

Fourier Spectra for Non-Homogeneous Patterns

An optimal optical field size exists for any component of the power spectrum that one may wish to consider.

INTRODUCTION

THE APPLICATION of optical fourier transforms to the processing of photographic data in the earth sciences in general has been discussed by a number of authors (Pincus and Dobrin, 1966; Dobrin, 1968; Nyberg, Orhaug and Svensson, 1971). The use of such transforms in obtaining the power spectra of ocean surface waves was worked out by Stilwell (1969). Almost all the patterns being analyzed are non-homogeneous; however, the problems peculiar to the analysis of non-homogeneous patterns have not been ade-

$$\Delta k \Delta x \geq \frac{1}{2} \quad (1)$$

where Δk is the root-mean-square (*rms*) dispersion in the fourier transform of a function whose *rms* dispersion is Δx .

Applied to a photograph, the fourier transform of the area of the photo acts as a filter on the fourier transform of the pattern photographed. In one dimension, the transform of a rectangular photo is the classical one for diffraction from a rectangular slit.

$$F(k) = \frac{\sqrt{2}}{\pi} \frac{\sin \frac{1}{2} kl}{k} \quad (2)$$

where l is the width of the photo in the direc-

ABSTRACT: Non-homogeneous patterns are subject to intrinsic limitations on the degree of spectral resolution possible in pattern analysis by optical fourier transforms. The construction of nearly optimal spectra for such patterns deals with ocean surface waves in shoaling water as an example.

quately dealt with previously. In a recent paper (Polis, 1974) the present author addressed himself to the determination the optimum field size for the analysis of a particular component of the ocean-wave spectrum in a non-homogeneous regime. The present paper deals with the construction of optimal power spectra from photographs of non-homogeneous patterns, using ocean surface waves as an example.

GENERAL PRINCIPLES

It is well known in optics that, as was first pointed out by Rayleigh (1879, 1880), the resolving power of an optical system is ultimately limited by the size of its apertures. This principle was generalized by Heisenberg (1927) who postulated the uncertainty relation of quantum mechanics. In terms of fourier transforms, the uncertainty relation is

tion under consideration. This pattern, taken in its entirety, has an infinite width; however, in the optical fourier transform technique, only the central lobe is used, *i.e.*, the portion $-\pi < \frac{1}{2} kl < \pi$. The *rms* width of this portion is given by

$$(\Delta k(l))^2 = \int_{-\frac{2\pi}{l}}^{\frac{2\pi}{l}} k^2 F(k) dk \left(\int_{-\frac{2\pi}{l}}^{\frac{2\pi}{l}} F(k) dk \right)^{-1} \quad (3)$$

$$\Delta k(l) = \frac{2\sqrt{\pi}}{l} \quad (4)$$

which is consistent with Equation 1.

A non-homogeneous pattern is one whose spectral properties change from point to point. Obviously, this is an imprecise definition as the spectrum of a mathematically

well-behaved spatial pattern is a uniquely defined function; however, this uniqueness applies only the global pattern.

One is usually concerned with the spatial properties of a small area and these will vary from location to location. For example, the spacing and direction of streets differs between the older, historically Mexican section of Los Angeles, and the more recent adjacent areas. Similarly, ocean surface waves will vary in their direction and wave length as they enter shallow water.

CONSTRUCTION OF OPTIMAL SPECTRA

Following the method outlined above one can determine the optimal field size for the analysis of a particular vector wave number; however, the field size will vary with the wave number under consideration. For surface gravity waves in shoaling water Polis (1974) obtained

$$l_{opt} = \frac{2}{k} \left[\frac{3\pi(\sinh 2kh + 2kh)}{m(\sin\theta \sin 2\theta - 2\cos\theta)} \right]^{1/2} \quad (5)$$

where k is the magnitude of the wave number, h the water depth, m the bottom slope ($m < 0$), and the θ the angle between the direction of wave propagation and the bottom gradient. Thus, l_{opt} is strongly de-

pendent on vector wave number. This means that in order to construct an entire optimal spectrum it is necessary to use a variable field size.

A given field size will be optimal for an entire set of vector wave numbers. The loci of this set in k -space can be found by inverting the equation corresponding to Equation 5 for the particular phenomenon under consideration. Here Equation 5 is transcendental, and must be solved numerically; however, for purposes of illustration, we can consider the approximate form of Equation 5 which applies to shallow water. For $kh \ll l$,

$$l_{opt} = \frac{4}{k} \left[\frac{3\pi kh}{m} \right]^{1/2} f(\theta) \quad (6)$$

where

$$f(\theta) = (\sin\theta \sin 2\theta - 2\cos\theta)^{-1/2}. \quad (7)$$

Inverting Equation 6 one finds that a given field size is optimal for two lines in k -space symmetric about origin and given by

$$k_{opt} = \frac{48\pi h}{ml^2} f^2(\theta). \quad (8)$$

If the non-homogeneity causes a variation in the vector wave number of a particular

