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Propagation of Variance and Covariance*

Computed root-mean-square errors cannot detect the rapid accumulation of systematic effects caused by random errors.

Introduction

The method of propagation of variance and covariance has been used extensively in computational photogrammetry for evaluating the accuracy of the quantities determined in a least-squares solution. In general, the normal equations in a least-squares adjustment problem may be represented by the following matrix equation:

$$\boldsymbol{N}\Delta = \boldsymbol{C}$$

where Δ is a matrix of unknown parameters. The variance-covariance matrix of the Δ-matrix, denoted by σ_Δ, may be computed according to the following expression:

$$\sigma_\Delta = \sigma_o^2 \boldsymbol{N}^{-1} \qquad (1)$$

where σ_o^2 is the variance of unit weight.

Equation 1 provides a practical and convenient method of evaluating the accuracy of the computed pass point coordinates in analytical aerotriangulation, or the accuracy of the computed interior orientation parameters in camera calibration. The method is of particular importance as an accuracy analysis tool in simulation studies.

Abstract: *In a direct least-squares solution, the variance-covariance matrix of the unknown parameters may be computed by multiplying the inverse of the coefficient matrix of the normal equation by the variance of unit weight, i.e.,* $\sigma_o^2 N^{-1}$. *In an iterative least-squares solution, which is generally applied in problems of analytical photogrammetry, this formulation is theoretically valid only if the corrections to all the approximations become zero in the last iteration. Experimental evidence showed that this formulation could not detect any rapid accumulation of systematic effects caused by random errors in the measured parameters. Experimental results also showed that* $\sigma_o^2 N^{-1}$ *could provide reliable estimates on the RMS errors of the computed parameters if the correction parameters converge to a value which is less than the computed RMS errors.*

Unfortunately, Equation 1 is theroretically valid only for direct linear least-squares solutions; that is, solutions in which the observation equations are exactly linear and where unknown parameters can be determined directly by solving the normal equations. Most photogrammetric problems necessitate the use of iterative least-squares solutions because of the non-linearity of the observation equations. Initial approximations must be derived for the unknown parameters, and the least-squares solutions solve for the most probable corrections to these approximations. At the end of each iteration, the corrections are applied to the

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approximations. The solution is then reiterated until the corrections become negligibly small. For this type of iterative solutions, Equation 1 is theoretically valid only if the solution converges with the corrections becoming zero in the last iteration.

In practical computations, an iterative solution is usually terminated when the corrections become smaller than the accuracy requirements of the solution. In a weakly conditioned photogrammetric solution, such as that created by poor geometry or low-accuracy controls, some or all of the correction parameters may never approach zero. After a certain number of iterations, the correction parameters may simply oscillate between certain boundary limits from iteration to iteration. If the iteration procedure is allowed to continue, such a solution will eventually begin to diverge rapidly. Equation 1 is not theoretically valid for this type of solution. Yet, it is in this type of solution that some accuracy esimates of the computed parameters are most urgently needed.

This paper analyzes the validity of Equation 1 by reviewing the theoretical basis of the method of propagation of variance and covariance. Some experimental results on the application of Equation 1 to problems in coordinate transformation, camera calibration and aerotriangulation are presented. Based on these experimental results, conclusions will be made on the validity of $\sigma_o^2\mathbf{N}^{-1}$ as a reliable accuracy estimator in solutions which fail to converge with the correction parameters approaching zero.

Statistical Definition of Variance and Covariance

Let x be a discrete random variable which can take on any one of the values $x_1, x_2, x_3 \ldots x_n$ with the probability $p_1, p_2, p_3 \ldots p_n$, respectively. Then the expected value of x, denoted by the symbol $E(x)$, is defined as

$$E(x) = \sum_{i=1}^{n} x_i p_i \ . \tag{2}$$

The expected value of x is commonly called the population mean of x. If x is a continuous random variable which has a probability distribution function $f(x)$, then the expected value is defined as

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx \ . \tag{3}$$

From these basic definitions, it can be easily derived that if b is a scaler, then $E(bx) = bE(x)$.

The variance of the random variable x is defined as the second moment about the population mean, i.e.,

$$\sigma_x = E\left\{ (x - E(x))^2 \right\} . \tag{4}$$

The covariance σ_{xy} between two random variables x and y is defined as:

$$\sigma_{xy} = E\left\{ (x - E(x))(y - E(y)) \right\} . \tag{5}$$

Let $\mathbf{Z}$ be a column matrix of random variables $Z_1, Z_2, Z_3, \ldots Z_m$; i.e.,

$$\mathbf{Z}_{(m,1)} = \left[Z_1, Z_2 \ldots Z_m \right]^T .$$

Then, it can be easily derived from the above fundamental definitions that the variance-covariance matrix for $\mathbf{Z}$ may be expressed as follows:

$$\sigma_Z = E\left\{ (\mathbf{Z} - E(\mathbf{Z}))(\mathbf{Z} - E(\mathbf{Z}))^T \right\} \tag{6}$$

where σ_Z is an $(m \times m)$ matrix.

Propagation of Variance and Covariance

DIRECT LINEAR LEAST-SQUARES SOLUTIONS

Consider a simple problem in linear regression. Let x and y be two variables which are known to be related to each other.

$$y = a_o + a_1 x \ .$$

In conducting an experiment to determine the values of the co-efficients a_o, and a_1, the variable y is measured at various values of x. It is assumed that the measurement of x can be made exactly, and that only random errors are present in the measurements of y. The observation equation for the pair of measurements (x_i, y_i) can then be expressed as

$$[v_i] + [1\ x_i]\begin{bmatrix} a_o \\ a_1 \end{bmatrix} = \begin{bmatrix} y_i \end{bmatrix} \tag{7}$$

where v_i is the random error in the measurement y_i. Equation 7 can be expressed in matrix notation as

$$\boldsymbol{v}_i + \boldsymbol{B}_i \cdot \boldsymbol{A} = \boldsymbol{Y}_i\ . \tag{8}$$

