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Photogrammetric Interpolation*

Different interpolation methods should be used for different photogrammetric tasks.

Abstract: *A number of typical tasks of photogrammetry are defined as interpolation problems. Experiences in some of these are used to advocate a more differentiated judgement of interpolation methods. Some are compared with each other and an attempt is made to show that it is quite useful to rely on a number of different interpolation methods for different photogrammetric tasks.*

Introduction

This paper intends to draw attention to the fact that interpolation in photogrammetry is too important a problem to be approached with an attitude that "linear interpolation is good enough," that "one method is perfect in all applications," or that "the best method of interpolation is the one which costs the least effort." Instead, in evaluating the problem one has to differentiate many aspects.

This is attempted first by a number of theoretical considerations. These are then supported from specific photogrammetric interpolation experiences concerning projection errors in Dutch test field photography, film deformation correction, *SLAR* mapping, and interpolation for Digital Terrain Models *(DTM)*.

Interpolation

A phenomenon is known at a number of discrete points in n-dimensional space (Reference space). These points are called reference or data points. Interpolation consists of estimating the same phenomenon at intermediate points using the given data. The phenomenon under consideration may be described by a scalar, or by a vector of dimension $m > 1$ (see Figure 1, upper).

This definition is rather practical and does not specifically refer to "smoothing," "filtering" or "regression." These concepts will become relevant if the data in the reference points are measured, and thus composed of the "signal" and an uncorrelated measuring error, "noise." In this case it becomes a meaningful problem to separate in the data points the observational errors from the signal, and to obtain at non-data points estimates of the signal only ("filtering of the noise" or smoothing (Figure 1, lower)).

Interpolation as defined here is part of the more general mathematical theory of approximation, of which it represents a particular application. Further pertinent terminology refers to "curve" and "surface fitting" and to "prediction" (to denote interpolation and extrapolation).

These concepts will be explained by a number of tasks of practical photogrammetry. In this context, it is the term "interpolation" which is used to denote problems covering also smoothing, filtering, or surface fitting. Although this nomenclature might not satisfy representatives of other fields of science, it is

* Paper presented at Federation Internationale des Geometres, XIV Congress, Washington, D.C., September 7-16, 1974.

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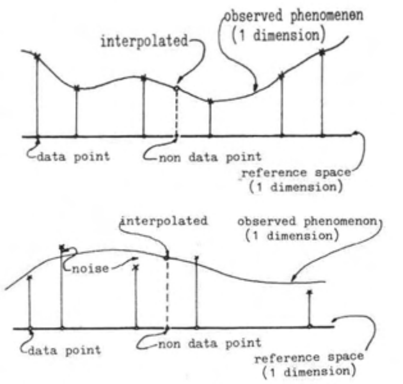

FIG. 1. Explanation of the concepts used for the definition of interpolation (upper) and interpolation and filtering (lower).

the one traditionally understood in photogrammetry.

PHOTOGRAMMETRIC INTERPOLATION TASKS

Table 1 summarizes a number of photogrammetric tasks which involve interpolation. It also specifies the dimension of the reference space as well as observed phenomenon. It turns out that correction of radial symmetric lens distortion is one of the simplest interpolation problems. The reference space is one dimensional: the radial distance. The observed phenomenon is the one-dimensional radial lens distortion. A more complicated task is interpolation of Δx, Δy, Δz block- or strip-deformation in the three dimensional coordinate system of the strip, block, or terrain.

In a number of cases, such as lens distortion and film deformation the phenomenon to be studied is observed directly. In others, the observations are indirect, as in strip- and block deformations. Typically, these indirect observations are obtained as residuals after a transformation by the method of least squares in which the mathematical model was imperfect. In this context it is relevant to note that transformation and interpolation are tasks performed sequentially. They can to some extent substitute each other. In the example of absolute orientation of a well-controlled photogrammetric model, a conformal transformation followed by an interpolative correction of model deformation might be as effective as a sole transformation with more than just the conformal parameters. A perfect mathematical model for a least squares adjustment can result in purely uncorrelated residuals. If, however, the mathematical model is simplified, there will be a "signal" left in the residuals, so that post-processing of the least squares adjustment results can be useful. Least squares adjustment, filtering and interpolation can be combined into one algorithm. This is called by Moritz "least squares collocation."