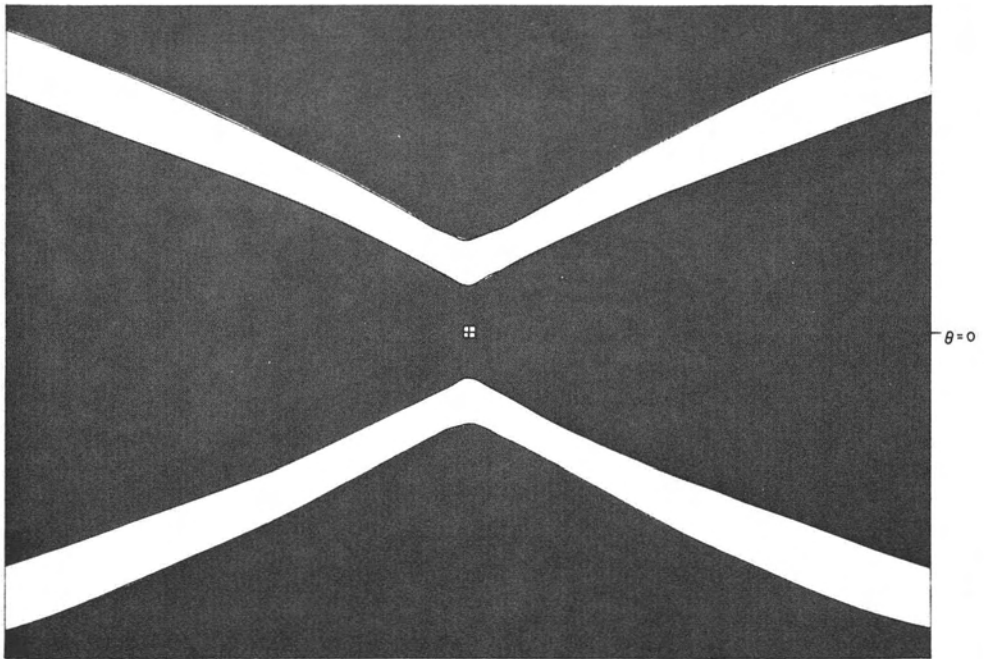


FIG. 1. Sample spatial filter for the optimal analysis of shoaling wave spectra in shallow water. The pass band has for its lower bound $k_{opt}(l_2, \theta)$, and for its upper bound $k_{opt}(l_1, \theta)$, as calculated from Equation 8, with $l_2 > l_1$. This filter would be used in conjunction with a photo with field size, l , intermediate between l_1 and l_2 .

component such as occurs in the shoaling of waves, then one may proceed to optimize the spatial analysis of the pattern using the method employed by Polis (1974). Namely, one estimates the *rms* variation in the wave number δk of a given component caused by the non-homogeneity over a variable distance, l . This variation will usually increase monotonically with l . At the same time Δk from Equation 4 is a monotonically decreasing function of l . If $\Delta k = \delta k$, one has the optimum pattern size for the analysis of the given component $l_{opt}(k)$. For values of $l > l_{opt}(k)$, the non-homogeneity will be limiting, whereas for $l < l_{opt}(k)$, the uncertainty principle will be limiting. One consequence of this is that there are ultimate limits of resolution for each vector wave number in a spatial pattern, *viz.*, the value of k corresponding to $l_{opt}(k)$.

One may be able to make some marginal improvements in the resolution by going from a uniform coherent illumination of a rectangular photograph to a coherent beam with a carefully tailored cross-sectional distribution of intensity in order to obtain a filter function superior to that given by Equation 2; however, improvements along these lines are ultimately limited by the uncertainty principle, Equation 1. In some instances Δk will be so large compared to the central value of k that the whole notion of a spectrum for the local area is meaningless, *e.g.*, if $\Delta k/k \geq 1$. Thus, one must be careful in interpreting the spectra of non-homogeneous patterns, as nothing in the physical technique of optical fourier transforms will prevent the generation of meaningless spectra. The analysis must be limited to the domain of k -space where the results are physically meaningful.

Actually, because only limited resolution is possible, smearing of the spectrum will occur over a band $\Delta k(l_{opt})$ wide in k -space. This means that if one wants to piece a spectrum together as a mosaic, there is no point in using bands much smaller than $\Delta k(l_{opt})$ for any particular band centered on k . If there is reason to believe that the spectrum is relatively smooth, then one might be able to use bands much wider than this. In general, it turns out to be impossible to construct a mosaic from bands of constant Δk using, say, successively larger rectangular photographs. This is because of the usually non-linear relationship between k_{opt} and l demonstrated in Equation 8 for waves or in its analogue for some other phenomenon.

It is simplest to take as the boundaries of each mosaic section, lines given by Equation 8 for values of l intermediate between the successive field sizes used. If this is done,

one will have a pass-band for each field size similar to that shown in Figure 1. This is simply a spatial filter, except that the inverse transform is not taken after filtering. Using a series of such filters in conjunction with successively larger fields it is possible to construct a nearly optimal spectrum for any pattern whose spatial variation can be described theoretically.

This leaves a very large question for further research on the analysis of non-homogeneous patterns, *viz.*, how does one determine the optimal field sizes and appropriate spatial filters for patterns about which we have no *a priori* knowledge? Such patterns do, after all, constitute the majority of those encountered in remote sensing.

CONCLUSION

In analyzing non-homogeneous patterns by the optical fourier transform technique, an optimal field size exists for any component of the power spectrum one may wish to consider. If one seeks to analyze patterns smaller than this, the spectral resolution will be poor, whereas for larger field sizes the inhomogeneity of the pattern will mask the local features one is trying to investigate. Such considerations place ultimate limits on the possible spectral resolution of local patterns. We have discussed the analysis of patterns for which theoretical models exist; however, further work is needed for the optimal analysis of non-homogeneous patterns which are not adequately described theoretically.

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