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PH. HOTTIER
Institut Geographique National
Paris, France

Accuracy of Close-Range Analytical Restitutions: Practical Experiments and Prediction

The distinction between precision and accuracy; the prediction of accuracy; and the effects of measurement redundance, geometry, and the use of non-metric cameras are discussed.

INTRODUCTION

CLOSE-RANGE PHOTOGRAMMETRY does not consider large object distance such as in aerial photogrammetry, which involves the problems of refraction, but only the range 0 to 200 meters.

This report is limited to analytical restitutions, because analog restitutions are now being employed to a lesser degree in precision civil engineering applications due to the lower (likely 3 to 5 times) accuracy and also to the strict geometry conditions (only the normal case can be used on the modern analog stereoplotters). Nevertheless, in my opinion it also would be interesting to conduct similar studies for analog or semi-analytical restitutions.

One may wonder why so general a problem has not been exhaustively treated in the past. The principal reason was probably the lack of civil engineering applications, the development of which is recent. There was no necessity to compare photogrammetric accuracy with the accuracy of other metric techniques, and to produce in this particular area efficient criteria and predictors of accuracy. Another reason is the multiplicity of parameters (geometry, physical characteristics of the photogrammetric system, redundance of the measurements) to be considered, and the necessity to conduct numerous and costly experiments.

ACCURACY AND PRECISION

The two concepts (accuracy and precision) should be carefully distinguished.* Let $\dot{X}$ be the true value of some physical quantity, and $\hat{X}$ an estimation of $\dot{X}$ based upon a particular measuring system S ($\hat{X}$ included in S). One always states a difference between the expectation of $E_s\hat{X}$ of $\hat{X}$ within the measuring system and $\dot{X}$:

$$E_s\,\hat{X} \neq \dot{X}$$

Then we define the error of $\hat{X}$ as follows:

$$e = \hat{X} - \dot{X} = (\hat{X} - E_s\,\hat{X}) + (E_s\,\hat{X} - \dot{X}) = \epsilon + \beta$$

where

$$\epsilon = \hat{X} - E_s\,\hat{X};\ E\epsilon = 0$$
$$\beta = E_s\,\hat{X} - X. \qquad (1)$$

* ISP Commission VI 1964 Glossary of some terms and expressions in the theory of errors.

ABSTRACT: *Some essential points relating to precision and accuracy (distinction between precision and accuracy, correct evaluation of accuracy) are reviewed; then, based on experimental results from several sources, a quantitative study of the accuracy of analytical restitutions in the case of the photo-pair is presented. The principal themes are the following:*

The effect of measurement redundance upon the accuracy (repetition of the settings, use of several neighboring targets to define an object point, and use of several frames at each station can on an average increase the accuracy by 50 per cent, whatever be the base-to-mean-object-distance ratio, the maximal accuracy for a certain kind of photogrammetric system has been found to correspond to a measurement equivalent normal law with a 1.2 μm standard deviation (which is the RMS bias of the measurements) and the minimum accuracy to a normal law with a 2.5 μm standard deviation);

The effect of the geometrical characteristics of the system (base-to-object-distance ratio, camera axis convergence, and number and disposition of the control points);

Accuracy prediction (two predictors are presented; the Karara/Abdel Aziz predictor, which reduces the problem to the central point of the object volume, and a predictor obtained from simulation. These predictors are correct on the condition that a good estimation of the standard deviation of the equivalent normal law of the comparator observation of the system (camera plus comparator) is employed; and

Non-metric camera accuracy (for the best of them, it seems to be the same as for metric cameras).

RÉSUMÉ: *Après avoir rappelé un certain nombre de points relatifs à la précision et à l'exactitude (distinction entre précision et exactitude, évaluation corrçcte de l'exactitude), on présente, basée sur des résultats expérimentaux provenant de plusieurs sources, une étude chiffrée assez complète de l'exactitude des restitutions analytiques dans le cas de couple (stéréogramme); les principaux thèmes sont les suivants:*

L'effet de la redondance des mesures sur l'exactitude: répétition des pointés au comparateur, emploi de plusieurs cibles voisines pour définir un point, prise de plusieurs clichés à chaque station, peuvent accroître en moyenne l'exactitude de 50%, quelque soit le rapport base sur éloignement; l'exactitude maximale, dans le cas d'un système assez répandu (chambres métrique + comparateur Zeiss), correspond à une loi équivalente de mesures-comparateur d'écart-type 1.2 μm qui est le biais moyen quadratique des mesures); l'exactitude minimale à une loi d'écart-type 2.5 μm.

L'influence des caractéristiques géométriques du système: rapport base/éloignement; nombre et disposition des points d'appui: la multiplication des points d'appui: peut accroître l'exactitude de 20 à 30%.

La prédiction de l'exactitude; deux prédicteurs sont étudiés: celui de Karara-Abdel Aziz qui assimile l'exactitude obtenue dans le volume objet à celle du point central, et celui obtenu par simulation; ces prédicteurs sont corrects à condition qu'on possède une bonne estimation de l'écarttype de la loi normale équivalente des observations du système (chambre + comparateur).

Exactitude des chambres non métriques: pour les meilleures, elle est très voisine de celle des chambres métriques.

The quantity ϵ is the residual random error of the measuring system S. As a rule the distribution of the random error ϵ is normal:

$$\epsilon \in \mathcal{N}(0, \sigma_{\hat{x}}).$$

The precision of $\hat{X}$, within the measuring system S, indicates the closeness of $\hat{X}$ to $E_s \hat{X}$ and is characterized by the standard deviation of ϵ, $\sigma_{\hat{x}}$. It is a well known experimental fact that, for the standard deviation $\sigma_{\bar{x}}$ of the arithmetic mean $\bar{X}$ computed from a large number of measurements in a stable measuring system, we have

$$\sigma_{\hat{x}} \simeq 0.$$

The quantity β is the bias (rather than the systematic error, the meaning of which is too restrictive). The principal reasons for the bias are systematic effects, lack of definition of the measured quantity, and resolving power of the measuring procedure.

The accuracy of the estimation $\hat{X}$ indicates the closeness of $\hat{X}$ to $\dot{X}$. It is characterized by RMS error $\sqrt{E_s\, e^2}$, with

$$E_s\, e_{\hat{x}}{}^2 = E\,(\epsilon + \beta)^2 = E\epsilon^2 + \beta E\epsilon + \beta^2 = \sigma_{\hat{x}}{}^2 + \beta^2.$$

If we consider all the measuring procedures which give approximately the same precision (for example all the comparators with the same trade-work), we have

$$Ee_{\hat{x}}{}^2 = \sigma_{\hat{x}}{}^2 + E\,\beta^2. \tag{2}$$

The quantity $\sqrt{E\beta^2}$ may be called the RMS bias.

From Equation 2 it can be seen that the RMS error is statistically superior to the RMS bias,* which characterized the average maximum accuracy of all measuring procedures with the same features, i.e.,

$$\text{RMS error} \geqslant \text{RMS bias} \tag{3}$$

It is important to notice that, in any case, even though the systematic effects are well corrected, there is in the RMS bias some irreducible part due to the lack of definition of the measured quantity and to the resolving power of the measuring procedure. It can be observed otherwise that for many precise equipments the RMS bias is often approximately equal to the standard deviation σ of an elementary measurement of $\dot{X}$ (estimated from the repetition of measurements under the same conditions with the same operator. For example, for comparators and theodolites, for the arithmetic mean of n elementary measurements we obtain the RMS error

$$\sqrt{1 + \frac{1}{n}}\ \sigma > \sigma).$$

Finally, we can say that if we consider a one-dimensioned physical quantity whose true value is $\dot{X}$, the estimator $\hat{X}$ of $\dot{X}$ from a particular type of measuring procedure is a normal variable

$$\hat{X} \in \mathcal{N}(\dot{X}, \sqrt{\sigma_{\hat{x}}{}^2 + E\,\beta^2}) \tag{4}$$

where $\sigma_{\hat{x}}$ is the precision of $\hat{X}$ and $\sqrt{E\,\beta^2}$ is the RMS bias.

