DR. YOUSSEF I. ABDEL-AZIZ
*Cairo University*
*Cairo, Egypt*

# Asymmetrical Lens Distortion

## The parameters can be estimated from the laboratory data obtained during camera calibration.

ABSTRACT: *In laboratory camera calibration, radial lens distortion is provided along four perpendicular diagonals in the image plane. According to Harris, Tewinkel and Whitten (Coast and Geodetic Survey Technical Bulletin No. 21), one can use this information to estimate the symmetrical lens distortion and the radial component of asymmetrical lens distortion. In the Harris-Tewinkel-Whitten model, the tangential component of asymmetrical lens distortion is considered to be non-existent (based mainly on the lack of data at that time). This article presents formulation by which symmetrical and asymmetrical (radial and tangential) lens distortion parameters can be estimated from the radial lens distortion along the four diagonals.*

### INTRODUCTION

IN LABORATORY camera calibration, according to the ASP Manual of Photogrammetry[1], the calibration report provides the user with the following information:

- Point of symmetry.
- Tangential distortion (maximum values of tangential distortion).
- The radial lens distortions along at least four diagonals.

In photogrammetric applications the user treats the calibrated data by one of these methods:

- Estimate the average radial lens distortion and neglect the asymmetrical lens distortion.
- Estimate the average values of the radial lens distortion and the radial components of asymmetrical lens distortion by Harris, Tewinkel, and Whitten model[3].
- Use the information as it is and by interpolation one can estimate the corrections for any point on the film plane.

The corrections of the image coordinates by applying any one of the above methods (including the interpolation method) are based mainly on one assumption — that the tangential components of asymmetrical lens distortion do not exist.

In high-precision photogrammetric applications this assumption cannot be accepted for two simple reasons:

- The tangential components of asymmetrical lens distortion of some cameras are large, as one can see from the maximum value of tangential distortion components given in the calibration report.
- The tangential components of asymmetrical lens distortion can be estimated from the calibration data.

In the next sections one can see that asymmetrical, radial, and tangential lens distortion parameters can be estimated from the values of radial lens distortion along the four diagonals.

### THE ASYMMETRICAL LENS DISTORTION MODELS

In photogrammetric literature two models have been adopted to represent the asymmetrical lens distortion. The first model is called the thin-prism model[2], according to which the asymmetrical lens distortion components $\Delta r_r$ and $\Delta r_t$ take this form:

$$\left.\begin{aligned} \Delta r_r &= P \cos(\phi - \phi_o) \\ \Delta r_t &= P \sin(\phi - \phi_o). \end{aligned}\right\} \qquad (1)$$

The second model is called Conrady's model[2], according to which the asymmetrical lens distortion components $\Delta r_r$ and $\Delta r_t$ take the form:

$$\left.\begin{aligned}\Delta r_r &= 3\ P\ sin\ (\phi - \phi_o)\\ \Delta r_t &= P\ cos\ (\phi - \phi_o)\end{aligned}\right\} \quad (2)$$

$$P = J_1 r^2 + J_2 r^4 \ldots \quad (3)$$

where $\Delta r_r$ is the radial component of assymmetrical lens distortion, $\Delta r_t$ is the tangential component of assymmetrical lens distortion, and $\phi_o, J_1, J_2 \ldots$ etc., are constants.

In most analytical photogrammetry applications the above Equation 3 takes the form:

$$P = Jr^2. \quad (4)$$

Brown[2], and many others, adopted Conrady's model as the model for asymmetrical lens distortion and it will be adopted in this paper. According to Brown[2], the correction of the image coordinates takes the form:

$$\left.\begin{aligned}\Delta x &= P_1 (r^2 + 2x^2) + 2P_2\, xy\\ \Delta y &= P_2 (r^2 + 2y^2) + 2P_1\, xy\end{aligned}\right\} \quad (5)$$

where

$$P_1 = -J_1\ sin\ \phi_o \quad (6)$$

$$P_2 = J_2\ cos\ \phi_o \quad (7)$$

$$r^2 = x^2 + y^2 \quad (8)$$

and $x$, $y$ are the image coordinates with the point of autocollimation as an origin.

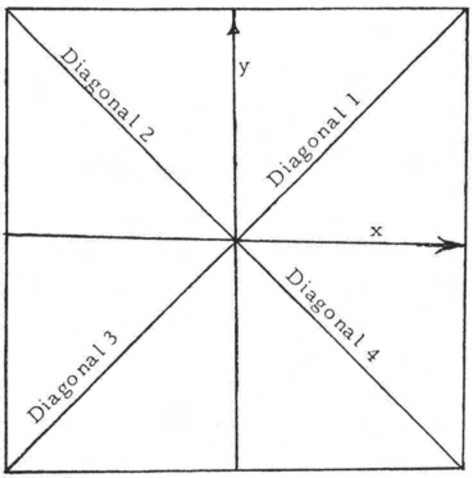


FIG. 1. The location of the four diagonals in the image plane.