The complete set of observation equations for m measurements of y can then be simply expressed as:

$$\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_m \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_m \end{bmatrix} \cdot A = \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ \cdot \\ Y_m \end{bmatrix}$$

or,

$$\boldsymbol{V} + \boldsymbol{B} \cdot \boldsymbol{A} = \boldsymbol{Y}. \tag{9}$$

Let σ_Y denote the variance-covariance matrix of the $\boldsymbol{Y}$-matrix; i.e.,

$$\sigma_Y = \begin{bmatrix} \sigma_{y_1}{}^2 & \sigma_{y_1 y_2} & \sigma_{y_1 y_3} & \cdots & \sigma_{y_1 y_m} \\ \sigma_{y_1 y_2} & \sigma_{y_2}{}^2 & \sigma_{y_2 y_3} & \cdots & \sigma_{y_2 y_m} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \sigma_{y_1 y_m} & & \cdots & & \sigma_{y_m}{}^2 \end{bmatrix} \tag{10}$$

where $\sigma_{y_i y_j}$ denotes the covariance between the two measurements y_i and y_j. If the measurements are all mutually independent, then $\sigma_{y_i y_j} = 0$ when i is not equal to j. The corresponding weight matrix of $\boldsymbol{Y}$, denoted by $\boldsymbol{W}$, is then defined as:

$$\boldsymbol{W} = \sigma_o{}^2 \sigma_Y{}^{-1} \tag{11}$$

where $\sigma_o{}^2$ is the variance of unit weight.

The normal equation for this regression problem is:

$$\boldsymbol{N} \cdot \boldsymbol{A} = \boldsymbol{C} \tag{12}$$

where

$$\boldsymbol{N} = \boldsymbol{B}^T \boldsymbol{W} \boldsymbol{B} \tag{13}$$

and

$$\boldsymbol{C} = \boldsymbol{B}^T \boldsymbol{W} \boldsymbol{Y}. \tag{14}$$

The most probable value of the coefficients a_o and a_1, i.e., A, can then be computed as

$$\boldsymbol{A} = \boldsymbol{N}^{-1} \boldsymbol{C}. \tag{15}$$

The following paragraphs show that the variance-covariance matrix for the computed parameter $\boldsymbol{A}$, denoted by σ_A, may be simply expressed as

$$\sigma_A = \sigma_o{}^2 \boldsymbol{N}^{-1}\ . \tag{16}$$

According to the fundamental definition of the variance-covariance matrix as stated in Equation 6, the variance-covariance matrix σ_A can be expressed as:

$$\sigma_A = E\ \{(\boldsymbol{A} - E(\boldsymbol{A}))\ (\boldsymbol{A} - E(\boldsymbol{A}))^T\}\ .$$

Substituting Equations 15 and 13 into the above expression yields

$$\sigma_A = E\left\{(N^{-1}+B^TWY - E(N^{-1}B^TWY))\,(N^{-1}+B^TWY - E\,(N^{-1}B^TWY))^T\right\}.$$

As the matrices N^{-1}, B^T and W are matrices of constants and Y is the only matrix of variables,

$$E\,(N^{-1}B^TWY) = N^{-1}B^TWE\,(Y).$$

Therefore,

$$\sigma_A = E\left\{(N^{-1}B^TW\,(Y - E\,(Y))\,(Y - E\,(Y))^T\,WB\,N^{-1}\right\}$$

$$= N^{-1}B^TW\,E\left\{(Y - E\,(Y))\,(Y - E\,(Y))^T\right\}WB\,N^{-1}.$$

By definition,

$$\sigma_Y = E\left\{(Y - E\,(Y))\,(Y - E(Y))^T\right\}.$$

Furthermore, from Equation 11, $\sigma_Y = \sigma_o^2\,W^{-1}$. Therefore,

$$\sigma_A = \sigma_o^2\,N^{-1}B^TW\,W^{-1}\,WB\,N^{-1}$$

$$= \sigma_o^2\,N^{-1}\,(B^TW\,B)\,N^{-1}$$

$$= \sigma_o^2\,N^{-1}N\,N^{-1}$$

Hence,

$$\sigma_A = \sigma_o^2\,N^{-1}$$

which is as stated in Equation 16.

This derivation is based primarily on the fact that the Y-matrix in the observation Equations 9 is a matrix of random variables. Its elements consisted only of the measured values of the variable y. It is shown in the following paragraphs that this condition is true in an iterative solution only if the solution converges with the correction parameters approaching zero.

ITERATIVE LEAST-SQUARE SOLUTIONS

As an example of iterative solutions, consider the solution model for the simultaneous adjustment of photogrammetric blocks in aerotriangulation. The solution is based on the following pair of collinearity equations which express the relationship between the image coordinates (x_{ij}, y_{ij}) of point i on photo j and the corresponding ground coordinates (X_j, Y_j, Z_j) of point j:

$$x_{ij} - x_p + \frac{f[m_{11}(X_j - X_i^c) + m_{21}(Y_j - Y_i^c) + m_{31}(Z_j - Z_i^c)]}{[m_{13}(X_j - X_i^c) + m_{23}(Y_j - Y_i^c) + m_{33}(Z_j - Z_i^c)]} = 0 \qquad (17)$$

$$y_{ij} - y_p + \frac{f[m_{12}(X_j - X_i^c) + m_{22}(Y_j - Y_i^c) + m_{32}(Z_j - Z_i^c)]}{[m_{13}(X_j - X_i^c) + m_{23}(Y_j - Y_i^c) + m_{33}(Z_j - Z_i^c)]} = 0 \qquad (18)$$

where x_p and y_p are the image coordinates of the principal point; f is the focal length X_i^c, Y_i^c, and Z_i^c are ground coordinates of the ith camera position; and the m_{ij}'s are functions of the three rotation parameters $(\omega_i, \phi_i, \kappa_i)$ of the camera.