As a matter of course, Equation 4 is only a mathematical model. In particular, the difficulty is to assign a meaning to the symbol $\dot{X}$, the "true value" of the physical quantity (what are the "true values" of the coordinates of a geodetic signal, of a physical object point, of an image point?). The best way is probably to consider $\dot{X}$ as the mean of estimations coming from a great number of "optimal" measuring procedures. Then other problems occur: the true value depends on the chosen type of measuring procedure. For example, it can be thought that the center of an irregular, or even regular, spot is appreciated in different ways when using microscopic or long distance procedures, and in analytical aerotriangulation one cannot be sure that the true geodetic definition of an object point is identical to the photogrammetric one, particularly when there are no targets.

Anyway, there is a gap between the symbol $\dot{X}$ and the reality owing to the lack of definition of the physical quantity and the resolving power of the measuring procedure. It is of no physical interest, and perhaps impossible, perhaps a statistical nonsense, to want to attain $\dot{X}$. What is of interest is to eliminate random error and systematic effects. In other words, we can say that the maximum accuracy (systematic effects and random errors eliminated) is characterized by the RMS bias, and that all estimations of maximum accuracy are equivalent to describing the true value of the quantity.

* In French, the RMS bias characterizes what is called *justesse* (accuracy = *exactitude*, precision = *precision*).

Evaluation of Accuracy

Two methods are at our disposal: We can evaluate accuracy by using check measurements and computing from these check measurements the value of adequate accuracy criteria; and we can use accuracy predictors, on the condition that they exist and that their reliability has been proved. In photogrammetry, where the accuracy problems are very complex, the two methods should in my opinion be employed together, and the values supplied from the predictor always checked with a minimum of check measurements.

Now, the methodologic base is, of course, the check measurements. Thus, I desire to insist a little upon this problem bearing in mind the photogrammetric area. The principle is to compare the results obtained from one particular measuring procedure with the results from a "more accurate" measuring procedure. Of course, it is implied that the definitions of the measured quantities are the same for the two measuring procedures, which is generally the case for precise works of analytical photogrammetry where the measured object is equipped with standard targets.

One of the most frequently used among check measuring procedures is triangulation or micro-triangulation. It is generally possible with these methods to determinate a net the errors of which are negligible in comparison to the photogrammetric errors. For example, at the Institut Geographique National (IGN) in Paris a careful micro-triangulation for a three-dimensioned vertical test field (10 m by 12 m by 2 m) has been characterized by the following RMS errors:

30 μm for the x axis (parallel to the test field and horizontal),
60 μm for the y axis (horizontal and perpendicular to the test field), and
70 μm for the z axis (vertical).

Such an accuracy is enough so as to appreciate spatial photogrammetric errors of magnitude greater than 0.3 to 0.4 mm. I indicate here a very interesting triangulation method*: Recording from two (or more) stations the perspective bundles with a theodolite used as a camera (no necessity of spirit levels adjustment, as reference to the physical vertical is not useful for determination of relative positions of a set of points) associated with the well known computational method of relative orientation (scaling from a calibrated tape with targets) (Figure 1). It has been shown from experiments that this method is at least as and probably more accurate than conventional micro-triangulation for distances between 4 and 8 meters. Its essential advantage is that it is considerably more simple, rapid, and economic.

Unhappily it is not always possible to emplace an accurate reference test field, particularly for very close ranges (<2.5 m)[10]. However, there is the possibility of considering the points in the test field in the photogrammetric adjustment as observed quantities rather than fixed; but appreciation of the accuracy of the photogrammetric procedure alone is then delicate.

Criteria of Accuracy

The problem is to estimate the accuracy in a given object volume with convenient criteria (Figure 2). It is desirable that such criteria have the following properties: Simplicity (complex criteria indeed could be developed, but would not be practical for daily use by engineers); reliability for the whole volume; and, if possible, independent of the object

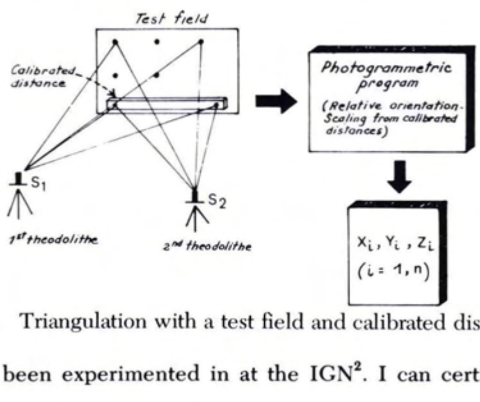

Fig. 1. Triangulation with a test field and calibrated distance.

* This method has also been experimented in at the IGN[2]. I can certify its excellence and convenience.

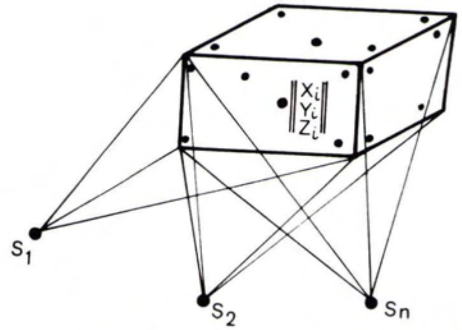

FIG. 2. Estimating accuracy for an object volume.

space (stations, orientation of the optical axis, object volume) in order to be able to compare results obtained from different working scales and from all available focal lengths.

There is a solution in the symmetrical case of a pair (at least for identical left and right focal lengths). I do not know if it can be extended to other geometries.

THE RMS SPATIAL RESIDUALS

We consider n check points in the studied volume, that is, points whose true coordinates are known but not used in the photogrammetric computations.

Then, if X_{iT}, Y_{iT}, and Z_{iT} are the true coordinates of the check point M_i $(i = 1, M)$, and X_{iPH}, Y_{iPH}, and Z_{iPH} its photogrammetric coordinates, an estimation of the RMS spatial residual is

$$RXYZ = \sqrt{\frac{1}{n} \sum_{1}^{n} \left[(X_{iPH} - X_{iT})^2 + (Y_{iPH} - Y_{iT})^2 + (Z_{iPH} - Z_{iT})^2\right]} \qquad (5)$$

(the true RMS residual $\dot{R}XYZ$ is in fact the limit of $RXYZ$ for n infinite, with points in every part of the volume).

It is interesting to determine the maximum spatial residual among the n check points:

$$RMXYZ = \text{Max} \sqrt{(X_{iPH} - X_{iT})^2 + (Y_{iPH} - Y_{iT})^2 + (Z_{iPH} - Z_{iT})^2}. \qquad (6)$$

Of course, if necessary, analogous quantities can be estimated for the three axes. For example, in the X-direction.

$$RX = \sqrt{\frac{1}{n} \sum (X_{iPH} - X_{iT})^2} \text{ and } RMX = \max |X_{iPH} - X_{iT}|. \qquad (7)$$

In addition to its simplicity, such a criterion can be correctly estimated provided that:

(a) The number n of check points is sufficient. It is presumably rather complex, and perhaps impossible, to give for the quantity $\dot{R}XYZ$ the exact confidence limits on a given per cent level.

Then we proceed in the simplistic following way: We assume the variables $(X_{iPH} - X_{iT})$, $(Y_{iPH} - Y_{iT})$, and $(Z_{iPH} - Z_{iT})$ are normal and independent, and have the same standard deviation σ. Under these conditions, and if we call s an estimation of σ, we have

$$s^2 = \frac{1}{3n} \sum_{1} \left[(X_{iPH} - X_{iT})^2 + (Y_{iPH} - Y_{iT})^2 + (Z_{iPH} - Z_{iT})^2\right] = \frac{1}{3} R^2 XYZ.$$

Hence the confidence limits are the same for σ and $\dot{R}XYZ$. They are given by well known statistical tables, e.g., on the five per cent level we find

n	Confidence limits of $\dot{R}XYZ$	
3	0.67 $RXYZ$	1.92 $RXYZ$
6	0.75 $RXYZ$	1.50 $RXYZ$
8	0.78 $RXYZ$	1.40 $RXYZ$
10	0.80 $RXYZ$	1.34 $RXYZ$
15	0.83 $RXYZ$	1.24 $RXYZ$
25	0.86 $RXYZ$	1.20 $RXYZ$

It is seen that, even for a small number of check points, the estimation of the RMS residual is not so bad, i.e., it is satisfactory when $n > 15$.