A CLASSIFICATION OF INTERPOLATION METHODS

Interpolation methods could be classified according to the purpose for which they would be suited, e.g., according to Rice[12] whether they are for mathematical representation (derive values at non-data points); for data analysis (smoothing, extract signal,

TABLE 1. PHOTOGRAMMETRIC TASKS INVOLVING INTERPOLATION.

Task	Dimensions of Reference Space	Dimension of Phenomenon
Determination of refractive index	3 (x, y, z coordinates, plus event. time?)	1 (refractive index)
Lens distortion correction in photograph (a)	2 (x, y image coordinates; or radial distance r and azimuth)	2 (tangential and radial distortion; or Δx, Δy image errors)
Lens distortion correction in photograph, only radial (b)	1 (radial distance r)	1 (radial distortion)
Film deformation correction	2 (x and y image coordinates)	2 (Δx, Δy film deformations)
Rectification	2 (x and y image coordinates)	2 (Δx, Δy image deformations)
Instrumental error correction		
(a) comparator	2 (x, y coordinates)	2 (Δx, Δy)
(b) plotter	3 (x, y, z model coord.)	3 (Δx, Δy, Δz)
Model deformation correction	3 (x, y, z model coord.)	3 (Δx, Δy, Δz model deformations)
External strip adjustment, planimetry and height	3 (x, y, z strip coordinates)	3 (planimetry: Δx, Δy; & height: Δz)
External block adjustment planimetry + height	3 (x, y, z block coordinates)	3 (planimetric + height deformations)
Digital terrain model (DTM)	2 (x, y reference plane)	1 (z . . . height)

analyse trend); for data compression (elimination of redundant information); or for easy manipulation and evaluation.

Photogrammetric interpolation tasks often combine some or all of these four objectives. In such a case another classification might be appropriate, depending on whether interpolation is with a single, global function; interpolation is by piecewise, locally defined functions; or interpolation is pointwise.

For example, interpolation with a global function is applied in strip adjustment. All data points are used simultaneously to define the interpolating function.

This might be acceptable for strip adjustment with only a few control points, but for a large number of control points, a low order function cannot conform to all data points. High order functions, on the other hand, tend to be unstable, if an orthogonalization procedure is not used. A solution to this dilemma is interpolation with piecewise functions[6]. The reference space is subdivided into patches, and for each patch another interpolation function is defined. Often the necessity arises of enforcing some continuity between neighbouring patches to avoid cracks. This does not necessarily require explicit consideration of boundary or joining conditions[2]. Typical piecewise interpolation employs linear interpolation, piecewise polynomial interpolation, double linear interpolation[13], or spline functions.

Pointwise interpolation defines a new interpolation function for each non-data point, using the surrounding subset of data points. Pointwise interpolation is flexible, and does not require extensive computer memory, but is often slower than the other two classes of interpolation. Typical pointwise interpolation methods are with moving averages[3, 14], linear prediction[5], and weighted arithmetic mean.

A detailed description of each of these interpolation methods can be found in the references indicated. A short review is included in the appendix.

Rather than speaking about "methods of interpolation," Rice[12] refers to "algorithms." With these, he denotes basically a computer program performing an interpolation task. This algorithm, however, is composed of different constituents, namely: interpolation form (polynomials, piecewise functions, etc.); error measure (least squares, perfect fit at data points, etc.); and method of solving for the unknown. Rice[12] mentioned that the same constituents can produce a number of alternative algorithms and compared the situation with "cooking."

CONCEPTS FOR EVALUATION OF INTERPOLATION METHODS

General. At present there is a need for objective comparison of interpolation methods in photogrammetry. In the past, the parameters of a specific method have usually been optimised for a specific application. One of the reasons for this might be the great deal of intuition that is often used in choosing an interpolation method, and a lack of criteria beyond accuracy to differentiate between methods. An often encountered attitude is therefore to assert that all interpolation methods work equally well, so that the simplest and thus cheapest should be used. However, experience has shown that there are cases where significant differences exist in the performance of different interpolation methods, even with respect to overall accuracy. And in other cases, this accuracy might indeed not differ from one method to another. The performance of an interpolation algorithm may vary considerably as a function of the structure of the input data (distribution of data points).