By first-order Newton approximation, these collinearity equations are linearized to the form:

$$v_{x_{ij}} + b_{11}\Delta X_i^c + b_{12}\Delta Y_i^c + b_{13}\Delta Z_i^c + b_{14}\Delta\omega_i + b_{15}\Delta\phi_i + b_{16}\Delta\kappa_i + b_{17}\Delta X_j + b_{18}\Delta Y_j + b_{19}\Delta Z_j = \epsilon_{x_{ij}} \qquad (19)$$

$$v_{y_{ij}} + b_{21}\Delta X_i^c + b_{22}\Delta Y_i^c + b_{23}\Delta Z_i^c + b_{24}\Delta\omega_i + b_{25}\Delta\phi_i + b_{26}\Delta\kappa_i + b_{27}\Delta X_j + b_{28}\Delta Y_j + b_{29}\Delta Z_j = \epsilon_{y_{ij}} \qquad (20)$$

where

$$\epsilon_{x_{ij}} = -\left[x_{ij} - x_p \quad + \frac{f[m_{11}(X_j^o - X_i^{co}) + m_{21}(Y_j^o - Y_i^{co}) + m_{31}(Z_j^o - Z_i^{co})]}{[m_{13}(X_j^o - X_i^{co}) + m_{23}(Y_j^o - Y_j^{co}) + m_{33}(Z_j^o - Z_i^{co})]}\right. \tag{21}$$

$$\epsilon_{y_{ij}} = -\left[y_{ij} - y_p \quad + \frac{f[m_{12}(X_j^o - X_i^{co}) + m_{21}(Y_j^o - Y_i^{co}) + m_{32}(Z_j^o - Z_i^{co})]}{[m_{13}(X_j^o - X_i^{co}) + m_{23}(Y_j^o - Y_i^{co}) + m_{33}(Z_j^o - Z_i^{co})]}\right. \tag{22}$$

where X_j^o, Y_j^o, Z_j^o are approximate ground coordinates of point j; $X_i^{co}, Y_i^{co}, Z_i^{co}$ are approximate ground coordinates of the ith camera position; and $\Delta X_i^c, \Delta Y_i^c$, etc., are the most probably corrections to the approximations.

Equations 19 and 20 may be expressed in matrix notation as:

$$\underset{(2,1)}{v_{ij}} + \underset{(2,6)}{\dot{B}_{ij}}\,\underset{(6,1)}{\dot{\Delta}_i} + \underset{(2,3)}{\ddot{B}_{ij}}\,\underset{(3,1)}{\ddot{\Delta}_j} = \underset{(2,1)}{\epsilon_{ij}}\,. \tag{23}$$

Equation 23 differs from the observation equation of the linear regression problem, i.e., Equation 8, in one significant aspect. The elements of the constant matrix ϵ_{ij} are functions of both the measured quantities (x_{ij}, y_{ij}) and the approximations $(X_j^o, Y_j^o$, etc.), as can be clearly observed in Equations 21 and 22.

The complete set of collinearity equations for a photogrammetric block may be expressed as:

$$V + B\Delta = \epsilon\,. \tag{24}$$

The corresponding normal equation will be as follows:

$$N\Delta = C \tag{25}$$

where

$$N = B^T WB \text{ and } C = B^T W\epsilon\,.$$

The weight matrix W of the measured image coordinates is again defined as

$$W = \sigma_o^2\,\sigma_{xy}^{-1} \tag{26}$$

where σ_{xy} is the variance-covariance matrix of the measured image coordinates. The following paragraphs show that the convergence of the Δ-matrix is the necessary condition for the expression $\sigma_\Delta = \sigma_o^2 N^{-1}$ to be valid.

The normal Equation 25 is in exactly the same form as the normal Equation 12. Therefore, the derivation of σ_Δ can proceed in exactly the same manner as that presented above until the following expression is reached:

$$\sigma_\Delta = \mathrm{N}^{-1} B^T WE\,\{(\epsilon - E(\epsilon))\,(\epsilon - E(\epsilon))^T\}\,WB\,N^{-1}\,.$$

Because, by definition,

$$E\,\{(\epsilon - E(\epsilon))\,(\epsilon - E(\epsilon))^T\} = \sigma_\epsilon\,,$$

$$\sigma_\Delta = N^{-1} B^T W \sigma_\epsilon WB\,N^{-1}\,. \tag{27}$$

If the Δ-matrix became zero in the iterative solution, Equation 24 becomes

$$V = \epsilon.$$

That is, the residual terms $\epsilon_{x_{ij}}$ and $\epsilon_{y_{ij}}$ in Equations 19 and 20 are caused by random errors in the measured parameters. Under this condition of convergence, it is obvious that the following relationship is true:

$$\sigma_\epsilon = \sigma_{xy}\,.$$

From Equation 26, $\sigma_{xy} = \sigma_o^2 W^{-1}$. Therefore,

$$\sigma_\epsilon = \sigma_o^2 W^{-1}. \tag{28}$$

By substituting Equation 28 into 27, it can then be easily shown that

$$\sigma_\Delta = \sigma_o^2 N^{-1}.$$

Let X^o denote the matrix of all the approximations, and X denotes the matrix of the unknown parameters; i.e.

$$\mathbf{X} = \mathbf{X}^o + \Delta.$$

Then, because X^o has stabilized to become a constant term, the variance-covariance matrix of the computed values of the unknowns (σ_X) is equal to the variance-covariance matrix of the corrections; i.e.,

$$\sigma_X = \sigma_\Delta.$$

However, if the iterative solution fails to converge with Δ approaching zero, it would no longer be true that $V = \epsilon$. In such a solution, the residual terms $\epsilon_{x_{ij}}$ and $\epsilon_{y_{ij}}$ in the collinearity equations will include measurement errors, approximation errors, as well as errors introduced by the linearization of the original observation equations. It can no longer be stated that $\sigma_\epsilon = \sigma_{xy}$, and Equation 27 cannot be further simplified, i.e.,

$$\sigma_\Delta = N^{-1} B^T W \sigma_\epsilon W B N^{-1}.$$

There is no simple statistical method available for directly computing σ_ϵ from the solution. In fact, there is no assurance that the elements in the ϵ-matrix actually follow a normal distribution.