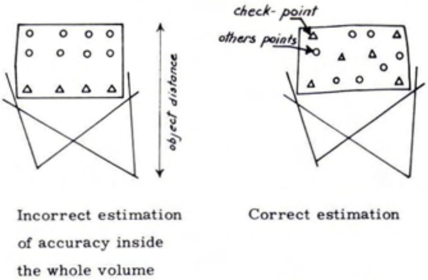

FIG. 3. Distribution of check points.

(b) The distribution of the n points inside the volume is regular (Figure 3). In particular, any extrapolation about accuracy outside the check point volume is not valid.

(c) The volume is not too deep. If the volume is too deep, then evaluation accuracy has to be estimated for successive slices.

At last, we demonstrate the fundamental announced property of the RMS spatial residuals: An RMS spatial residual is proportional to the RMS error of the comparator measurements.

It can be shown in the following way: for each check-object point M_i

$$\text{(true coordinates } X_{iT} = \begin{Vmatrix} X_{1iT} \\ X_{2iT} \\ X_{3iT} \end{Vmatrix} ;$$

$$\text{computed coordinates: } X_{iPH} = \begin{Vmatrix} X_{1iPH} \\ X_{2iPH} \\ X_{3iPH} \end{Vmatrix}),$$

it is obvious that $X_{iPH} = X_{iT}$ plus a linear function of the measurement errors (measurement errors for the image points relating to M_i, but also for the image points of any control point used in the computation).

Then, if we name R_{Mi} the RMS spatial residual at the point M_i, we have (all the measurements are assumed to be independent)

$$R^2_{Mi} = E\,(X_{iPH} - X_{iT})^T\,(X_{iPH} - X_{iT}) = q^2_{Mi}\,\sigma^2$$

and for the RMS spatial residual RXYZ worked out with all the n check points

$$RXYZ = \sqrt{\frac{1}{n} \quad \sum_{1} \quad R^2_M}$$

$$RXYZ = q\,\sigma \tag{8}$$

with q depending only on the object volume.

ESTIMATION OF THE RMS SPATIAL RESIDUALS FROM CONTROL POINTS

The previous criteria can be used without difficulty in laboratory experiments. However, it is more difficult in the field where it is often costly to provide enough control points for good determinations and enough check points for good estimation of the obtained accuracy.

However, for lack of check points or check measurements one may use the RMS residuals computed with control points. We will name them $R'XYZ$, $R'X$, etc. Unfortunately, if the number n of control points is below 25 to 30, there is (statistically) a sensible overestimation of accuracy (See Appendix A), that is,

$$R'XYZ < RXYZ.$$

Nevertheless, it seems possible and reliable, at least in the case of the pair, to compute a corrective coefficient K so as to have

$$RXYZ = K\,R'XYZ \text{ (statistically)} \tag{9}$$

K depending on the number n of control points, the computational method, and the number r of unknowns estimated in the least-square adjustment. If from each control point we obtain p observation equations, we compute K from

$$K = \sqrt{\frac{pn}{pn - r}} \tag{10}$$

For example, for the method of the resections in space with direct linear transformation, followed by the intersecton of homologous rays, we have $r = 11$, $p = 2$, and

$$K = \sqrt{\frac{2n}{2n - 11}} .$$

Verification of validity is given in the table below (See Appendix A for more details)

n	$\overline{\left(\frac{RXYZ}{R'XYZ}\right)}$	$\sqrt{\frac{2n}{2n-11}}$
7	1.97	2.16
10	1.46	1.49
15	1.26	1.26
28	1.18	1.12

($\overline{(RXYZ/R'XYZ)}$ is the mean of elementary ratios $RXYZ/R'XYZ$ corresponding to four different couples at least.)

The reader will find in Appendix A justification of the method on the basis of assumptions probably well verified in the usual practice.

CASE OF THE STEREOPAIR

Let B be the base, O the mean object distance, and p the common value of left and right focal lengths (Figure 4). It is obvious that, on a first approximation, $RZYX$, the RMS residual, is for a particular geometry in inverse ratio to p (of course, we suppose all the other parameters of the system, that is, objective, emulsions, comparator, etc., to be invariant); and $RXYZ$, for a given ratio B/O (and the same axis orientations), is proportional to O, on the condition that the ratio D/O (where D is the depth of the volume) is constant.

Subsequently, for $r = B/O$ fixed, we have

$$rXYZ = RXYZ\,\frac{P}{O} = \text{constant} \tag{11}$$

where $rXYZ$ is the normalized RMS residual. It is obtained from the RMS residual by multiplication with the mean scale of the pictures, and is expressed in micrometers.

It can be thought that such an expression depends exclusively on the ratio B/O (and perhaps on the camera axis orientations) when using a certain type of camera, comparator, and computational method. We will find it true from practical experiments.

Some authors use another kind of normalized criterion which in my opinion is imperfect. If D is the greatest dimension of the measured object, they consider

$$r'\,XYZ = \frac{RXYZ}{D}$$

and name it "relative accuracy". Such a criterion is of course convenient for the customer, but it does not give any precise indication regarding the photogrammetric accuracy. For example, let us consider $D = 10$ m, $O = 7$ m, and fixed positions for the two points of view. The work can be done with many available focal lengths, and thus we can obtain

focal length P	RMS residual $RXYZ$	"Relative accuracy" $RXYZ/D$
50 mm	0.56 mm	1/ 20000
100 mm	0.28 mm	1/ 40000
200 mm	0.14 mm	1/ 80000
300 mm	0.09 mm	1/120000

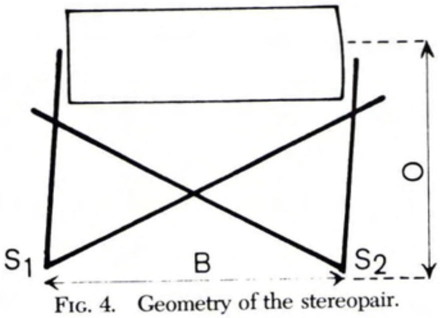

FIG. 4. Geometry of the stereopair.

These possible relative accuracies are very different, but in this case all of them, if focal length value is taken into account, point out identical quality of results (they correspond with the normalized criterion (Equation 11) of $rXYZ = 4\ \mu m$).

CRITERION OF THE PLATE RMS RESIDUAL

In the case of a simple resection in space, there is no possibility to use the criteria of the RMS spatial residual. If one uses the rigorous solution, one can adopt instead the standard error of unit weight, σ_o, and more precisely the estimation s_o of σ_o. With n control points and r unknowns we have:

$$s^2_o = \frac{1}{2n - r} \sum_{1}^{2n} v^2$$

(each point gives two observation equations . s_o is of course expressed in micrometers).

This criterion could be extended to the case of the stereopair and to the case of multi-station geometry; but it is easy to understand that it couldn't be completely satisfactory. In particular, the obtained accuracy cannot be deduced from the values of the RMS plate residuals for two reasons: One can have more than two rays for one object point and there is no evident relation between the RMS residuals of the different plates and a possible increase of accuracy due to the use of more than two rays; and the RMS plate residual can include slight systematic effects such as residual distortion for example, although these slight systematic effects often have little influence upon the final determinations. Then the obtained accuracy is better than one would expect from the examination of the s_o values.

Principal Parameters Influencing Accuracy

The principal parameters influencing accuracy may be classified into three groups.

