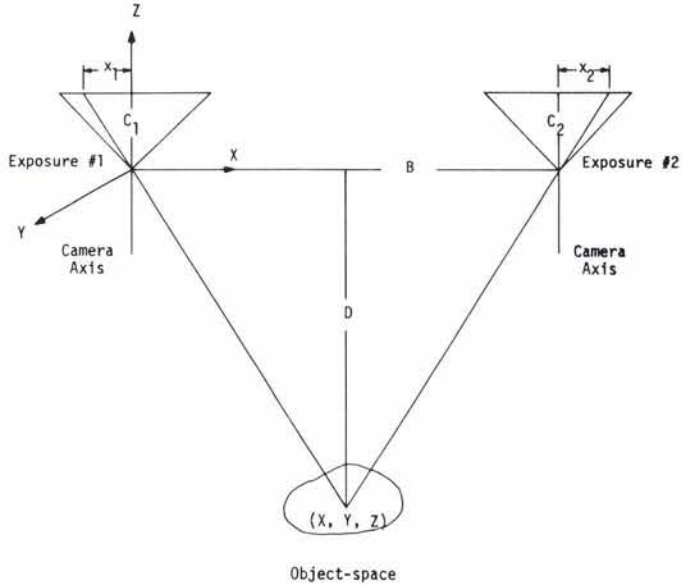

FIG. 1. Data acquisition arrangement for the normal photogrammetric situation.

The Accuracy of the Object-Space Coordinates in Normal Cases of Photogrammetry

According to Karara,[3] the accuracy of the object-space coordinates X, Y, and Z in the normal case of photogrammetry, referring to Figure 1 are:

$$mX = (D/C)\ mx \tag{1}$$

$$mY = (D/C)\ my \tag{2}$$

$$mZ = \frac{D/C}{B/D}\ \sqrt{2}\ mx. \tag{3}$$

The formulas above were developed under the following assumptions:

$$mx_1 = mx_2 = mx \tag{4}$$

$$my_1 = my_2 = my \tag{5}$$

$$C_1 = C_2 = C \tag{6}$$

where mX, mY, and mZ, are the standard errors of the object-space coordinates X, Y, and Z, respectively; mx_1, my_1 are the standard errors of x_1, y_1, respectively; mx_2, my_2 are the standard errors of x_2, y_2, respectively; (x_1, y_1) and $x_2, y_2)$ are the image coordinates on Photo 1 and Photo 2, respectively; B is the distance between the two exposures; D is the average distance between the object points andthe the object-space points obtained from tances of Photo 1, and Photo 2, respectively.

Equations 1, 2, and 3 give the accuracy of the object-space points obtained from normal case photography. The accuracy of the object-space points obtained from convergent photography is discussed in the next section.

The Accuracy of Object-Space Coordinates Obtained from Convergent Photography

The accuracy of the object-space coordinates obtained from the intersection of the rays from convergent photos can be determined from the intersection of their corresponding rays from pseudo-normal images.

The pseudo-normal image is an imaginary image obtained from the same position as convergent image, but with zero angle of convergence. The relationship between the image coordinates x, y of a convergent photo and their corresponding pseudo-normal image coordinates $\bar{x}$, $\bar{y}$ can be obtained from Figure 2 as follows:

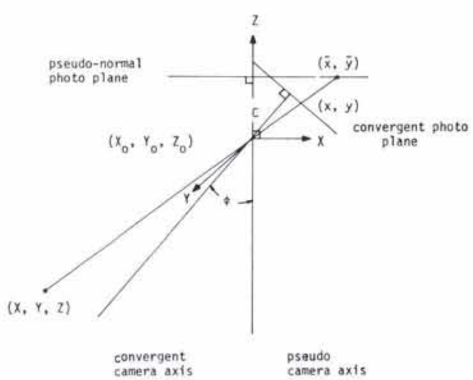

FIG. 2. The relationship between the image coordinates of the same point on a convergent photo and a pseudo-normal photo.

$$\begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix} = \lambda \begin{bmatrix} \cos\phi & o & -\sin\phi \\ o & 1 & o \\ \sin\phi & o & \cos\phi \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ -C \end{bmatrix} = \lambda' \begin{bmatrix} \bar{x} \\ \bar{y} \\ -C \end{bmatrix} \tag{7}$$

where X_o, Y_o, Z_o are the coordinates of exposure station, X, Y, Z are the object-space coordinates of a point, x, y are the image coordinates of that point on convergent photo, $\bar{x}$, $\bar{y}$ are the coordinates of that point if the photo was taken with zero angle of convergence (pseudo-normal case), ϕ is the angle of convergence, C is the photo principal distance, λ', λ are constants.