In addition to having performance criteria other than accuracy, evaluation of interpolation methods is made difficult by this dependence on input data structure. But for a given input, Rice[12] identifies a series of properties of interpolation algorithms to be used for comparative evaluation: speed (of solving for unknowns); flexibility (overall accuracy and maintaining shape of small features); smoothing power; constraint imposition (terrain break lines in DTM); memory requirement; smoothness (continuous, derivatives); and speed of evaluation (of interpolating function).

Speed of solving for the unknowns and speed of evaluation cannot be considered separately in pointwise interpolation, but can be separated in, for example, piecewise polynomials. Smoothing power refers to the capability of filtering a measuring error from the given data, whereas smoothness refers to the appearance of an interpolated curve or surface, and whether it is continuous or not. From practical experience it seems to be useful to add to these properties: usefulness for extrapolation, and reliability (sensitivity to right choice of parameters).

The Accuracy of Interpolation Methods. Generally, evaluation of interpolation methods is attempted on the basis of their accuracy, which can be described by a root mean square interpolation error. What is this error and how can it be obtained?

In a controlled experiment, the interpolation error can be found by using checkpoints, in which the interpolated and the known values are compared. Such an error is composed of the propagation of the measuring error into the interpolated value, and of the loss of information through the sampling of the phenomenon at discrete points.

In actual application, the interpolation error can be estimated by interpolation in data points without using the information of the data point, except for a comparison with the interpolated value. If sampling were regular, then this method might produce an error estimate which is too large, since the distance between any non-data point to the closest reference value is at least half as small as the distance of such check points to the closest data point. The error estimate can however be used for a comparative evaluation of different interpolation methods, or the optimization of parameters within a method[8]. Propagation of variances into an interpolated value does not result in an estimate of the interpolation error. Such propagation would only account for the effects of measuring errors (noise) which often will be much smaller than the effect of a limited sampling density.

An approach to evaluating the accuracy of interpolation can be derived from the theory of random functions[1]. Assuming that a phenomenon can be defined as a stochastic function $z = f(x)$, which is characterized by the covariance function $cov\ (x_i, x_j)$ between the random variables $z_i = f(x_i)$, $z_j = f(x_j)$; $x_i, x_j \epsilon x$, where x is the continuous range of definition of the independent variables.

All interpolation methods mentioned in the classification can be described as a linear relation between the interpolated value $\overline{z}_p$ and the values in the data points $z_1, z_2 \ldots z_n$:

$$\overline{z}_p = a_1 \cdot z_1 + a_2 \cdot z_2 + \ldots + a_n \cdot z_n = \underline{a}^t \cdot \underline{z}$$

The coefficients a are specific for each interpolation method (and also specific for each position of the non-data point). Variance propagation is applied to the expression

$$\epsilon = z_p - \overline{z}_p$$

with the result:

$$\sigma^2_{\underline{z}} = cov(x_p, x_p) + \underline{a}^t \cdot \underline{Cov}\,(x_i, x_j) \cdot \underline{a} - 2 \cdot \underline{a}^t \cdot \underline{cov}(x_p, x_i)$$

Here, $cov\ (x_p, x_p)$ is a scalar, namely the variance of the phenomenon.

$\underline{Cov}\,(x_i, x_j)$ is a matrix of covariances among all n data points ($i = 1, \ldots, n; j = 1, \ldots, n$).

$\underline{cov}\ (x_p, x_i)$ is a vector of covariances between the interpolated and given values. The elements of this vector depend on the position of the non-data point.

Clerici and Kubik[1] used $\sigma^2_{\underline{z}}$ for a comparative evaluation of linear prediction and linear interpolation. The conclusion was that there is hardly a significant difference between the accuracy of the two methods. This conclusion is, however, based on the assumption that the phenomenon can fully be described by the covariance function. Often, this might not be the case. This might explain why, in contradiction to the conclusion reached by Clerici and Kubic[1], significant differences between linear interpolation and other methods were encountered in an experiment with *DTM* data[9].

TABLE 2. SUBJECTIVE EVALUATION OF A NUMBER OF INTERPOLATION METHODS; F . . . FAVOURABLE; M . . . MEDIUM; U . . . UNFAVOURABLE. SQUARE-GRID INTERPOLATION ASSUMED.