GEOMETRICAL CHARACTERISTICS AND COMPUTATIONAL METHODS OF THE PHOTOGRAMMETRIC SYSTEM

It is obvious that accuracy depends on the camera focal lengths, positions and number of stations, density of control net or control measurements, and probably, but I ignore to what extent, on the choice of the computational method.*

As to geometry, focal length apart, one can consider two cases.

Double station geometry. Probably the most frequently used geometry as a rule is well characterized by the ratio $r = B/O$ (B = base; O = mean object distance). Needless to say, strong values of r imply as a matter of course a convergence of the two camera axes.

The choice of the computational procedure[3] depends on the presence or absence of a control net. If there is a control net, it is possible to have recourse to resections in space of each bundle followed by intersections of homologous rays. At least 10 to 15 control points are necessary to obtain correct accuracy.

If there is only a control distance for scaling, one must use the relative orientation procedure followed by scaling to the control distance. The total procedure is statistically less accurate than than the previous one. In particular, model distortions could occur due to a failure of the photographic plate to lie firmly against the camera frame.

Multi-station geometry. As far as I know, only one organization[8] currently uses multi-station geometry with no control net but only some scaling distances. The computational procedure seems to be a generalization of the one used for the couple (relative orientation plus scaling).

PHYSICAL CHARACTERISTICS OF THE PHOTOGRAMMETRIC SYSTEM

Among the principal physical characteristics of the photogrammetric system are

- the quality of the camera objectives and the lens distortion;
- the plate flatness (support and emulsion flatness);
- the definition of the object, e.g., natural details or targets (here I mention another possibility,

* For an equal number of degrees of freedom, one can resort to several approximate applications of the least-square method (although, theoretically, there is only one rigorous least-square solution, it is complex to program).

photogrammetry by the means of homologous lines. Note that a natural linear or curved line is generally better defined than successive points belonging to it.); and
- the accuracy of comparator* measurements.

REDUNDANCE OF MEASUREMENTS

The redundance of measurements can take place in three ways: number of comparator measurements for each image point of each picture; number of frames at each station,** and number of stations; and number of targets for each object point (an object point is defined as the barycenter of n neighbouring targets).

Prediction of Accuracy

THE METHOD OF ACCURACY PREDICTORS

Accuracy predictors are formulas or diagrams which give accuracy as a function of the principal parameters of a photogrammetric system. The aim of accuracy studies is precisely to provide simple and reliable accuracy predictors. Accuracy predictors can exist only for simple configurations of the data acquisition system (for example the symmetric case of the pair) which happily are the most frequent in practice.

The method of mathematical models, though more complex, presents on the other hand more possibilities.

THE METHOD OF MATHEMATICAL MODELS

The method of mathematical models consists in building fictitious data, that is, the image coordinates of each object point, and then in simulating the image coordinate error for each image point. After computation, photogrammetric spatial coordinates are compared to the original "true" coordinates, and the accuracy is estimated and compared with the accuracy estimated from practical experiments. If the mathematical model is correct, it can be used to predict accuracy in any particular case.

The essential part of the mathematical model is of course the simulation of the image coordinate error. This can be done in two ways.

Analytic formulation of each kind of error. This is the case for different mathematical models considered in aerial photogrammetry. For example Meier,[9] considered the coordinate errors in the left and right hand photos as a function of film deformation ds_1, irregularities in the emulsion and the film support dr_1, as well as the optical errors dr_2, with the formulation

$$ds_1 = F_0 + F_1 s$$
$$dr_1 = (U_0 + U_1 s)\ \frac{r}{p}$$
$$dr_2 = k_1 + k_2 \left(\frac{r^2}{p}\right)$$

where s negative side, p principal distance, and r radial distance.

To these systematic errors, it would be convenient to include the setting measurement error, but Meier[9] didn't (it should be noted that it was the purpose in Meier[9] to simulate analog plotting with the Planimat).

Synthetic formulation. It is assumed as a first (and often, sufficient) approximation, that the distribution of the errors is normal $\mathcal{N}(0, \sigma)$, as are those for the image coordinates x and y which are otherwise considered independent.

The second model is of course simpler than the first one. We will see that it seems very satisfactory for the case of a pair in analytic close-range photogrammetry; therefore, we need no longer consider the first model.

Now, if the existence of a mathematical normal model can be proved for a given topology (that is for focal lengths, number of stations, and number of image measurements fixed), it is possible to use it for the following aims:

(a) Determination of the optimal geometry for a given topology, that is, determination of the position of stations which provide the best accuracy. It should be noted that here the choice of the standard deviation σ of the normal law is of no matter. As a matter of fact,

* We suppose all the measurements (x and y) to have the same accuracy σ.

** We suppose that only one camera is used at each station.

the RMS spatial residual $RXYZ$ is proportional to the standard deviation σ of the measurements, that is, $RXYZ = q\sigma$ and, if we consider two different geometries 1 and 2 and attempt to estimate the accuracy gain g (reference geometry 1), we have

$$g = \frac{R_1XYZ - R_2XYZ}{R_1XYZ} = 1 - \frac{q_1}{q_2}$$

where g is independent of σ.

(b) Determination of an optimum computational method for a given geometry. Again, there is no need to know the value of the standard error.

(c) Prediction of the absolute accuracy of one particular geometry for a given topology (fixed number of stations), fixed physical characteristics (emulsion, objective quality, and comparator accuracy), and fixed number of image measurements. But this is only possible if we have a good estimation of the standard deviation σ, an estimation which should be derived only from practical experiments.

As I stated earlier, the normal model is satisfactory for the simpler photogrammetric systems. I don't know if this is true for n-station geometries, but it seems likely enough.

The Stereopair (Symmetrical Case)

Introduction

In the symmetrical case, the two camera axes make the same angle with the base. Of course, this has to be understood in a comprehensive way: practically, the angles should not differ by much more than 10 grads. Even in this simple case, it is difficult to find in the literature practical accuracy studies with some statistical value. Indeed a lot of time, instrumentation, and money are necessary to perform such studies.

Subsequently I chiefly use two data sources:

(a) Accuracy studies performed by the IGN[5,6].

The IGN studies employed the following photogrammetric system:

- metric cameras, with Gevapan 33 emulsion;
- symmetrical case;
- Zeiss Asco-Record comparator; and a
- 10 m by 12 m by 2 m (width, height, depth) test-field with the following accuracies for microgeodesic target determinations:
  30 μm for the x-axis (parallel to the base),
  60 μm for the y-axis (horizontal, perpendicular to the base), and
  70 μm for the z-axis (vertical).

(2) studies performed by the Karara/Abdel Aziz team.[1,2]

The questions which will be examined here more or less exhaustively are the following:

- The effect of measurement redundance (evolution of accuracy with repetition of comparator measurements per image point, with the multiplication of neighboring targets defining an object point, and with the multiplication of frames per station; maximum accuracy; independence of the parameter measurement redundance and of the parameter geometry; and evaluation of the RMS bias of the photogrammetric system);
- The effect of the geometry (ratio of the base to the mean object distance, and camera axis convergence; and evolution of accuracy with the number of control points if a control net is used in the system);
- The prediction of accuracy (simulation accuracy predictor, and Karara/Abdel Aziz predictor, comparison and validity); and
- The use of non-metric cameras.

I won't study the effect of computational methods on accuracy.

The most important results (IGN),[5,6] relating to the first two questions are summarized in Tables 1 and 2 which record gains in accuracy when varying different parameters. The accuracy criterion is always the RMS spatial residual $rXYZ$, referred to the image plane and expressed in micrometers. The reference case is always for a given geometry and camera, the one most economically and quickly applied (only one right and left setting per object point).

If $rXYZ$ and $rMXYZ$ are the RMS and maximal spatial residuals for the reference case, and $r'XYZ$ and $r'MXYZ$ are the corresponding quantities for the studied case, the accuracy gain g is defined as

$$g = \frac{rXYZ - r'XYZ}{rXYZ} \times 100 = 100 \left(1 - \frac{r'XYZ}{rXYZ}\right)$$

$$g = 100 \left(1 - \frac{r'MXYZ}{rMXYZ}\right) \qquad (12)$$

EFFECT OF MEASUREMENT REDUNDANCE

Principal results. The principal results of the effect of measurement redundance appear in Tables 1 and 2. It should be pointed out that a considerable amount of data were employed. On the whole, 60 plates have been studied, and 80,000 individual settings executed with the comparator.