Experimental Verification

The validity of the method of propagation of variance and covariance was studied by the method of simulation for several common adjustment problems in computational photogrammetry. All the problems to be presented in the following paragraphs required iterative solutions. In all instances, the iterative solution was terminated when the computed estimate of the standard error of unit weight between two successive iterations was less than 0.01 σ_o; where σ_o is the predefined standard error of unit weight, i.e., $|(m_o)_i - (m_o)_{i-1}| < 0.01\ \sigma_o$.

In each simulation solution, the expected root-mean-square *(RMS)* errors of the adjusted parameters were computed according to the method of propagation of variance and covarience; i.e., Equation 1. The true errors in the computed values of the unknown parameters were also determined to provide a check on the validity of the method.

Coordinate Transformation

Let x_j, y_j, z_j represent the model coordinates of a point j; and X_j, Y_j *and* Z_j be the corresponding ground coordinates of point j. The transformation of the model coordinates into the ground coordinate system requires the determination of seven transformation parameters: scale, translations in the X, Y and Z directions, and rotation about the X, Y and Z axes.

Table 1 lists the expected *RMS* errors against the true errors of the computed parameters. Twenty-five ground control points were arranged in a 5 × 5 rectangular pattern over an area of 125 km × 125 km. The model coordinates were assumed to have a relative accuracy of ±0.1 m; and the ground control coordinates had an *RMS* error of ±1 m, ±10 m, and ±1000 m for Cases 1, 2 and 3 respectively.

In Cases 1 and 2, the corrections in the last iteration were negligibly small and could be considered as zero. The true errors were all less than three times the expected *RMS* errors (3σ).

In Case 3, the corrections in the last iteration were relatively large but were all much smaller than the expected *RMS* errors. The true errors were also within the three-sigma level.

Camera Calibration

Table 2 lists the results from three simulated cases of camera calibration. The camera was assumed to have the same geometry as the return-beam-vidicon (RBV) television cameras being flown on board the ERTS-1 satellite. It had a 126-mm focal length, and a narrow (16.2°) field of view. The camera had 81 reseau points arranged in a 9 × 9 rectangular pattern at its focal plane. The directional angles to these reseau points were measured through the lens with a travelling telescope. These directional angles were used as controls in the determination of

TABLE 1. TRUE ERRORS VS. RMS ERRORS IN COORDINATE TRANSFORMATION

Parameters .	Final Correction	Final Value	Expected RMS Error	True Error
		Case 1. $\sigma_{X_j} = \sigma_{Y_j} = \sigma_{Z_j} = \pm 1$ m		
Scale	$.14 \times 10^{-6}$	0.50000	± .00002	0.
X-translation	−.01	219.4	.4	−.6
Y-translation	.03	220.5	.4	+ .5
Z-translation	.015	29.1	.4	−.9
ω-rotation	.01″	29° 59′ 59″	1″	−1″
φ-rotation	.01″	29° 59′ 58″	1″	−2″
κ-rotation	.06″	60° 0′ 0″	1″	0
		Case 2. $\sigma_{X_j} = \sigma_{Y_j} = \sigma_{Z_j} = \pm 10$ m		
Scale	$.86 \times 10^{-7}$	.50001	± 0.00008	.00001
X-translation	.03	217.3	1.7	−2.7
Y-translation	.02	220.8	1.7	+0.8
Z-translation	+.008	27.6	1.7	−2.4
ω-rotation	0″	29° 59′ 58″	5″	−2″
φ-rotation	.01″	29° 59′ 48″	4″	−12″
κ-rotation	.15″	60° 0′ 0″	4″	0
		Case 3. $\sigma_{X_j} = \sigma_{Y_j} = \sigma_{Z_j} = \pm 1000$m		
Scale	$.15 \times 10^{-3}$	.5009	± .0007	.0009
X-transalation	46	−33	167	−253
Y-translation	33	255	160	35
Z-translation	−39	−141	167	−171
ω-rotation	−41″	29° 57′ 55″	8′ 2″	−2′ 5″
φ-rotation	−1′ 47″	29° 40′ 59″	6′ 10″	−19′ 1″
κ-rotation	1′ 44″	60° 0′ 7″	6′ 17″	7″

All translation parameters are expressed in meters.

the interior geometry of the camera. The coordinates of the reseau points were assumed to be measured with an *RMS* error of ±2 μm. The horizontal (α_j) and vertical (β_j) angles of the light ray incident on the reseau point *j* were assumed to be measured with an *RMS* error of ±2 sec ±4 sec, and ±10 sec of arc for Cases 1, 2 and 3, respectively.

In spite of the poor geometry caused by the narrow field angle of the lens, the true errors in the computed parameters were all less than 3σ.

TABLE 2. TRUE ERRORS VS. RMS ERRORS IN CAMERA CALIBRATION

($x_p = y_p = 0.0, f = 126.00$ mm.)

Case	RMS Error of Directional Controls	Parameters	Expected RMS Error (± mm)	True Error (mm)
1	± 2 sec.	x_p	±.06	.002
		y_p	.06	.014
		f	.03	−.04
2	± 4 sec.	x_p	.08	−.006
		y_p	.08	−.028
		f	.04	−.054
3	±10 sec.	x_p	.16	.16
		y_p	.16	.22
		f	.07	−.17

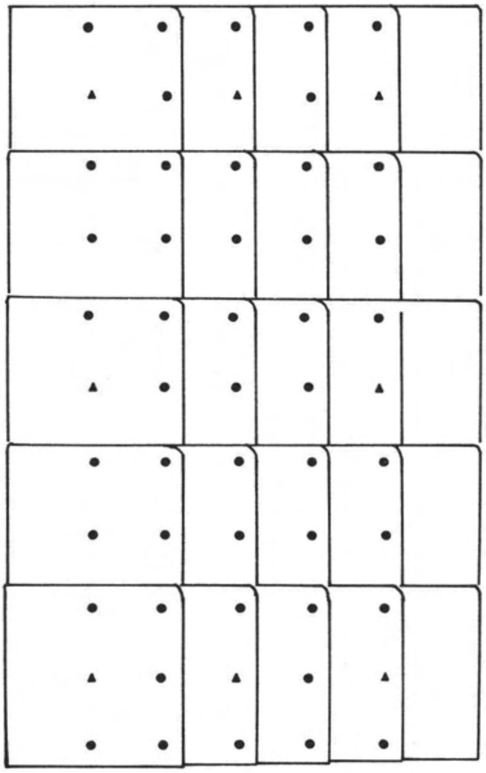

FIG. 1. A 25-photo block.