	Linear interpolation	Polynomial	Meshwise bi-linear polynomial	Double linear interpolation	Weighted arithmetic mean	Pointwise linear prediction	Zonewise linear prediction	Moving average	Piecewise polynomial
Speed	F	F	F	F	F	U	M	U	U
Accuracy	U	U	M	M	U	F	F	F	F
Smoothing power	U	M	U	U	M	F	F	F	M
Constraint imposition	U	M	U	U	M	F	F	F	M
Memory requirement	F	F	F	F	F	M	U	F	F
Smoothness	M	F	M	M	M	F	F	F	F
Speed of evaluation	F	F	F	F	F				F
Use for extrapolation	U	U	U	U	F	F	F	U	U

Properties of Some Interpolation Methods. Preferably, a judgment of interpolation methods should consider the structure of input data to which the methods are to be applied. A thorough effort in this direction should be undertaken, since at the present time only a subjective judgment for square-grid interpolation can be presented in Table 2. Its main purpose is to give an example of an attempt to evaluate interpolation methods. It can only be intended as a basis for discussion rather than as an authoritative classification.

SOME EXPERIENCES WITH INTERPOLATION PROBLEMS

Filtering of Projection Errors in Test Field Photography. For the Dutch Photogrammetric Society, a large set of aerial photographs was obtained over the testfield "Flevopolder," with different cameras and at different flying heights. The purpose was to study the errors of the central projection, comparing photogrammetric points with the "theoretical" points derived from terrestrial surveys by a resection in space. Consequently a set of projection errors was obtained for each of the given photographs, representing a two-dimensional phenomenon on a two-dimensional reference space. Part of the data analysis was to study the trend and the amount of random noise in these errors.

The details of the projection will be published. What is of interest here, is the interpolation aspect of it. The usefulness of four different algorithms was compared for the purpose. Since one objective was data analysis, use of a number of methods without smoothing power was not possible (e.g., linear interpolation). So the methods selected were a single regression polynomial with 1-to-10 coefficients, independent for Δx and Δy errors; a meshwise third-order polynomial according to Jancaitis and Junkins[2]; a moving average of order 1-to-10; and linear prediction.

It was soon found that, for the present project, these methods all produced the same results, which are shown in Figure 2. A similar result was obtained by Kupfer[7], comparing a single polynomial with linear prediction in an application to test photography of the Rheidt test area near Bonn, Germany: a third-order polynomial with 10 coefficients was found to be sufficient to describe the signal in the data. "Sufficient" was defined by a lack of correlation in the residuals left after filtering.

The conclusion which one might be tempted to draw from these results is that there are no differences between the accuracy or power of interpolation methods. It will be demonstrated that such a conclusion would be premature. One can state safely only that the specific data do not show such a difference.

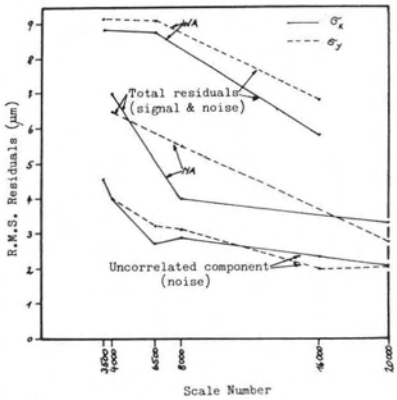

FIG. 2. Separation of projection errors into signal and noise.

The interpolation aspect of the study demonstrates the importance of the concept of filtering in photogrammetry. By means of the correlation function, presence of a signal in the data can be verified. In the particular case of the analysis of projection errors, one might use a "signal" found in the data analysis step for correction of projection errors in future flight missions. Such an objective would mean mathematical representation. The pointwise interpolation algorithms are inappropriate for this purpose. Instead of such a pointwise numerical trend, a mathematical global function can be used. Since the global polynomial proved sufficient in the data analysis step, it would be the obvious function to be used for correction of projection errors.