The target-centers of the test field (10 m by 12 m by 2 m) were determined from a conventional micro-triangulation (accuracy as noted earlier); two metric cameras were employed (TMK with 60 mm focal length, and UMK with 100 mm focal length); and for each camera three different geometries were used (the three base-to-object-distance ratios are 0.86, 0.33, and 0.14, with a convergence of the two camera axes of 40 grads for the first ratio and zero for the other two). For each geometry and camera, five stereopairs were acquired without moving the camera (five frames per station). Only one emulsion was used (Gevapan 30, 125 ASA).

The measurements were performed with a Zeiss Asco-Record monocomparator.

For each measurement combination (N_1 settings per image point; N_2 targets per object point; and N_3 frames per station. On the whole N_1 by N_2 by N_3 left measurements and the same number for right station) the computational procedure was as follows:

- average of the N_1 settings per image point;
- superposition of the N_3 frames of a station by means of a homographic transformation computed with the image points; and
- analytical restitutions of the two summarized frames by the means of the direct linear transformation method (resections in space with direct determination of eleven prameters for each bundle, followed by the determination of spatial coordinates). Fifteen object control points* have been used for each computation.

Finally one takes the average of the determinations of the N_2 neighbouring targets relating to each object point. Then the RMS and maximum spatial residuals are estimated from at least 15 check points.

Tables 1 and 2 show that:

- The maximum average gain of accuracy is about 30 per cent when increasing the number of settings per image point. Furthermore, there appears to be no correlation between the gain in accuracy, the camera type, and the geometry.
- The maximum average gain of accuracy when increasing the number of targets defining an object point is about 40 per cent. There is no apparent correlation between the gain in accuracy, the camera type and the geometry.
- The maximum average gain of accuracy when increasing the number of frames per station is about 40 per cent. Again, there is no apparent correlation between the gains in accuracy, the camera type, and the geometry.

Furthermore, the two parameters, number of targets per object point and number of frames per station, are more efficient than the parameter number of settings per image point. For a given total number of measurements it is preferable to choose a combination (frames, targets, settings) with a large number of targets or frames.

Now we possess the answers relating to the three ways of varying the redundance of measurements per object point. Thus, it can be deduced from our observations (this is summarized in the lower part of Table 1) that absolute maximum accuracy gain is obtained only when mixing the three parameters of measurement redundance, and that practically for a correct choice of the combination (number frames per station, number of targets per object point, and number of settings per image point) there is no additional gain beyond seven to nine comparator measurements whatever the camera and the geometry (See, for example, Figure 5. Other examples can be found in Hottier[5,6]).

The maximum accuracy gains that were obtained are in the lower part of Table 1. They average 51.3% for *rXYZ* and 54.8% for *rMXYZ*.

* Of course, if it is decided to define an object point with N_2 neighbouring targets, the same option is valuable for the control points: 15 control points with $N_2 = 2$ mean then 15 pairs of targets.

TABLE 1. GAINS OF ACCURACY (PHOTOPAIR: SYMMETRICAL CASE) WITH REDUNDANCE OF MEASUREMENTS PER OBJECT POINT (PERCENTAGES)

Cameras		TMK (1-5)		TMK (6-10)		TMK (11-15)		UMK (1-5)		UMK (6-10)		UMK (11-15)	
ratio r = *B*/*O*		0.86		0.33		0.14		0.84		0.33		0.14	
		rXYZ	*rMXYZ*	*rXYZ*	*rMXYZ*								
Minimum accuracy (1 frame, 1 target, 1 setting)		8.2 umit:μm	21.3	13	24.3	35.6	99.4	7.2	15.3	10.7	22.6	26.9	55.9
Gain of accuracy with number of settings/image point (1 frame/station; 1 target/object point)	2	19.5%	19.2%	17.0%	20.0%	4.5	13.2	12.5	16.3	18.7	13.3	13.4	− 7.0
	3	21.9	24.4	26.0	30.0	10.4	10.5	13.0	18.9	22.4	28.8	18.7	−14.8
	4	23.2	29.1	27.7	36.2								
	5	28.0	39.0	29.2	24.7								
	6	30.5	40.4										
	7	28.0	35.7										
Gain of accuracy with number of targets/object point (1 frame/station; 1 setting/image point)	2	32.9	40.4	18.4	13.2	7.9	16.6	9.7	3.3	− 1.0	− 5.0	6.7	10.5
	3	47.6	59.1	23.1	18.5	14.3	28.6	22.2	28.1	30.8	47.8	31.3	41.3
	4	54.9	71.4	27.6	36.6								
	5	53.7	69.9	21.5	33.7								
	6	57.3	73.2										
	7	54.9	71.8										
Gain of accuracy with number of frames/station (1 target/object point; 1 setting/image point)	2	28.0	42.7	30.7	21.8	15.7	10.4	9.7	8.5	1.0	− 4.0	26.9	20.4
	3	46.3	55.9	44.6	43.2	35.1	38.5	30.6	38.6	33.6	33.2	29.8	41.3
	4	54.9	68.5	48.4	51.0	28.1	29.3	25.0	34.0	37.4	38.9	36.6	34.9
	5			52.3	51.4	26.7	33.1	31.9	40.5	49.5	61.9	35.1	24.9
Maximum accuracy gain (At least 2 frames/station, 2 targets/object points, 2 settings/image point)		61%	77%	54	49	52	47	42	41	49	60	50	55

TABLE 2. AVERAGE ACCURACY GAINS WITH REDUNDANCE OF MEASUREMENTS PER OBJECT POINT WITHOUT CONSIDERING THE RATIOS $r = B/O$

		TMK		UMK		TMK + UMK		Theoretical gain
		rXYZ	rMXYZ	rXYZ	rMXYZ	rXYZ	rMXYZ	
Gain of accuracy with number	2	13.7%	17.5%	14.9	14.8	14.3	16.1	29%
of settings/image point	3	19.4	21.6	18.3	23.8	18.8	22.7	42
(1 frame/station; 1 target	4	25.4	32.6			26.4	32.6	50
object point)	5	28.6	31.8			28.6	31.8	55
Gain of accuracy with number	2	19.7	23.4	5.1		19.7	23.4	29
of targets/object point	3	28.3	35.2	28.1	39.1	28.2	37.1	42
(1 frame/station; 1 setting/	4	40.9	54.0			46.9	59.6	50
image point)	5	38.0	51.8			38.6	51.8	55
Gain of accuracy with number	2	24.8	25.0	18.3	14.4	21.5	19.7	29
of frames/station	3	42.0	45.9	31.3	37.7	36.6	41.8	42
(1 target/object point;	4	43.8	49.6	33.0	35.9	38.4	42.7	50
1 setting/image point)	5	39.5	42.2	38.8	42.4	39.1	42.3	55

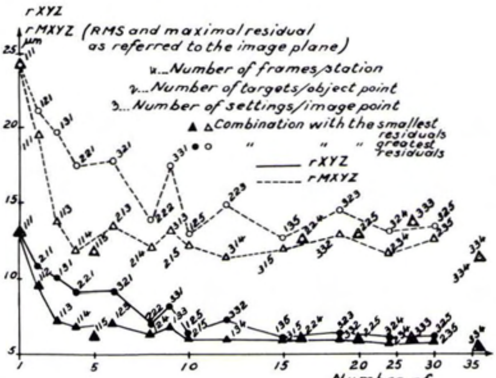

FIG. 5. Accuracy as a function of number of measurements per object point.

We have also collected in Table 3 the accuracy gains supplied from the measurement combination of one setting per image point, three targets per object point, and three frames per station.