Conrady's model, as well as thin prism model, shows one important fact, that the radial component and the tangential component of asymmetrical lens distortion are correlated, and if the radial component is known, one can determine the tangential component. Based on such a correlation, one should be able to determine the parameters of asymmetrical lens distortion $P_1$ and $P_2$ in Equation 5 from the radial component of the lens distortions along the four diagonals.

## Estimation of the Asymmetrical Lens Distortion Parameters from the Radial Lens Distortion Along the Four Diagonals

In any report on camera calibration, the user is provided with a table similar to Table 1 in which the values of the radial lens distortion $\Delta r_1$, $\Delta r_2$, $\Delta r_3$, and $\Delta r_4$ along the four diagonals *1*, *2*, *3*, and *4* are given at different radii (Figure 1). The values of the radial lens distortion given in Table 1 at any diagonal $i$ can be expressed mathematically in the form:

$$\Delta r_i = \Delta r_s + \Delta r_r \quad (9)$$

where $\Delta r_i$ is the total radial lens distortion at radii $i$, $\Delta r_s$ is the radial component of symmetrical lens distortion, and $\Delta r_r$ is the radial component of asymmetrical lens distortion.

Substituting the value of $\Delta r_r$ from Equation 2 into Equation 9, one obtains:

$$\Delta r_i = \Delta r_s + (3P)\ sin\ (\phi - \phi_o). \quad (10)$$

Substituting the value of $P$ from Equation 4 into Equation 10 one gets:

$$\Delta r_i = \Delta r_s + (3Jr^2)\ sin\ (\phi - \phi_o). \quad (11)$$

The radial components of the lens distortion $\Delta r_1$, $\Delta r_2$, $\Delta r_3$, and $\Delta r_4$ at radius $r$ along any diagonal can be obtained by substituting the values of $\phi$ in Equation 10 to be equal to *45*,

TABLE 1. THE CALIBRATION DATA PROVIDED BY CAMERA CALIBRATION LABORATORY FOR LENS DISTORTION

| *Cone Angle* | *Radius* | *Lens Distortion (um)* $\Delta r_a$ | $\Delta r_b$ | $\Delta r_c$ | $\Delta r_d$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 7.5 | 19.738 | −6.0 | −6.3 | −7.2 | −6.2 |
| 15.0 | 40.171 | −14.5 | −15.0 | −15.2 | −15.1 |
| 22.45 | 61.944 | −9.9 | −11.8 | −13.8 | −12.4 |
| 30.0 | 86.549 | 0.1 | −1.8 | −7.3 | −4.7 |
| 35.0 | 104.962 | 11.0 | 7.4 | 0.3 | 5.1 |
| 40.0 | 125.774 | 19.9 | 14.3 | 4.5 | 10.2 |
| 45.0 | 149.881 | 29.3 | 21.2 | 8.1 | 14.6 |

*90 + 45, 180 + 45* and *270 + 45* for the diagonals *1, 2, 3*, and *4* respectively. Accordingly,

$$\Delta r_1 = \Delta r_s + (3Jr^2) \sin (45 - \phi_o) \quad (12)$$

$$\Delta r_2 = \Delta r_s + (3Jr^2) \cos (45 - \phi_o) \quad (13)$$

$$\Delta r_3 = \Delta r_s - (3Jr^2) \sin (45 - \phi_o) \quad (14)$$

$$\Delta r_4 = \Delta r_s - (3Jr^2) \cos (45 - \phi_o). \quad (15)$$

Using the values of $\Delta r_1$, $\Delta r_2$, $\Delta r_3$, and $\Delta r_4$ from Table 1, one can calculate the values of three function $f(r)$, $f_1(r)$, and $f_2(r)$ at different radii using the formulas:

$$f(r) = (\Delta r_1 + \Delta r_2 + \Delta r_3 + \Delta r_4) / 4 \quad (16)$$

$$f_1(r) = (\Delta r_1 + \Delta r_4 - \Delta r_2 - \Delta r_3) / 4 \quad (17)$$

$$f_2(r) = (\Delta r_1 + \Delta r_2 - \Delta r_3 - \Delta r_4) / 4. \quad (18)$$

The values of $f(r), f_1(r)$, and $f_2(r)$ calculated by using Equations 16, 17 and 18 are given in Table 2.