From Equation 7 one gets:

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ -C \end{bmatrix} = \frac{\lambda}{\lambda'} \begin{bmatrix} \cos\phi & o & -\sin\phi \\ o & 1 & o \\ \sin\phi & o & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ -C \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} X \\ y \\ -C \end{bmatrix} = \frac{\lambda'}{\lambda} \begin{bmatrix} \cos\phi & o & \sin\phi \\ o & 1 & o \\ -\sin\phi & o & \cos\phi \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ -C \end{bmatrix} \tag{9}$$

From Equation 8 one gets:

$$\bar{x} = -C \; \frac{x \cos\phi + C \sin\phi}{x \sin\phi - C \cos\phi} \tag{10}$$

$$\bar{y} = -C \; \frac{y}{x \sin\phi - C \cos\phi}. \tag{11}$$

Applying the law of propagation of errors to Equations 10 and 11 one gets the following two equations:

$$m\bar{x} = \frac{(C \cos\phi + \bar{x} \sin\phi)\, mx}{x \sin\phi - C \cos\phi} \tag{12}$$

$$m\bar{y} = \frac{\left[(\bar{y} \sin\phi \; mx)^2 + (C \; my)^2\right]^{1/2}}{x \sin\phi - C \cos\phi} \tag{13}$$

The two equations above can be put into the form:

$$m\bar{x} = \frac{(1 + (\bar{x}/C) \tan\phi)\, mx}{1 - (x/C) \tan\phi} \tag{14}$$

$$m\bar{y} = \frac{\left[(x/C) \tan\phi \; mx)^2 + (\sec\phi \; my)^2\right]^{1/2}}{1 - (x/C) \tan\phi} \tag{15}$$

The value of x/C in Equations 14 and 15 can be obtained from Equation 9 as follows:

$$\frac{x}{C} = \frac{\bar{x} \cos\phi - C \sin\phi}{\bar{x} \sin\phi + C \cos\phi} \tag{16}$$

Let

$$\bar{x}/C = \tan\alpha. \tag{17}$$

Then

$$\sin\alpha = \bar{x}\,(\bar{x}^2 + C^2)^{1/2}. \tag{18}$$

Accordingly,

$$\frac{x}{C} = \frac{\sin\alpha \cos\phi - \sin\phi \cos\alpha}{\sin\alpha \sin\phi + \cos\phi \cos\alpha} \tag{19}$$

$$= \tan(\alpha - \phi). \tag{20}$$

Substituting the values of $\bar{x}/C$ and x/C from Equations 20 and 17 into Equations 14 and 15 one gets:

$$m\bar{x} = \frac{(1 + \tan\alpha \tan\phi)\, mx}{1 - \tan(\alpha - \phi) \tan\phi} \tag{21}$$

$$m\bar{y} = \frac{\left[(\bar{y}/C \tan\phi \; mx)^2 + (\sec\phi \; my)^2\right]^{1/2}}{1 - \tan(\alpha - \phi) \tan\phi} \tag{22}$$

Theoretical Accuracy of the Object-Space Coordinates

The accuracy of the object-space points can be determined by using the image coordinates of the pseudo images. The determination of the accuracy of the object-space coordinates is based on the following assumptions:

a. The values of mx, my are equal, and have the same values for all the points. (This assumption has been adopted in most applications of analytical photogrammetry.) According to such an assumption one has:

$$mx_1 = mx_2 = my_1 = my_2 = m. \tag{23}$$

b. The photos have a symmetrical configuration as in Figure 3. (This configuration