Thus, under normal conditions (that is, with a reasonable number of measurements), one statistically obtains a 50 per cent gain about the RMS residual and the maximum residual. The gain seems to be independent of the geometry and the camera type (metric camera).

It is interesting to note that it is not sufficient to increase the frames per station if maximum accuracy is desired; it is necessary to mix the increase in frames and increase in targets per object point. The two parameters present a combined effect and probably concern two different errors sets.

Independence of the accuracy gain and of the geometry. This independence can be empirically determined. It also can be derived theoretically on the assumption that the distribution of errors is normal. This assumption is confirmed by the experiments described earlier.

Under these conditions the RMS residual is proportional to the standard deviation, that is, $RXYZ = q\,\sigma_o$ where σ_o is a function of the equipment (camera, emulsion, and comparator) and q is a function of geometry. Then, if we increase the number of measurements, σ_o becomes $\sigma_1 < \sigma_o$, and the gain in accuracy is

$$g = 1 - \frac{q\,\sigma_1}{q\,\sigma_o} = 1 - \frac{\sigma_1}{\sigma_o}\,. \tag{13}$$

The gain in accuracy is independent of the geometry.

Comparison between experimental and theoretical gain and evaluation of the RMS bias *of the system*. Theoretically, for n comparator measurements, the gain in accuracy should be

$$g = 1 - \frac{1}{\sqrt{n}} \cdot \qquad (14)$$

The last two columns of Table 2 show that there is no agreement between the theoretical and experimental values. Moreover, there is a stabilization of the accuracy beyond 7 to 9 measurements. This is due to the bias of the measurements. If we designate by $\sqrt{E\beta^2}$ the RMS bias of the different photogrammetric systems (which differ according to the camera, the comparator operator, the redundance of measurements, etc.) used in these experiments, we now consider that after n measurements the standard deviation of the equivalent normal error law is

$$\sqrt{E\ \beta^2 + \frac{\sigma_o^2}{n}} \text{ instead of } \sqrt{\frac{\sigma_o^2}{n}}$$

where σ_o is the RMS error of the measurements.

Of course, $\sqrt{E\ \beta^2}$ depends on the measuring procedure, and we can distinguish:

$\sqrt{E\ \beta_1^2}$, the RMS bias relating to the increase in settings per image point (one target per object point, one frame per station);

$\sqrt{E\ \beta_2^2}$, the RMS bias relating to the increase in the number of targets defining an object point (one setting per image point, one frame per station);

$\sqrt{E\ \beta_3^2}$, the RMS bias relating to the increase in the frames per station (one setting per image point, one target per object point); and

$\sqrt{E\ \beta^2}$, the RMS bias relating to the photogrammetric systems of maximum accuracy.

Then for a given n-measurement procedure, for example the first one, we have (from Equation 13)

$$g_n = 1 - \sqrt{\frac{E\beta_1^2 + \sigma_o^2/n}{E\beta_1^2 + \sigma_o^2}} \qquad (15)$$

instead of Equation 14.

From Table 2 (third column) and Equation 15 it is possible to estimate $E\beta_1^2$, that is,

$$\frac{\sqrt{E\beta_1^2}}{\sigma_o} = \sqrt{\frac{(1 - g_n)^2 - 1/n}{1 - (1 - g_n)^2}} \qquad (16).$$

The estimates found for the three ratios (RMS bias/σ_o) and computed from the different available values of n are the following:

n	$\sqrt{E\beta_1^2}/\sigma_o$ (Settings)	$\sqrt{E\beta_2^2}/\sigma_o$ (Targets)	$\sqrt{E\beta_3^2}/\sigma_o$ (Frames)
2	0.94	0.64	0.55
3	0.98	0.61	0.34
4	0.83	0.39	0.46
5	0.79	0.55	0.52
RMS Average	0.89	0.55	0.47

These estimates should be interpreted with regard to the estimated value of σ_o, the standard deviation of the normal equivalent low for the reference measuring procedure (one setting per image point, one target per object point, one frame per station) which is estimated as $\sigma_o = 2.46\ \mu$m.

It is also possible to estimate the residual RMS bias when obtaining the maximum accuracy. This last one, given above, is obtained with mixed combinations (settings, targets, frames). From Table 3 ($g_9 = 47\%$), and estimation of maximum accuracy gain ($g = 51.3\%$), we obtain

$$0.49 < \sqrt{E\beta^2} < 0.56.$$

Finally, the measurements residual RMS bias of the photogrammetric systems considered (TMK or UMK camera, Gevapan 33 emulsion, Zeiss Asco-record monocomparator, at least three targets per object point or three frames per stations) is

$$\sqrt{E\beta^2} \simeq \frac{1}{2}\sigma_o \simeq 1.23\ \mu\text{m}$$

which is very close to the standard deviation of the comparator individual settings* (which has been estimated to be $\simeq 1\ \mu$m from repeated measurements with the same operator).

Practical choice of the combination (settings, targets, frames). This question is

* See Appendix B for an attempt at theoretical justification of this fact.

TABLE 3. ACCURACY FOR THE MEASUREMENT COMBINATION, ONE SETTING PER IMAGE-POINT, THREE TARGETS PER OBJECT-POINT, AND THREE FRAMES PER STATION.

Camera	$R = B/O$	RMS residual ($rXYZ$) accuracy gain (%)	Maximal residual ($rMXYZ$) accuracy gain (%)
TMK	0.86	61	72
	0.33	49	39
	0.14	53	49
UMK	0.24	38	39
	0.39	29	45
	0.14	51	56
Average gain (%)		47	50

economically important. We saw that increasing the number of measurements produces a gain in accuracy at the very most of 50 per cent (at least for the photogrammetric systems considered). However, for attaining the maximum accuracy, at least 7 to 9 measurements per image point (14 to 18 per object point) are necessary (Figure 6).

The corresponding number of measurements can appear quite large. However, it is not overly costly; even with a high number of measurements, the cost of the comparator measurements in the total budget, for reasonable amount of object points ($30 < n < 150$), is not more than 10 per cent of the total budget (ground operations, data preparation and treatment, etc.).

On the other hand, if few comparators are available, a severe loss of time can occur. It is chiefly for this reason that it is important to reduce the number of measurements.

From Figure 6, which condenses results from several pairs, it is clear that the improvement is rapid from one to three-to-four measurements (the gain of accuracy is then 40 per cent), but rather slow beyond that number.

Then combinations such as one setting, one target, and three frames, or one setting, two targets, and two frames give accuracies (accuracy gain, 40 per cent) approaching the maximum accuracies (maximal accuracy gain, 50 per cent).

THE EFFECT OF GEOMETRICAL CHARACTERISTICS

I distinguish here two sorts of characteristics.

(a) Geometric characteristics.

In the symmetrical case (Figure 7), the only one considered here, the geometry is characterized by two parameters: the base to object distance ratio, $r = B/O$; and the convergence of the two optical axes, 2Φ.

For large values of r, r and Φ are not independent. For example, if $0.7 < r < 1$, the angular field of presently available metric cameras will have a convergence of about $2\Phi = 40$ grads.

(b) In case of the use of a control net, the number and the disposition of control points.

The effect of geometry: experimental results. Experimental results are shown in three graphs (Figures 8, 9, and 10) which give both the RMS and maximum residuals $rXYZ$ and $rMXYZ$ (as referred to the image plane), and similar quantities for the X and Z components,

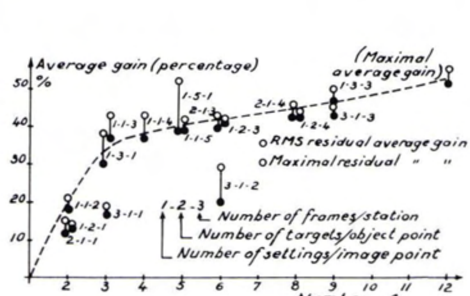

FIG. 6. Per cent gain in accuracy as a function of the number of measurements per image point.