The mathematical formulas of $f(r)$, $f_1(r)$, and $f_2(r)$ can be obtained by substituting the values of $\Delta r_1$, $\Delta r_2$, $\Delta r_3$, and $\Delta r_4$ from Equations 12, 13, 14 and 15 into the above Equations. Accordingly,

$$f(r) = \Delta r_s \quad (19)$$

$$F_1(r) = (-3Jr^2 \sin \phi_o) / \sqrt{2} \quad (20)$$

$$f_2(r) = (3Jr^2 \cos \phi_o) / \sqrt{2}. \quad (21)$$

The function $f(r)$ gives the symmetrical lens distortion at any radius $r$. The two functions $f_1(r)$, and $f_2(r)$ will be used for estimating asymmetrical lens distortion parameters $P_1$ and $P_2$ in Equation 5.

The two functions $f_1(r)$ and $f_2(r)$ in Equations 20 and 21 can be put into the form:

TABLE 2. THE VALUES OF SYMMETRICAL LENS DISTORTION (IN MICROMETERS) f(r) AND THE TWO FUNCTIONS $f_1(r)$ AND $f_2(r)$ AT DIFFERENT RADII

| *Radius* | $f(r)$ | $f_1(r)$ | $f_2(r)$ |
|---|---|---|---|
| 0 | −0 | 0 | 0 |
| 19.738 | −6.425 | 0.325 | 0.275 |
| 40.171 | −14.950 | 0.150 | 0.200 |
| 61.944 | −11.975 | 0.825 | 1.125 |
| 86.549 | −3.425 | 1.125 | 2.575 |
| 104.962 | 5.950 | 2.1 | 3.250 |
| 125.774 | 12.225 | 2.825 | 4.875 |
| 149.881 | 18.3 | 3.65 | 6.950 |

$$f_1(r) = K_1 r^2 \quad (22)$$

$$f_2(r) = K_2 r^2 \quad (23)$$

where

$$K_1 = (-3J_1 \sin \phi_o) / \sqrt{2} \quad (24)$$

$$K_2 = (3J_1 \cos \phi_o) / \sqrt{2}. \quad (25)$$

Using the values of $f_1(r)$ and $f_2(r)$ at different radii from Table 2, one can use the least-squares solution to obtain the most probable values of $K_1$ and $K_2$ in Equations 22 and 23.

The most probable value of $K_1$ and $K_2$ can be obtained from the expressions:

$$K_1 = (\Sigma r_i^2 f_1 (r_i)) / \Sigma r_i^4 \quad (26)$$

$$K_2 = (\Sigma r_i^2 f_2 (r_i)) / \Sigma r_i^4 \quad (27)$$

The estimated values of $K_1$ and $K_2$ obtained by substituting the values of $f_1(r)$ and $f_2(r)$ and $r$ from Table 2 into Equations 26 and 27 are:

$$K_1 = 0.170 \times 10^{-4}\ mm^{-1}$$

$$K_2 = 0.308 \times 10^{-4}\ mm^{-1}.$$

Knowing the values of $K_1$ and $K_2$ one can estimate the values of $P_1$ and $P_2$ by using Equations 24, 25 and 7 and 8. Accordingly, the values of $P_1$ and $P_2$ are:

$$P_1 = (\sqrt{2}\ K_1) / 3 \quad (28)$$

$$P_2 = (\sqrt{2}\ K_2) / 3. \quad (29)$$

The estimated values of $P_1$ and $P_2$ obtained by substituting the calculated values of $K_1$ and $K_2$ into equations and are:

$$P_1 = 0.807 \times 10^{-5}\ mm^{-1}$$

$$P_2 = 0.145 \times 10^{-1}\ mm^{-1}.$$

Using the estimated values of $P_1$ and $P_2$ one can correct for asymmetrical radial and tangential lens distortion by using Conrady's model.

As a summary one can follow these procedures for estimating the asymmetrical lens distortion parameters $P_1$ and $P_2$:

★ Calculate the values of $f(r), f_1(r)$ and $f_2(r)$ by using Equations 16, 17 and 18.
★ Estimate the values of $K_1$ and $K_2$ using Equations 26 and 27.
★ Estimate the values of $P_1$ and $P_2$ using Equations 28 and 29.

## Modification of Harris-Tewinkel-Whitten Model for Asymmetrical Lens Distortion

The model developed by Harris, Tewinkel, and Whitten[3] has been built to correct for asymmetrical radial components of lens distortion. This model is still the only model in analytical photogrammetry for correction of the asymmetrical lens distortion by using the radial lens distortion along the four diagonals provided by laboratory camera calibration. The asymmetrical lens distortion parameters in this model are defined by three parameters $a$, $b$, and $c$. The mathematical model for estimation of such parameters is described in detail in reference[3].