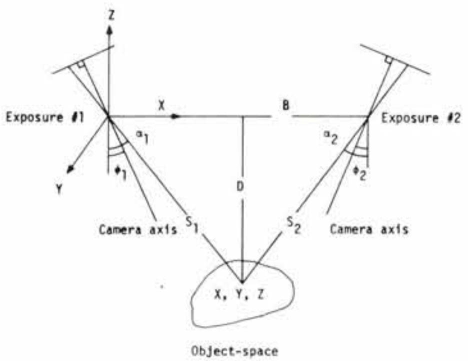

FIG. 3. Data acquisition arrangement for the symmetrical situation ($\phi_1 = \phi_2 = \phi$ and $\alpha_2 = \alpha_2 = \alpha$).

has been adopted by Karara,[3] Malhotra and Karara,[6] Kenefick[4] and many others in their investigations.) According to the symmetrical configuration one has:

$$\phi_1 = \phi_2 = \phi \qquad (24)$$

$$\alpha_1 = \alpha_2 = \alpha. \qquad (25)$$

In the error analysis in this section one takes the central point of the object for comparison between different models, and for estimation of the accuracy of the object-space coordinates.

The accuracy of the image coordinates of the central point of the object according to the above assumptions are:

$$mx_1 = mx_2 = \frac{(1 + tan\ \alpha\ tan\ \phi)m}{1 - tan\ (\alpha - \phi)\ tan\ \phi} \qquad (26)$$

$$my_1 = my_2 = \frac{sec\ \phi\ m}{1 - tan\ (\alpha - \phi)\ tan\ \phi} \qquad (27)$$

Substituting the values of mx and my from Equations 26 and 27 into Equations 1, 2, and 3, one has:

$$mX = \frac{D}{C}\ \frac{(1 + tan\ \alpha\ tan\ \phi)\ m}{1 - tan\ (\alpha - \phi)\ tan\ \phi} \qquad (28)$$

$$mY = \frac{D}{C}\ \frac{(sec\ \phi)\ m}{1 - tan\ (\alpha - \phi)\ tan\ \phi} \qquad (29)$$

$$mZ = \sqrt{2}\ \frac{(D/C)}{(B/D)}\ \frac{(1 + tan\ \alpha\ tan\ \phi)\ m}{1 - tan\ (\alpha - \phi)\ tan\ \phi} \qquad (30)$$

Equations 28, 29, and 30 give the expected values of mX, mY, and mZ in the symmetrical case as a function of base (B) height (D), angle of convergence (ϕ), camera principal distance (C), and the standard deviation of image coordinates.

The experimental investigation for the validity of the developed formulas was performed by applying them to some experiments of reliable investigators. The results of these investigations are reported in the next two experiments.

Experiment 1. Malhotra and Karara[6] and Kenefick[4] studied the effect of the angle of convergence in case of maintaining the scale of convergent photo constant. The distance S between the central point of the object and the two cameras is always constant and the cameras' axes are always directed at the central point as in Figure 3. The expected accuracy in such a situation can be obtained from the developed equations by introducing:

$$S_1 = S_2 = S$$

$$D = S\ cos\ \phi$$

$$B = 2\ S\ sin\ \phi.$$

Substituting the above values into Equations 28, 29 and 30 yields the expressions:

$$mX = (S/2C)(sec\ \phi)m = K_1(sec\ \phi)m \qquad (31)$$

$$mY = (S/C)m = K_2 m \qquad (32)$$

$$mZ = (S/C\ 2)(cosec\ \phi)m = K_3\ (cosec\ \phi)m. \qquad (33)$$

The positional accuracy mT is equal to

$$mT = (mX^2 + mY^2 + mZ^2)^{1/2} \qquad (34)$$

One can see from Equations 31, 32 and 33 that by increasing the angle of convergence ϕ, mX increases, mY remains constant, and mZ decreases. By substituting the values of mX, mY, and mZ from Equations 31, 32, and 33 into Equation 34 one gets:

$$mT^2 = \left[K_1^2\,(sec^2\ \phi) + k_2^2 + K_3^2(cosec^2\ \phi)\right]m^2. \qquad (35)$$