Planimetric Mapping with Side-Looking Radar Imagery. As part of a large mapping project in Colombia, South America, it was necessary to perform a planimetric triangulation with *SLAR* imagery covering approximately 400,000 km² and 41 ground control points[10]. The task was split into an internal and external adjustment. First, *SLAR* strips were transformed into a common block system using piecewise third-order polynomials with continuous 1st derivative. The internally adjusted block of *SLAR* images was then transformed into the set of ground control points. Discrepancies in these points were used to compute corrections in radargrammetric points (external adjustment). The corrections were interpolated by global polynomials, pointwise linear prediction,

TABLE 3. RESULTS OF INTERPOLATIVE CORRECTION OF SLAR BLOCK DEFORMATIONS IN X AND Y, USING A NUMBER OF DIFFERENT METHODS. THE VALUES IN MM AT IMAGE SCALE, COMPUTED FROM 41 CONTROL- AND 610 TIEPOINTS.

	After 4-parameter fit of SLAR block into all control points	Linear prediction, with a 10% filter	Moving average, weights equal, order 1	Moving average, equal weights, order 3	Arithmetic mean, weight $1/d^2$	Regression polynomial 6 coefficients	Regression polynomial 10 coefficients
Residuals in control points							
RMSE X	3.93	0.55	3.69	0.42	0.50	1.42	0.83
RMSE Y	3.50	0.69	2.48	0.58	0.75	1.58	1.00
Corrections in non-data points							
RMSE X		3.58			3.34	5.76	4.66
RMSE Y		3.25			3.00	3.92	7.18

arithmetic mean, and moving averages of orders 1 and 3. The results are shown in Table 3 and are self-explanatory.

The lack of check points prohibited a reliable estimate of the accuracy. But interpolation in each control point, without using it in the computation, leads to an upper bound for the residual errors in non-data points *(RMSE X = ±2.15 mm; RMSE Y = ±2.14 mm)*. However, this upper bound is far from the actual accuracy, since control spacing is very large.

The data point distribution created the problem that in some areas extrapolation rather than interpolation was to be done. Figure 3 shows clearly how a third-order polynomial degenerates in areas of no control, while linear prediction produces corrections of the order of magnitude of the discrepancies in control points. This suggested that, for the given project, it had to be an interpolation method such as linear prediction, rather than global polynomials.

In conclusion, an interpolation which might turn into extrapolation requires great care in selecting the algorithm. If no check points are available as in *PRORADAM*, the danger of extrapolation remains hidden. To be safe, linear prediction can always be used in this case. It has the property of producing zero-corrections in areas of no control. Similar to this is the arithmetic mean, which, however, is often less accurate and without a quantitative filter control. A global function can only be used for interpolation with large areas of no control, if constraints can be imposed on the function pulling it towards zero in areas of no control.

Correction of Film Deformation with Reseau.[4,17,18] The importance of filtering the

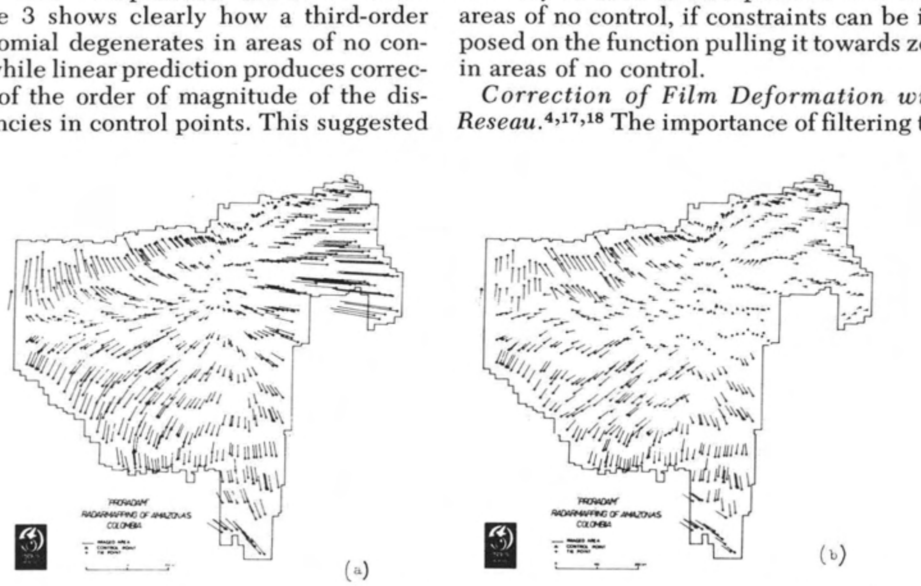

FIG. 3. Interpolated corrections for block deformations of *SLAR* imagery obtained (a) with a 10-parameter polynomial and (b) with linear prediction. Extrapolation occurred in the northeast and southwest of the area.

measuring error, or at least an uncorrelated component of the observations, is demonstrated by the results obtained with two methods of correcting film deformation with reseau photography. The observed coordinates of a reseau are compared with the theoretical coordinates. The differences are the sum of film deformation and the measuring error, and represent the data points for the interpolation of corrections in photogrammetric points.