BLOCK AEROTRIANGULATION

Figure 1 shows the location of the pass points and the control points in a 25-photo (5 × 5) block. There are 47 pass points and 8 control points. The photographs had a nominal scale of 1:4000. The block was triangulated by a simultaneous solution program called SAPGO-A. The image coordinates were weighted with a standard error of ±5 μm and the ground coordinates of the control points were weighted with a standard error of ±0.01 m.

Table 3 lists the true errors vs the expected *RMS* errors in the computed coordinates of six pass points. All the true errors were less than 3σ.

LUNAR PHOTOTRIANGULATION

The data for a strip of 11 photos was generated to simulate the metric photography from the Apollo 15 mission. The camera had a focal length of 76.7 mm and the nominal flight altitude was 110.3 km. There were 25 pass points in each photo and adjacent photos had 60 percent overlap. The image coordinates of the pass points were perturbed with a standard error of ±5μm.

The strip was first triangulated using all the controls which were available on the Apollo 15 photography: tracking data, laser altimeter measurements, and attitude data from a stellar camera. The tracking data was assumed to provide the velocity vector with an *RMS* error of ±0.5 m/sec. Assuming that the position vector of the first photo was error free, an *RMS* error of ±0.5 m/sec. in the velocity vector was equivalent to an *RMS* error of ±20 m, ±29 m, ± 35 m, ±40 m, ±45 m, ±50 m, ±54 m, ±57 m, ±61 m, and ±64 m. for the coordinates of the 2nd, 3rd, . . . and 11th photo, respectively. The laser altimeter measurements were assumed to be ±3 m. The stellar camera was assumed to provide attitude data with an *RMS* error of ±10 sec for both the ω- and κ-rotation and ±20 sec. for the ϕ-rotation.

TABLE 3. TRUE ERRORS VS RMS ERRORS IN BLOCK AEROTRIANGULATION

Point	Coord.	Final Correction (meters)	Expected RMS Error (± meters)	True Error (meters)
1	X	−.01	.04	−.03
	Y	−.03	.05	−.01
	Z	−.3	.4	−.2
3	X	.003	.02	−.01
	Y	.004	.03	.01
	Z	−.2	.4	−.1
5	X	.000	.04	.02
	Y	.02	.05	−.06
	Z	−.3	.4	−.2
16	X	−.01	.03	.01
	Y	.01	.04	.01
	Z	.02	.3	.4
18	X	.003	.02	.01
	Y	−.01	.02	−.01
	Z	.01	.3	.3
20	X	−.01	.03	.02
	Y	.01	.04	−.02
	Z	.02	.3	0.2

Five different sets of fictitious data were generated for the same strip. The perturbations in the measured parameters of each data set were generated independently. Each data set was triangulated independently.

Tables 4, 5, and 6 list the last corrections, true errors and expected *RMS* errors for the *X*, *Y* and Z coordinates of the exposure stations. In all five instances, the true errors were within three times the expected *RMS* errors (3σ). With the exception of the Z coordinates in Case 5 of Table 6, the last corrections were all within the expected *RMS* error (1σ).

Figure 2 is a graphical plot of the true errors in the computed coordinates of the pass points located at the center of each photo. With the exception of three data points, all the true errors were within three times the expected *RMS* errors (3σ).

To provide a more rigorous test of the method of propagation of variance and covariance, the five data sets for the same 11-photo strip were triangulated as cantilever extensions. In each instance, the six orientation parameters of the first photo and the rectangular coordinates of

TABLE 4. EXPECTED RMS ERROR VS. TRUE ERROR IN X-COORDINATE OF EXPOSURE STATIONS (TRACKING + ALTIMETER + ATTITUDE CONTROLS WITH $\sigma_V = \pm 0.5$ *m/sec*)

Station	Expected RMS Errors (± meters)	True Error in Case 1	2	3	4	5	Last Correction in Case 1	2	3	4	5
		(meters)									
1	–	–	–	–	–	–	–	–	–	–	–
2	10	−5	−10	4	26	−6	−3	−3	5	−6	1
3	11	+3	−13	1	6	6	−6	−3	6	−6	2
4	12	25	20	18	3	−10	−9	−2	5	−4	3
5	13	−1	−10	−19	−4	−6	−10	2	7	−8	2
6	13	−2	−32	−28	2	14	−10	6	7	−2	7
7	14	−7	−17	−31	2	21	−5	−2	4	−7	5
8	15	3	−14	−39	7	−4	−9	−6	8	−6	5
9	15	−16	−25	−36	9	−25	−11	−7	8	−9	0
10	16	−16	−11	−38	13	−33	−12	−3	10	−9	1
11	18	13	0	−36	17	−8	−9	−2	5	−7	0