One can see from the values of $(d/d\phi)\ mT$ and $(d^2/d\phi^2)\ mT$ that the values of mT decrease by the increase of ϕ for $\phi<\gamma$; the value of mT reaches its minimum level at $\phi=\gamma$, and the values of mT increase by increasing the value of ϕ for $\phi>\gamma$ where $tan^2\ \gamma = (K_3/K_1) = 1/\sqrt{2}$. The values of mX, mY, and mZ at different angles of convergence ϕ reported by Kenefick,[4] Malhotra and Karara,[6] are given in Figures 4 and 5, respectively.

The results published by Malhotra and Karara, and Kenefick conform very much with the formulas in Equations 31, 32, and 33, the minimum value of mT obtained from Kenefick, and Malhotra and Karara's experimental results, given in Figures 4 and 5 are at

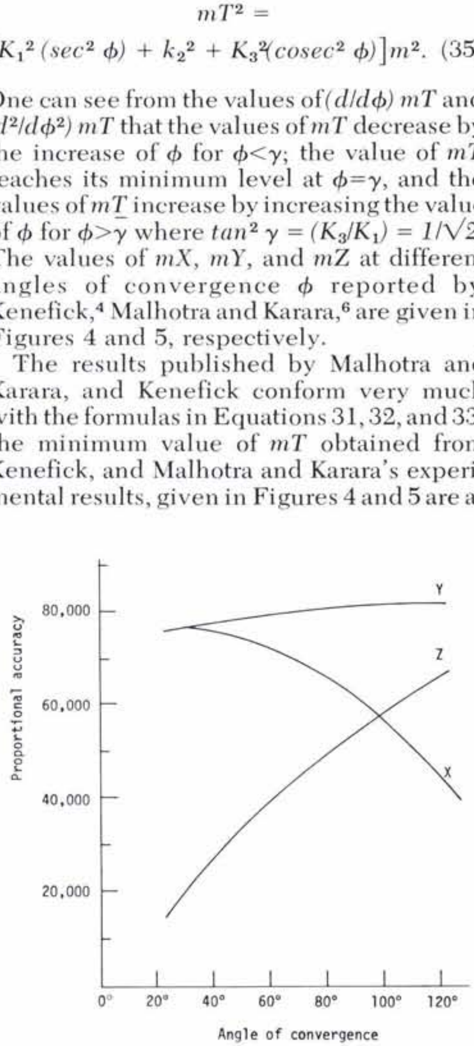

FIG. 4. Generalized curves showing the relation of standard deviation in the object space with the angle of convergence for a two-station reduction (Kenefick[4]).

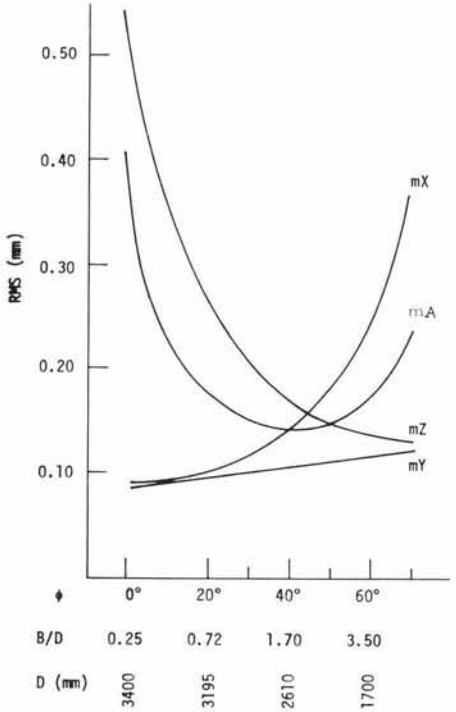

FIG. 5. Effect of base-to-distance ratio and convergence on precision of point determination by stereometric system (Malhotra and Karara[6]). ($mA = mT/\sqrt{3}$).

an angle of convergence of 42° ($2\phi = 84°$) and 40° ($2\phi = 80°$). According to the developed formula, the minimum value of mT is at a convergent angle of 40°.