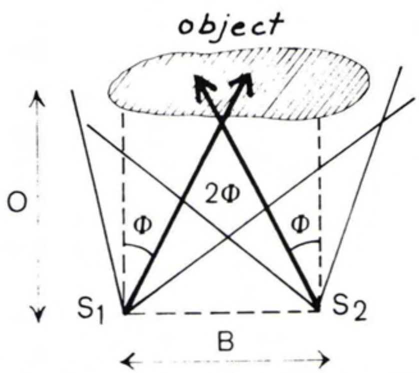

FIG. 7. Geometry of the symmetrical case.

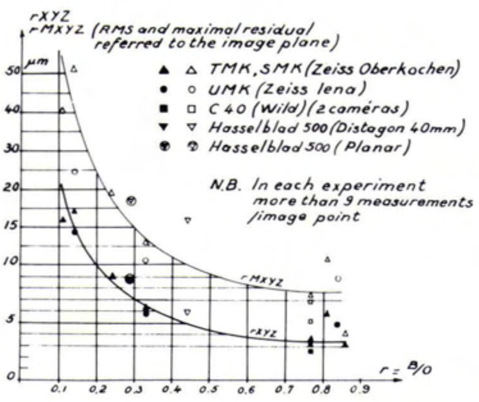

FIG. 8. Accuracy as a function of the ratio $r = B/O$.

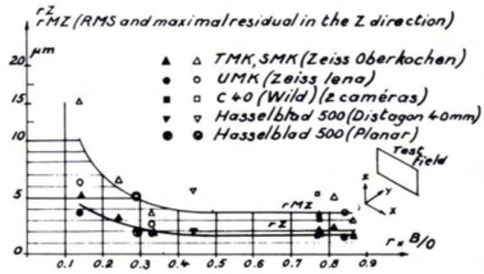

FIG. 9. Accuracy in X as a function of the ratio $r = B/O$.

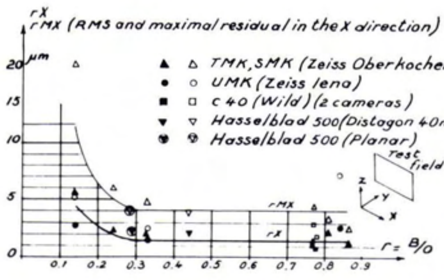

FIG. 10. Accuracy in Z as a function of the ratio $r = B/O$.

for different value of r, five distinct metric cameras (TMK, SMK, UMK, and two C40), and for maximum accuracy (for minimum accuracy, obtained with the combination one setting, one target, one frame, the ordinates should be multiplied by two).

There is a rapid gain in accuracy when r is increased from 1/7 (0.14) to 1/2 (0.5) with no convergence. At this point, there is a lack of experimental results. For $0.7 < r < 1$ (with $2\,\Phi \simeq 40$ grads), there is only a slight accuracy gain.

The maximum accuracy (obtained with $0.7 < r < 1$ and $2\,\Phi \simeq 40$ grads) can be characterized by the following values:

$rXYZ = 4.0\ \mu m$	$rX = 1.5\ \mu m$	$rZ = 1.8\ \mu m$
$rMXYZ = 8.0\ \mu m$	$rMX = 4.0\ \mu m$	$rMZ = 3.6\ \mu m$

The effect of the number and disposition of control points. (For more details see Appendix B). For a given stereopair computed with n control points (equidistribution in the volume), the following law can be proposed:

$$rXYZ = \sqrt{1 + \frac{r}{pn}}\ r_o XYZ \tag{17}$$

where r is the number of unknown parameters for the external orientation of the bundle and p is the number of observation equations per control point, r_oXYZ being the maximum accuracy that can be obtained from the data (plate coordinates of the image points) of the stereopair.

If Equation 17 is true, one must obtain, in a statistical way,

$$\frac{rXYZ}{\sqrt{1 + \frac{r}{pn}}} = ct = r_o XYZ.$$

The verification concerns six couples (three different ratios B/O) for different values of $n{:}n_1, n_2, \ldots, n_k$. At first, for each couple, r_oXYZ is estimated from

$$r_o XYZ = \frac{1}{k} \sum_{1}^{k} \frac{rXYZ_{(i)}}{\sqrt{1 + \frac{r}{pn_i}}}$$

Then, for each number n_i, the mean of the ratios $rXYZ/r_oXYZ$ is computed and compared to its theoretical value $\sqrt{1 + \frac{r}{pn_i}}$

(a) Collinearity equations: resections in space with the method of direct linear transformation and n control points. For $r = 11$ and $p = 2$, the following comparison can be made:

n	7	10	15	23	28
$(rXYZ/r_oXYZ)$	1.36	1.21	1.15	1.15	1.13
$\sqrt{1 + \frac{11}{2n}}$	1.34	1.24	1.17	1.11	1.09

(b) Relative orientation and adjustment to n control points. For $r = 7$ and $p = 3$, the following comparison can be made:

n	4	6	10	15	23	28
$(rXYZ/r_oXYZ)$	1.24	1.20	1.11	1.04	1.10	1.05
$\sqrt{1 + \frac{7}{3n}}$	1.26	1.18	1.11	1.07	1.05	1.04

Another experimental result is that accuracy is not increased when we define each object point with t neighboring targets instead of one. This, in our opinion, comes from the fact that the bias of the measuring system is not changed. It can statistically decrease only if the geometrical disposition of the control points is modified.

Finally, the maximum average accuracy gain is 20 to 30 per cent when the number of control points is increased.

PREDICTORS OF ACCURACY

Simulation predictor—Equivalent normal error law. Several fictitious pairs (plate coordinates contaminated with a normal error of 2 μm standard deviation) were created and computed under the same conditions as actual pairs in order to answer three questions:

(a) Does there exist an equivalent normal error law $\mathcal{N}(0, \sigma_o)$ which accounts for the experimental curves of Figures 8, 9, and 10?

(b) Is it possible from such curves to predict accuracy in large depth volumes (needless to say that in such a case the control net should cover the whole volume)?

(c) Does accuracy vary in a significant way with the convergence of the optical axes?

The corresponding accuracy predictors are shown in Figures 11 and 12. (The predictor of the RMS spatial residual $rXYZ$ comes directly from the smoothing of the different obtained values. It is the same for rX, rY, and rZ, but with the condition $r^2X + r^2Y + r^2Z = r^2XYZ$).

Now it is possible to give the following answers to questions (a), (b), and (c):

(a) For $0.1 < r < 1.0$, the real maximum accuracy curves (Figures 8, 9, and 10) can be approximated in a very good way by the accuracy curves built from fictitious data contaminated with normal errors of standard deviation

$$\sigma_o = 1.23\ \mu\text{m}. \tag{18}$$

This value has been found by computing the best affinity coefficient between the curve of Figure 8 and the curve of Figure 11. The fitness of the affinity adjustment is illustrated by the Figure 13.

As a consequence, the equivalent normal law of minimum accuracy (two times lower then maximum accuracy), which corresponds to the combination one setting, one target, one frame, has a standard deviation $\sigma'_o = 2 \times 1.23 = 2.46\ \mu$m.

It is interesting to compare the value of σ'_o with the value of σ, the standard deviation of the error measurements estimated from resections in space with the rigorous method (compensation of the measurements). From 12 different plates (one setting, one target, one frame), we obtain on an average the estimation $s = 2.76\ \mu$m with confidence limits at the five per cent level of 2.47 μm and 3.05 μm. This value is in good agreement with the value of σ'_o. Nevertheless, it can be pointed out that σ seems to be slightly, but significantly, superior to σ'_o (perhaps because of residual distortion).