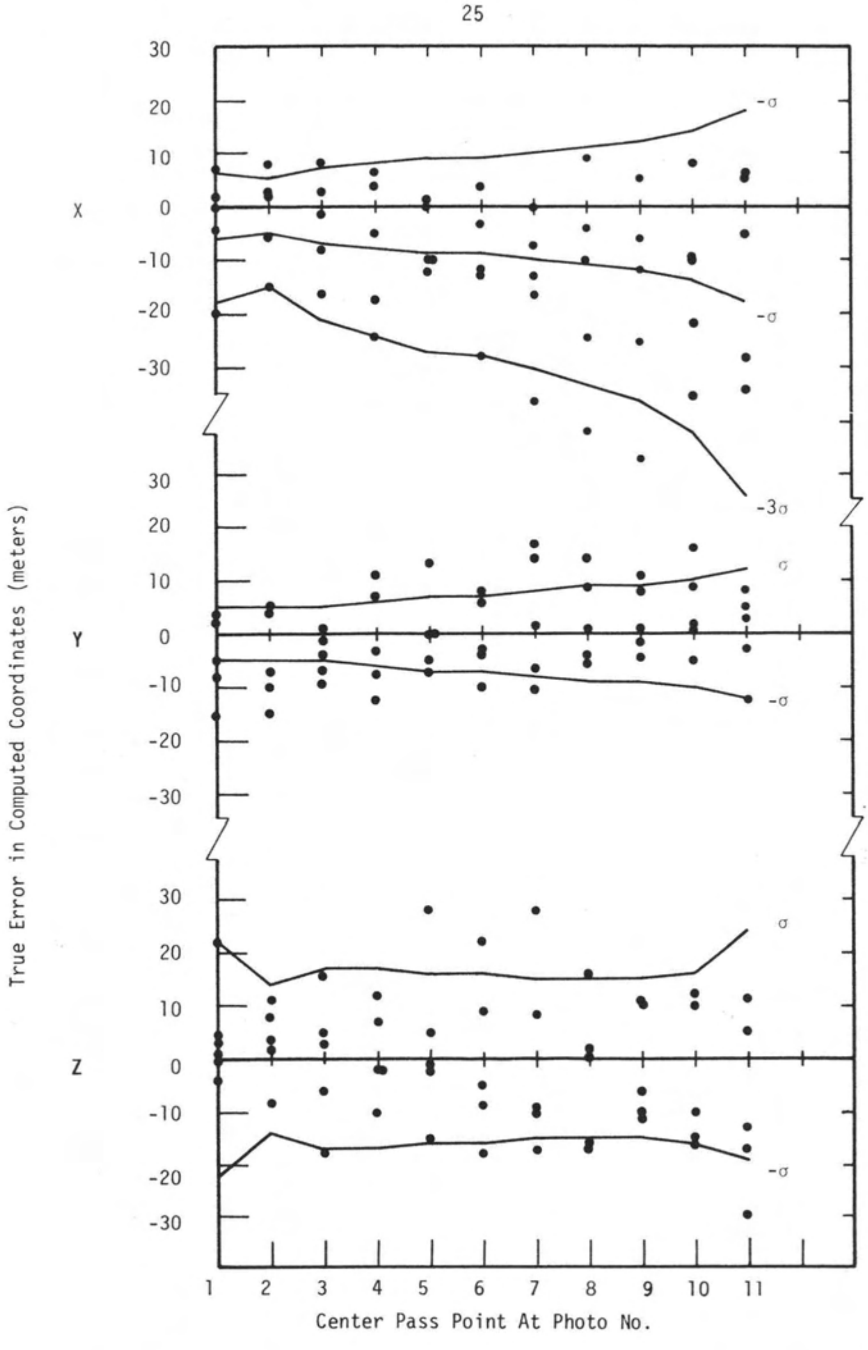

FIG. 2. True error in computed pass point coordinates of five cases (tracking + altimeter + attitude controls).

the second photo were held fixed in the solution. No other control data were used, and the image coordinates of the pass points were weighted with a standard error of ±5 μm.

Tables 7, 8 and 9 list the last corrections, true errors and expected errors for the coordinates of the exposure stations. It is obvious in these tests that the computed *RMS* errors completely broke down as an accuracy estimator. Many of the computed coordinates had true errors far

TABLE 5. EXPECTED RMS ERROR VS TRUE ERROR IN Y-COORDINATES OF EXPOSURE STATIONS (TRACKING + ALTIMETER + ATTITUDE CONTROLS WITH $\sigma_V = \pm 0.5$ *m./sec.)*

Station	Expected RMS Error (± meters)	True Error in Case 1 (meters)	2	3	4	5	Last Correction in Case 1	2	3	4	5
1	–	–	–	–	–	–	–	–	–	–	–
2	7	−14	4	−16	−5	+3	−2	3	0	1.0	1
3	7	11	13	−9	−3	−5	−4	2	0	0	1
4	8	4	11	4	−15	−13	−3	4	0	0	1
5	8	−17	5	−9	−24	−4	−5	3	0	−1	0
6	8	8	4	−4	−1	0	−5	5	0	−1	2
7	9	6	19	−6	−8	−8	−4	5	0	−1	3
8	9	12	27	−5	−9	−16	−5	8	0	−1	3
9	10	1	19	−1	−17	11	−6	8	0	0	3
10	11	2	16	16	−12	−8	−5	10	1	−1	5
11	11	16	17	10	−15	4	−5	9	0	−2	4

TABLE 6. EXPECTED RMS ERROR VS TRUE ERROR IN Z-COORDINATES OF EXPOSURE STATIONS (Tracking + Altimeter + Attitude Controls With $\sigma_V = \pm 0.5$ *m./sec.)*

Station	Expected RMS Error (± meters)	True Error in Case 1 (meters)	2	3	4	5	Last Correction in Case 1	2	3	4	5
1	–	–	–	–	–	–	–	–	–	–	–
2	6	10	−8	2	−4	9	−2	–	0	2	1
3	8	8	−20	−6	2	0	−4	−1	0	3	−3
4	9	13	−13	−4	0	8	−4	−2	−2	3	−5
5	9	23	−14	−1	0	11	−3	−5	−2	5	−8
−6	10	23	−15	−7	−5	8	−3	−7	−3	5	−13
7	11	12	−19	−9	−6	13	−5	−10	−2	4	−17
8	11	15	−19	−18	1	5	−7	−10	−2	4	−20
9	12	11	−12	−14	−14	6	−9	−8	−4	5	−22
10	14	11	−6	−18	−14	7	−8	−6	−4	8	−25
11	16	16	−8	−21	−23	11	−12	−5	−6	10	−24

TABLE 7. EXPECTED RMS ERROR VS. TRUE ERROR IN X-COORDINATES OF EXPOSURE STATIONS (CANTILEVER EXTENSION)