One method of correcting film deformation in non-data points was based on the assumption that it is equal in both a non-data point and the nearest reseau point, which in all cases is at a distance smaller than 7 mm in reseau photography. To reduce the effect of systematic measuring errors, the nearest reseau point was always measured after a measurement at a photogrammetric (non-data) point.

The other method of film deformation correction was with linear prediction, using exactly the same observations as before. Table 4 shows the results of the experiments after relative and absolute orientation: the photogrammetric model coordinates are compared with accurate terrestrial points. The root mean square coordinate differences without film deformation correction, and with corrections using the two above methods, are shown in the table. It is demonstrated that linear prediction produces significantly smaller root mean square differences than the simpler method, which does not allow for any smoothing of observed data.

In Table 4, results after linear prediction as obtained by two different authors are shown. The fact that these results are somewhat different indicate that the performance might be considerably altered through the parameters used within an interpolation method.

In conclusion, this interpolation demonstrated that there can be differences between accuracies obtained in one or another algorithm, as opposed to a conclusion drawn from a previous experience.

Interpolation in Square Grid DTM. In a controlled numerical experiment to relate the accuracy of a Digital Terrain Model *(DTM)* with the density of sampling the terrain along a regular grid, and with the type of terrain[9], it was also possible to compare a number of interpolation algorithms. The methods were compared in their application to six different types of terrain, eight different sampling densities, and also with a variation of the number of data points to be used in an interpolation of a new value. Table 5 illustrates the overall results of the comparison of methods. The numbers in this table are obtained as root mean square values of the results of over a million interpolations. Each interpolated value was compared with the "true" terrain height. In order to allow comparison of interpolation methods applied to

TABLE 4. ROOT-MEAN-SQUARE DISCREPANCIES BETWEEN 2 PHOTOGRAMMETRIC STEREMODELS AND TERRESTRIAL CONTROL, IN μM AT PHOTOSCALE, WITH AND WITHOUT USE OF RESEAU= SCALE 1:10500, 23 CONTROL POINT NESTS WITH 3 POINTS EACH.

	Without reseau[17]		With reseau, no filtering[18]		With reseau, linear prediction[4]		With reseau, linear prediction[18]	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
RMSE X	7.8	7.2	8.8	8.0	6.2	6.8	4.6	6.7
RMSE Y	7.9	9.1	8.0	7.0	5.8	5.5	5.5	6.0
RMSE Z	15.8	11.8	12.2	11.8	11.8	10.1	9.8	10.9

TABLE 5. RELATIVE COMPARISON OF INTERPOLATION METHODS, APPLIED TO SQUARE GRID DTM. VALUES ARE IN % RELATIVE TO LINEAR INTERPOLATION, AND REPRESENT INTERPOLATION ERRORS IN CHECKPOINTS.

Method of Interpolation Weights	Linear Interpol.	Bi-linear polynomial	Weighted arithm. mean $1/d^4$	Moving Average e^{-4d^2}	Meshwise polynomial q/d^4	Linear prediction $(a+d^2/4)^{-1}$
Number of data points used per interpolation						
4	1.00	0.89	0.92			0.88
16			0.97	0.76	0.76	0.76
36			1.03	0.77		0.76

Table 6. Comparison of Variable Computing Time per Interpolation of a Point. Values are Seconds, Valid for the PDP 11/45 of ITC.

No. of points used per interpolation						
4	0.04	0.04	0.04			0.04
16			0.05	0.21	0.25	0.12
36			0.08	0.43		0.23

different terrain and different sampling densities, the interpolation error of any method was always divided by the interpolation error after linear interpolation. Consequently, linear interpolation produces an interpolation error of "100 per cent." In general, other methods produce smaller interpolation errors. The differences in performance amount to 24 per cent in the experiment.