The accuracy of the object-space coordinates at angle of convergence of 45° ($2\phi = 90°$) estimated using the developed Equations 31, 32, and 33 are 1/53,000, 1/76,400 and 1/55,000 for X, Y, and Z, respectively. Their corresponding values obtained by Kenefick[4] are 1/56,100, 1/74,400 and 1/56,000 for X, Y and Z respectively. Moreover, the shape of the curves for X, Y, and Z obtained from the formula conforms with those given by Kennefick and Malhotra and Karara's experimental results.

Experiment 2. Malhotra and Karara[6] studied the effect of the angle of convergence in cases of maintaining base (B) and height (D) constant.

At fixed B and D, the accuracy on the object-space coordinates X, Y, and Z can be obtained from Equations 28, 29, and 30 by introducing B and D as constants. Referring to Figure 3,

$$mX = K_1 \frac{(1 + \tan \alpha \tan \phi)\, m}{1 - \tan (\alpha - \phi) \tan \phi} \tag{36}$$

$$mY = K_2 \frac{(\sec \phi)\, m}{1 - \tan (\alpha - \phi) \tan \phi} \tag{37}$$

$$mZ = K_3 \frac{(1 + \tan \alpha \tan \phi)\, m}{1 - \tan (\alpha - \phi) \tan \phi} \tag{38}$$

where K_1, K_2, and K_3 are constants. The values of mX, mY, and mZ at different angles of ϕ (angle of convergence) can be obtained from the two functions $x(\phi)$ and $y(\phi)$ where

$$x(\phi) = \frac{1 + \tan \alpha \tan \phi}{1 - \tan (\alpha - \phi) \tan \phi} \tag{39}$$

$$y(\phi) = \frac{\sec \phi}{1 - \tan (\alpha - \phi) \tan \phi}. \tag{40}$$

The properties of $x(\phi)$ and $y(\phi)$ at different values of ϕ can also be obtained by evaluating $(dx/d\phi)\phi$, $(dy/d\phi)\phi$, $(d^2x/d\phi^2)\phi$ and $(d^2y/d\phi^2)\phi$ at different values of ϕ.

Evaluating the above quantities at different ϕ one gets: $x(\phi)$ and $y(\phi)$ increase by increasing the angle of convergence for $\phi<\alpha$; $x(\phi)$ and $y(\phi)$ reach the highest values as ϕ reaches its critical value for $\phi=\alpha$; $x(\phi)$ and $y(\phi)$ decrease by increasing the angle of convergence for $\phi>\alpha$.

Accordingly, it is concluded that: mX, mY, and mZ increase by increasing the angle of convergence for $\phi<\alpha$; mX, mY, and mZ reach the highest value as ϕ reaches its critical value for $\phi=\alpha$; mX, mY, and mZ decrease by increasing the angle of convergence for $\phi>\alpha$.

The results published by Malhotra and Karara[6] conform with the results given by the formulae in Equations 36, 37, and 38. For example, Malhotra and Karara found that for $B/D = 0.5$ and 1.0, the accuracy of X, Y, and Z decrease by increasing the angle of convergence until the angle of convergence reaches a certain limit (the critical angle), then the accuracy of X, Y, and Z increase as the angle of convergence increases. This is exactly what one expects by using the above formulas.

CONCLUSIONS

The two experiments above prove the capability of Equations 18, 19 and 20 to estimate the accuracies of the object-space points in convergent photographs. In these two experiments the expected accuracies according to the developed formulas confirm the results published by the investigators. According to the formulas the accuracy of the object-space points is a non-linear function of the *base* and *height*. It is also a non-linear function of the angle of convergence. As a result, any statement defining the accuracy of the convergent photography as a function of the angle of convergence only is incomplete. Increasing the angle of convergence can increase or decrease the accuracies of the object-space points depending on how the base and height values change. Finally, the expected accuracy of the object-space points at any principal distance (base, height) can be obtained directly from the formulas. More investigations about these formulas are underway at the University of Illinois, U.S.A. and Cairo University, Egypt.

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