Station	Expected RMS Errors ± meters	True Error in Case 1 (meters)	2	3	4	5	Last Correction in Case 1	2	3	4	5
1	–	–	–	–	–	–	–	–	–	–	–
2	–	–	–	–	–	–	–	–	–	–	–
3	31	109	70	−19	−18	−5	16	−19	−15	47	−1
4	37	100	79	61	−137	2	16	−34	26	17	−6
5	35	74	−188	116	−184	12	7	−35	1	−19	−11
6	35	24	−356	171	−190	33	16	−13	−32	9	−5
7	36	−105	−467	248	−231	93	33	−7	−20	−19	−2
8	37	−168	−553	244	−325	137	59	33	2	−15	0
9	37	−299	−573	169	−454	192	71	79	19	−43	−3
10	43	−670	−637	111	−590	216	74	102	68	−83	−6
11	59	−360	−667	−4	−753	219	57	142	77	−103	−16

TABLE 8. EXPECTED RMS ERROR VS. TRUE ERROR IN Y-COORDINATES OF EXPOSURE STATIONS (CANTILEVER EXTENSION)

Station	Expected RMS Error ± meters	True Error in Case (meters)					Last Correction in Case				
		1	2	3	4	5	1	2	3	4	5
1	–	–	–	–	–	–	–	–	–	–	–
2	–	–	–	–	–	–	–	–	–	–	–
3	20	42	−15	−17	−57	1	4	−16	−6	21	−5
4	26	−8	−34	−28	−92	27	5	−27	15	24	−10
5	30	−92	−26	27	−127	61	−4	−17	6	17	−17
6	29	−139	−35	10	−131	84	−16	−11	−21	24	−26
7	30	−187	−46	−12	−127	107	−20	−12	−24	36	−34
8	30	−233	−13	−40	−110	133	−30	−1	−34	59	−50
9	32	−277	29	−58	−113	160	−44	19	−25	38	−60
10	37	−391	77	−52	−158	195	−52	4	−37	65	−81
11	46	−542	122	−103	−153	223	−37	58	−70	35	−99

TABLE 9. EXPECTED RMS ERROR VS. TRUE ERROR IN Z-COORDINATE OF EXPOSURE STATIONS (CANTILEVER EXTENSION)

Station	Expected RMS Error ± meters	True Error in Case (meters)					Last Correction in Case				
		1	2	3	4	5	1	2	3	4	5
1	–	–	–	–	–	–	–	–	–	–	–
2	–	–	–	–	–	–	–	–	–	–	–
3	12	−36	21	27	−27	11	−5	12	4	−15	5
4	19	−71	11	29	1	34	−5	21	−17	−16	11
5	22	−85	23	17	15	58	0	27	−17	−2	20
6	23	−104	87	−1	7	85	2	34	−3	−9	31
7	23	−114	178	93	−2	114	−4	38	8	−2	36
8	23	−101	285	−91	−34	125	−14	28	22	4	39
9	26	77	381	−141	−17	124	−26	9	36	18	40
10	37	−31	499	−168	−7	128	−36	−9	49	42	40
11	55	−7	613	−149	27	140	−39	−55	65	93	45

exceeding their expected *RMS* errors. It was observed that beyond Exposure Station 7, the last corrections generally exceeded the estimated *RMS* errors of these parameters.

Figures 3, 4 and 5 show the true errors in the computed coordinates of the pass points located at the center of the photos. These figures showed the rapid accumulation of systematic errors along the strip. Furthermore, the systematic pattern varied from strip to strip. The computed *RMS* errors failed to detect these systematic errors and clearly underestimated the inaccuracy of the solution.

CONCLUSIONS

Based on the above experimental results, the following conclusions can be drawn:

★ In photogrammetric adjustment problems where only low-order accuracy controls are available, the correction parameters generally will not reach zero in the iterative least-squares solution. However, the correction parameters should converge to a value less than the estimated *RMS* errors computed from the propagation of variance and co-variance.

★ Generally, if a well-distributed system of controls is available and if the correction parameters converge to a value which is less than the computed *RMS* errors, then the computed *RMS* errors should be reliable estimators for the accuracy of the computed parameters.

★ The computed *RMS* errors cannot detect the rapid accumulation of systematic effects caused by random errors. It is well-known that the double-summation effect of random errors produces systematic errors in phototriangulation.

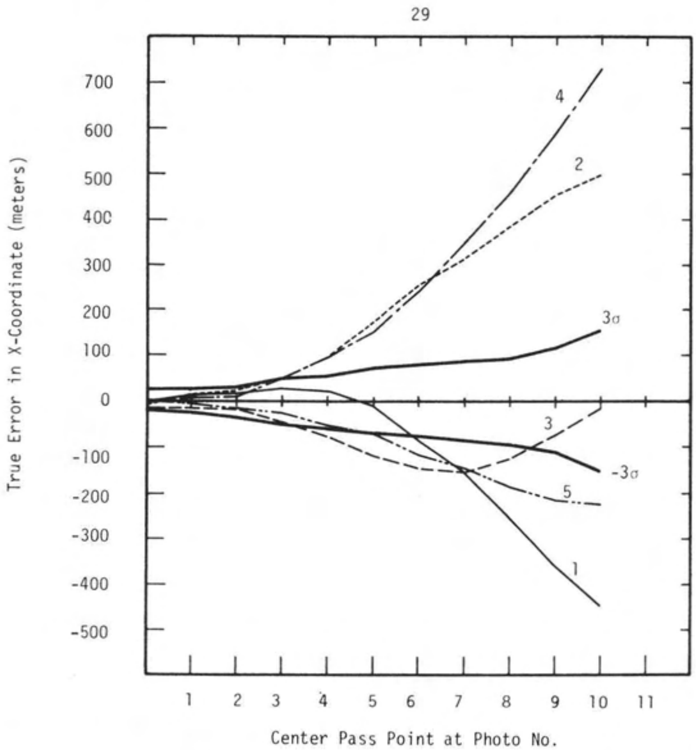

FIG. 3. True error in X-coordinate of computed pass points (cantilever extension).