It is beyond the scope of the present considerations to go into details of the study. Instead, these are given in reference 9, and only some conclusions relevant to the present topic are presented here.

Table 5 shows conclusively that linear interpolation produces larger errors than the more complex algorithms of linear prediction, moving averages or piecewise polynomials. On the other hand, it also indicates that there are two groups of methods performing differently: the "simple" methods (linear interpolation, bilinear polynomial, double linear interpolation, weighted arithmetic mean); and the "complex" methods. Within a group, differences are not distinct.

Table 6 then compares the efforts to be made using each of these methods. The splitting into two main groups also persists there: a slight reduction of the interpolation error can be obtained only with considerable increase of time of computation. It should be noted, however, that linear prediction must not be more expensive than linear interpolation, if it also is applied only with the four closest reference points, and not more. But in general it is also clear that the further benefit of reduced interpolation errors often might not be worth the extra effort. Only for specific purposes, it will have to be a "complex" method which is to be used, perhaps not for the interpolation error, but for other properties.

The efforts shown refer to data points on a regular square grid. It must be stressed that, for irregular distribution of data points, linear interpolation becomes as expensive as linear prediction: a point selection algorithm will demand considerable effort to define the three closest data points.

DTM interpolation does not necessarily require filtering, if it can be assumed that measuring errors are comparatively small. A possibility to filter might, however, be desirable in applications where measuring errors are significant, or where smoothing is essential (generalization).

An important problem in photogrammetric work with *DTMs* is the consideration of terrain break lines. Typically this is a matter of imposing constraints on the interpolated surface. A number of solutions exist to this problem. They have in common that a pointwise interpolation algorithm be used*. Successful attempts to consider terrain irregularities in piecewise interpolation have not yet come to the attention of the author. An unsuccessful effort was mentioned by Jancaitis and Junkins.†

Conclusions

The number and importance of photogrammetric tasks which make use of interpolation and filtering suggest that the theory and methods of interpolation can be an eminent tool for the photogrammetrist although approaches to interpolation often are intuitive and not systematic. Fortunately, however, there has been a new stimulus to study the problems of interpolation and filtering, namely the Digital Terrain Model *(DTM)*. The *DTM* itself does not pose the largest problems, nor does it require the most advanced theories of interpolation. These may rather be useful in problems of data analysis, thus filtering applied to strip and block adjustment, for example.

Although some applications may require very specific solutions, experience has shown that a majority of problems can be

* For example: Assmus, E., "Extension of Stuttgart Contour Program to treating Terrain Break Lines," IPS-Comm. III Symp., Stuttgart, W. Germany, 1974.

† Jancaitis, J. E. and Junkins, J. L., Personal communication.

treated with the same set of algorithms. An important element of these is the possibility of computing correlation functions. These may show quantitatively whether the data contain a correlated component. This is rather important for data analysis and helps to avoid interpolation and filtering being attempted in completely random noise. Further, the set of algorithms should contain a component to compute regression polynomials. This is required for simple smoothing problems (trend analysis), for mathematical representation, and to preprocess data for linear prediction. This latter method, then, should be available as a general purpose interpolation and smoothing algorithm. Although it is a computer-intensive method (expensive) it is useful because it allows for well-controlled smoothing and does not degenerate in cases of extrapolation. As a last algorithm it is recommended to have available piecewise polynomials (spline functions). If a more flexible mathematical representation of phenomenon is required, these might be of greater use than a global polynomial.

With this set of algorithms, a number of photogrammetric interpolation tasks have successfully been carried out at *ITC*. Some of these tasks have been described in this paper to justify conclusions on a comparison of interpolation methods. Interpolation errors have been shown to vary up to 24 per cent using different methods of interpolation. But an attempt was made to evaluate methods or interpolation not only on the basis of the interpolation errors but also according to criteria such as smoothing power, usefulness for extrapolation, etc. It is this spectrum of properties which should be used to decide on the choice of a particular method of interpolation, rather than the traditional considerations concerning only interpolation errors and efforts.

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Appendix — Interpolation Methods

The following description of interpolation methods assumes a one dimensional phenomenon on a two dimensional reference space.