In investigating the Harris-Tewinkel-Whitten model[3], it was found that one can correct for asymmetrical radial, and tangential distortion using Conrady's model if one has the asymmetrical lens distortion parameters $a$, $b$, and $c$.

The relationship between the asymmetrical lens distortion parameters $P_1$ and $P_2$ of Conrady's model (Equation 5) and the asymmetrical lens distortion parameters $a$, $b$, and $c$ can be defined as:

$$\left.\begin{aligned} P_1 &= (ca)\,/\,3 \\ P_2 &= (cb)\,/\,3. \end{aligned}\right\} \quad (30)$$

The mathematical proof of Equation 30 can be obtained by evaluating the values of the two functions $(\overline{\Delta r}_3 - \overline{\Delta r}_1)\,/\,2$ and $(\overline{\Delta r}_4 - \overline{\Delta r}_2)\,/\,2$ in both the Harris-Tewinkel-Whitten model and Conrady's model. In the Harris-Tewinkel-Whitten model, Equation 2.1 of reference three:

$$\left.\begin{aligned} (\overline{\Delta r}_3 - \overline{\Delta r}_1)\,/\,2 &= c_1 r^2 \\ (\overline{\Delta r}_4 - \overline{\Delta r}_2)\,/\,2 &= c_2 r^2. \end{aligned}\right\} \quad (31)$$

In Conrady's model according to Equations 12 through 15:

$$\begin{aligned} (\overline{\Delta r}_3 - \overline{\Delta r}_1)\,/\,2 &= (-3Jr^2)\, sin\,(45 - \phi_o) \\ (\overline{\Delta r}_4 - \overline{\Delta r}_2)\,/\,2 &= (-3Jr^2)\, cos\,(45 - \phi_o) \end{aligned} \quad (32)$$

where $c_1$ and $c_2$ are constant, $\overline{\Delta r}_1$, $\overline{\Delta r}_2$, $\overline{\Delta r}_3$, and $\Delta r_4$ are the radial components of asymmetrical lens distortion, i.e., $(\overline{\Delta r}_i = \Delta r_i - \Delta r_s)$.

From Equations 31 and 32 one can have these relationships:

$$J = (c_1^2 + c_2^2)^{1/2}\,/\,3 \quad (33)$$

$$tan\,(45 + \phi_o) = c_2\,/\,c_1. \quad (34)$$

According to the Harris-Tewinkel-Whitten model, the asymmetrical lens distortion parameters $a$, $b$, and $c$ can be estimated from the values of $c_1$ and $c_2$ as follows:

$$c = (c_1^2 + c_2^2)^{1/2} \quad (35)$$

$$tan\,\theta = c_2\,/\,c_1 \quad (36)$$

$$a = cos\,(\theta + 45) \quad (37)$$

$$b = sin\,(\theta + 45). \quad (38)$$

Using the above relationship in Equation 33 through 38 one has:

$$\theta = 45 + \phi \quad (39)$$

$$J = c\,/\,3 \quad (40)$$

$$a = -\,sin\,\phi_o \quad (41)$$

$$b = cos\,\phi_o. \quad (42)$$

Using the values of cos $\phi_o$, $sin\ \phi_o$, and $J$ from the above equations into Equations 6 and 7 one has:

$$P_1 = ca\,/\,3 \quad (43)$$

$$P_2 = cb\,/\,3. \quad (44)$$

As a result, one can estimate the asymmetrical lens distortion parameters in Conrady's model ($P_1$ and $P_2$) and correct for radial and tangential components of asymmetrical lens distortion, rather than using the Harris-Tewinkel-Whitten model asymmetrical lens distortion parameters $a$, $b$, and $c$ and correct only for asymmetrical radial lens distortion.

## Conclusions and Remarks

The developed model for asymmetrical lens distortion is based on one main assumption that Conrady's model is the proper model for representing asymmetrical lens distortion. This assumption has been accepted by most photogrammetrists.

The main advantage of the developed model is that it is the only model that makes use of laboratory camera calibration data for correcting the asymmetrical radial and tangential components of lens distortion.

Accordingly, it is recommended that such a model be used in camera calibration laboratory report to provide the user with asymmetrical lens distortion parameters.

## References

1. American Society of Photogrammetry, *Manual of Photogrammetry*, Third Edition, 1966.
2. Brown, D. C., "Decentering Distortion of Lenses", *Photogrammetric Engineering*, Vol 32, No. 3, 1966.
3. Harris, W. D., Tewinkel, G. C., and Whitten, C. A., "Analytic Aerotriangulation", Coast and Geodetic Survey *Technical Bulletin No. 21*, 1963.