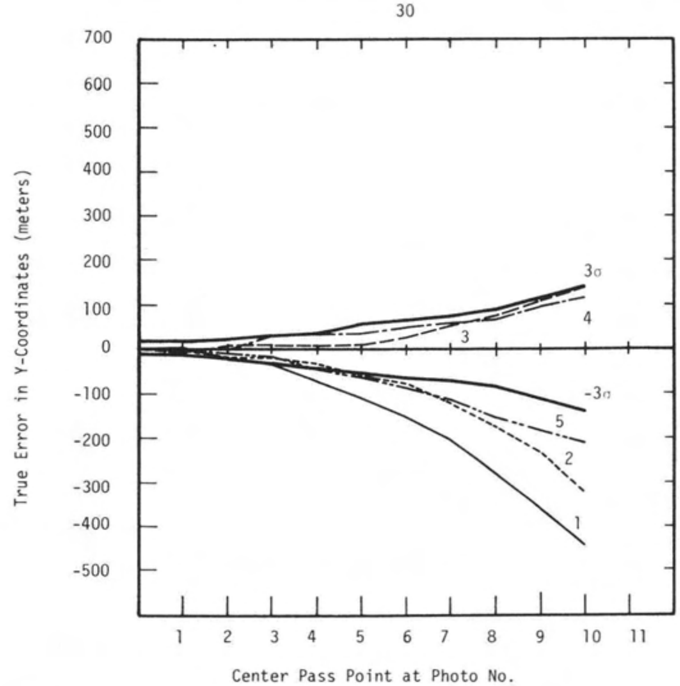

FIG. 4. True error in Y-coordinate of center pass points (cantilever extension).

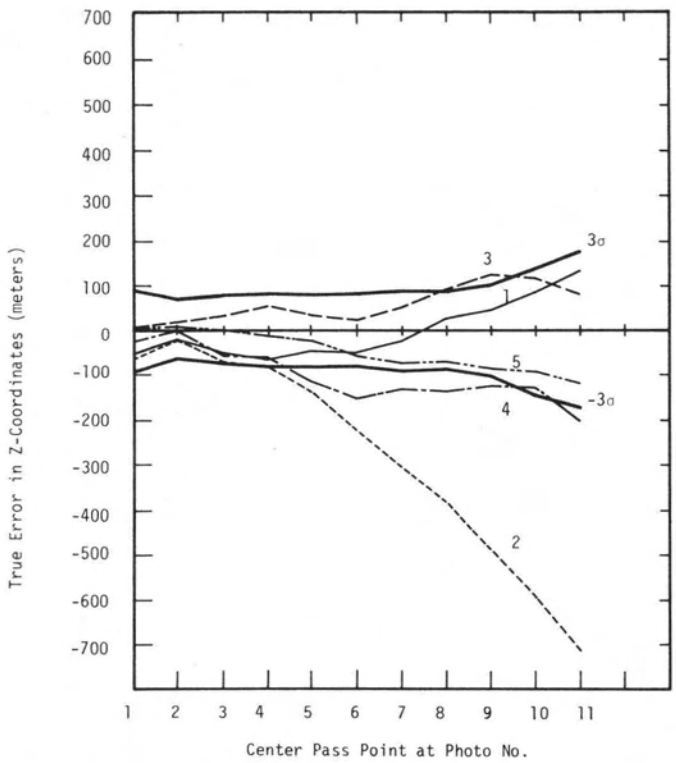

FIG. 5. True error in Z-coordinate of center pass points (cantilever extension).

★ Generally, if the corrections in the last iteration exceed the computed *RMS* errors even though the solution has stabilized, then the *RMS* errors are not reliable as estimator of the adjustment accuracy.

It is recommended that in simulation studies, true errors of the adjusted parameters should always be computed to check on the computed *RMS* errors. For problems in which the controls are sparsely distributed and/or are of low-order accuracy, several independent simulation cases should be performed to provide a check between the estimated *RMS* errors and the true errors. For actual problems in practice, check points should always be used as the primary verification of adjustment accuracy.

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(Continued from page 64)

FORUM

Dear Editor:

The recent article by Bob Reeves on "Education and Training in Remote Sensing" (*Photogrammetric Engineering*, June 1974) cites and briefly discusses an article I wrote (*Photogrammetric Engineering*, Sept. 1972) entitled "Remote Sensing Education in the U.S.A.". Although the citation is appreciated by this author, I believe that many, including Dr. Reeves, have not grasped the true focus of my article. Allow me to list some pertinent observations.

- My article was submitted to ASP in August 1971 and, for many of us authors who are not yet *household words*, it takes some time to see the fruits of our labor in print — in this instance some 13 months. Needless to say, as the field of remote sensing was rapidly developing at that time, many new programs and course offerings sprang up almost overnight, thereby making the list somewhat inconclusive.
- The focus of my article was on college and university courses, especially those that were listed in current school catalogs. For a student surveying the field, it is appropriate to review the various catalogs in search of a special program or unique course in remote sensing.
- The intent of my article was not to focus on traditional aerial photo interpretation courses (many other articles have done that in the past) but on those courses specifically involving other than black-and-white aerial photo study. Those courses in photo interpretation that also indicated some additional focus in unique sensor systems were separately treated.

I considered that my article was only an introductory incision into the field of remote sensing training and education. The article was directed to those scientists and resource managers, either practicing or students, who were in search of academic institutions in which they could obtain introductory formal training through lectures and labs or to receive an updating of their skills. The general response to the article was quite favorable as many believed that this article served as an excellent guide for prospective students seeking training in this dynamic field.

There were some, however, who were of the opinion that (a) a non-academician had no business in writing such an article and (b) surely far more courses in remote sensing were offered than those listed. To answer the second part of the criticism first (even though the printer did make a serious omission by deleting, for some reason, several graduate specializations which I had listed) the fact of the matter is, for the most part, the college catalogs did not list other additional courses, even though the latest catalogs were obtained. It was not the intention of my article to list industrial short courses, seminars, symposia, or research programs and grants.

As to the first charge, that a non-academician should not have even prepared such an article, I hasten to point out that still to this day, no other listing of remote sensing courses is readily available. True, as Dr. Reeves states " . . . the . . . Education Committee . . . is going to conduct a survey of . . . courses in remote sensing . . . ", but the initial demand for such a survey was during the formative stages of remote sensing.

I did not envision working against any committee or against any university image. However, I did see the need and attempted to fill the void.

— *Dean F. Eitel*

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