Linear Interpolation

The three closest data points with observed values Z_1, Z_2, Z_3 are used to define a new value Z_p by fitting a plane surface, so that

$$Z_p = a_0 + a_1 x + a_2 y \qquad (1)$$

Coefficients a_0, a_1, a_2 can be solved from the three data points.

Coordinates (x_i, y_i) give the location of a point i in the reference space.

Double Linear Interpolation

The four closest data points forming a quadrangle are selected. The four points define two triangles, each of which contains the new point. Two linear interpolations can be carried out, one linear interpolation in each triangle. The arithematic mean produces Z_p as:

$$Z_p = (Z'_p + Z''_p)/2$$

where Z'_p, Z''_p are produced according to (1).

BILINEAR POLYNOMIAL

The four closest data points define a quadrangle. This allows computation of Z_p using the bilinear polynomial

$$Z_p = a_0 + a_1x + a_2y + a_3xy$$

ARITHMETIC MEAN

From n data points, a new value Z_p is found from:

$$Z_p = \sum_{i=1}^{n} (x_i/d_i^k) / \sum_{i=1}^{n} 1/d_i^k$$

Here, $d_i^2 = (x_i - x_p)^2 + (y_i - y_p)^2$, and k is selected according to the intuition of the user. Weight $1/d_i^k$ can also be replaced by other functions.

MOVING AVERAGE

For each new data point, the n surrounding reference points are selected. The new point is chosen as the origin of planimetric coordinates. The absolute term of a polynomial of order m is computed from the n data points, giving each of them a different weight, e.g., according to distance from the new point. The computed absolute term is the interpolated Z_p, since $x_p = y_p = 0$. The described process is a weighted moving average of order m, using n points.

LINEAR PREDICTION

A (polynomial) regression function (trend $t(x,y)$) is computed from n data points. The residuals can be input only to linear prediction. From the residuals, a correlation function $cov(d,\alpha)$ is computed, or chosen *a priori*. The correlation function $cov(d,\alpha)$ describes the dependency of two residuals being a distance d apart and defining the direction α. Usually, dependency on α is not assumed, so that one uses $cov(d)$ only. A correlation matrix Cov is defined:

$$\underline{Cov} = \begin{bmatrix} 1 & cov(d_{1,2}) \ldots & cov(d_{1,n}) \\ cov(d_{2,1}) & 1 \quad \ldots & cov(d_{2,n}) \\ \cdots & \cdots & \cdots \\ cov(d_{n,1}) & cov(d_{n,2}) & 1 \end{bmatrix}$$

Distances d_{ij} are between data points i and j. Also a correlation vector $\underline{cov}$ is defined between the new and data points:

$$\underline{cov} = (cov(d_{1,p}), cov(d_{2,p}), \ldots, cov(d_{n,p}))$$

The new point obtains:

$$Z_p = t(x_p,y_p) + \underline{cov} * \underline{Cov}^{-1} \cdot \underline{\Delta Z}^t \,;\; \underline{\Delta Z} = (\Delta Z_1, \Delta Z_2 \ldots)$$

Careless application of linear prediction can be detrimental. Therefore, in depth study of literature (e.g.,[5,11]) should precede actual use of the method.

PATCHWISE POLYNOMIALS [SPLINE FUNCTIONS]

There are many ways to compute patchwise polynomials. Within the interpolation area, a not necessarily regular grid is chosen. Within each mesh of the grid, a different polynomial is defined. If the polynomials of order n join along the boundaries of adjacent meshes with all derivatives up to order $n-1$ being continuous, one speaks of "spline functions."

Continuity can be obtained by interpolating values and eventually also tangents of the phenomenon in the grid points. This represents a simple, fast, memory saving process and can be done, e.g., with a moving average. Next, the generated function values (and tangents) are used to define the polynomial in each mesh. Making an appropriate choice of the polynomials, and having sufficient values at the grid points, the generated polynomials will have continuous $(n-1)$ st, . . . 2nd, 1st, zero derivative.

The method evaluated in the section on Interpolation in square grid *DTM* applied 3rd order polynomials with 12 coefficients. In each grid point one function value and two tangents (t_x, t_y) were computed. These were per mesh the 12 given data points to define the 12 coefficients of the polynomial piece.