(b) The answer to question (b) is yes, at least for current uses of such curves. It is seen

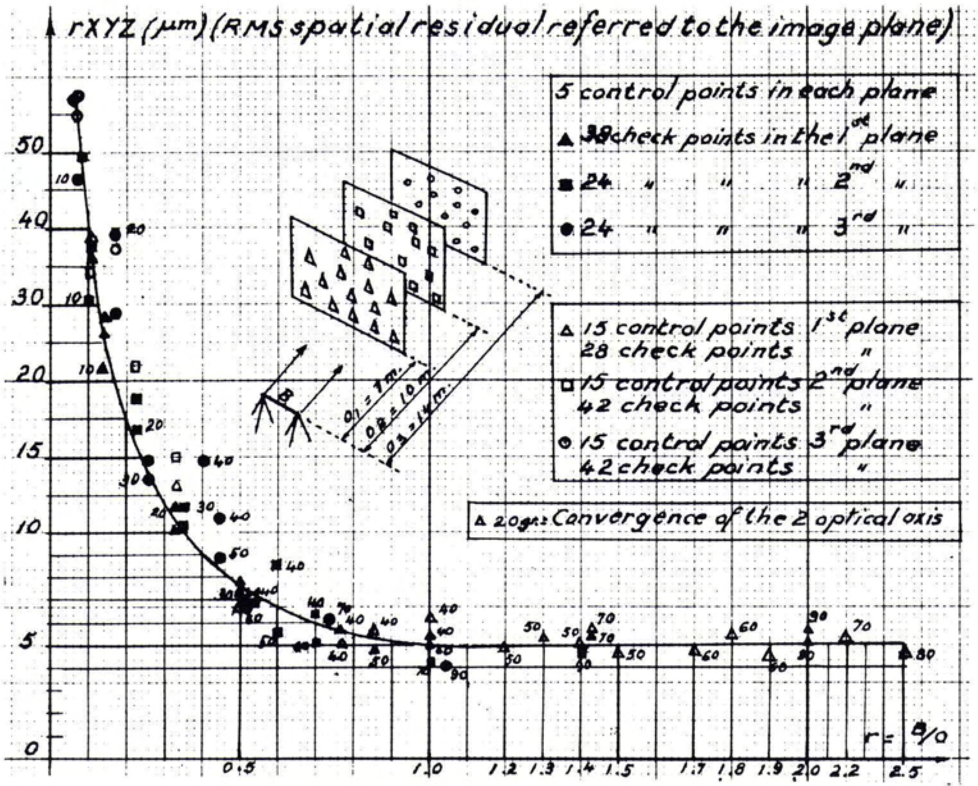

FIG. 11. Accuracy predictor (RMS spatial residual *rXYZ)* from simulation with a normal error distribution (standard deviation of 2 μm).

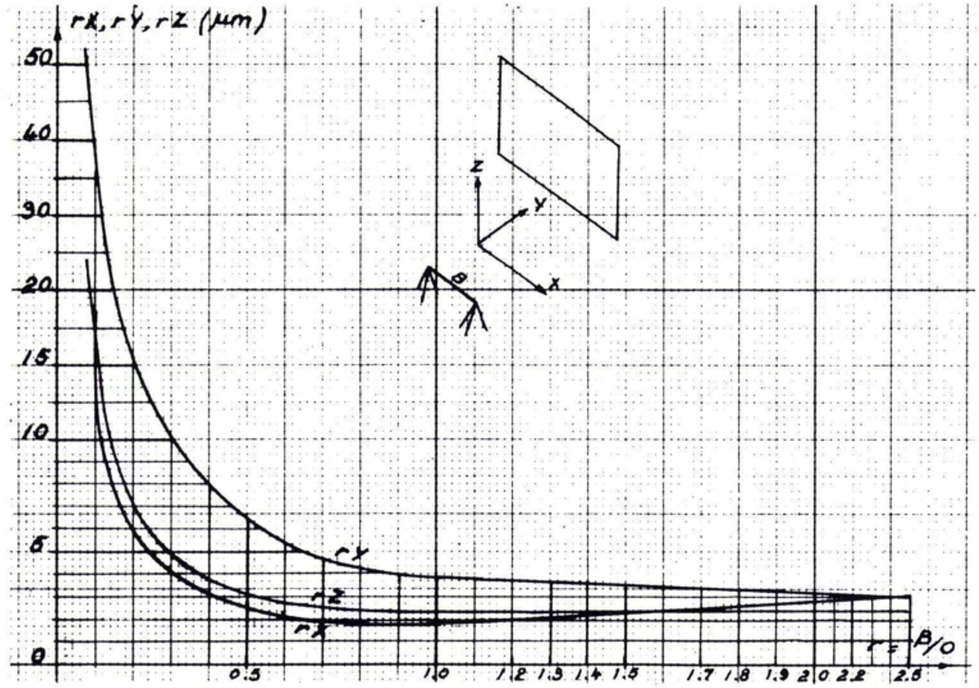

FIG. 12. Accuracy predictor (RMS residuals *rX*, *rY*, *rZ* as referred to the image plane) from simulation with a normal error distribution (standard deviation of 2 μm).

from the simulations (Figure 11) that, given a deep volume, it is possible to estimate with a reasonable precision the accuracy at any depth. It is sufficient to compute the corresponding base-to-object-distance ratio, and to read the values of RMS residuals. This supposes, of course, that there is no extrapolation outside the volume of the control net.

(c) The Answer seems to be no, at least for the ratio base-to-object-distance range 0.1 to 1.5.

Karara/Abdel Aziz predictors. Karara and Abdel Aziz[1,2] have studied the problem in another way by considering the accuracy only at the central object point (Figure 14), the implicit assumption being that the accuracy at the central object point can reasonably represent the accuracy of the whole overlap.

The formulas which give the errors σ_X, σ_Y, and σ_Z (X axis parallel to the base, Y axis perpendicular to the object) referred to the image plane (that is, multiplied by p/O, where p is the focal length and O the object distance) are

$$\sigma_X = \frac{1}{\sqrt{2}} \frac{1 + \tan\alpha \tan\varphi}{1 - \tan(\alpha - \varphi)\tan\varphi}\sigma$$

$$\sigma_Y = \frac{2}{r}\sigma_X$$

$$\sigma_Z = \frac{1}{\sqrt{2}} \frac{1/\cos\varphi}{1 - \tan(\alpha - \varphi)\tan\varphi}\sigma \quad (19)$$

where σ is the standard deviation of the error measurements, φ is the angle which the camera axis makes with the direction perpendicular to the base, and α is the angle which the line joining the central point C of the object space and the perspective center makes with the perpendicular direction to the base.

For the normal case we have ($\varphi = 0$)

$$\sigma_X = \frac{\sigma}{\sqrt{2}};\ \sigma_Y = \frac{\sqrt{2}\sigma}{r};\ \sigma_Z = \frac{\sigma}{\sqrt{2}};$$

$$\sigma_{XYZ} = \sqrt{1 + \frac{2}{r^2}}\,\sigma. \quad (20)$$

For the case where the camera axes are directed towards the central point ($\varphi = \alpha$, $\tan\varphi = r/2$),

$$\sigma_X = \frac{\sigma}{\sqrt{2}\cos^2\varphi};\ \sigma_Y = \frac{\sqrt{2}\sigma}{r\cos^2\varphi};\ \sigma_Z = \frac{\sigma}{\sqrt{2}\cos\varphi}$$

$$\sigma_{XYZ} = \frac{1}{\cos\varphi}\sqrt{\frac{1+\cos\varphi}{2} + \frac{2}{r^2}};\ \tan\varphi = r/2. \quad (21)$$

(Note that this last case is very important in practice when strong base-to-object-distance ratios are used.)

In Table 4 values of σ_{XYZ} are listed for values of r (ratio base-to-object-distance) between 0.1 and 2 and for $\sigma = 2\ \mu m$. This is done for the normal case ($\varphi = 0$) and for the case where the camera axes are directed towards the central point ($\tan\varphi = r/2$). Also in Table 4 the values of the RMS residual $rXYZ$ have been listed as determined from the simulation predictor (Figure 11).

The following points are to be observed:

- The central case gives theoretically (for the central point) a slightly lower accuracy, but it is not very sensible. For example for $r = 0.05$, we have $\sigma XYZ = 6.2\ \mu m$ instead of $6.0\ \mu m$ for a convergence of 31 grads between the two optical axes.
- There is good agreement between the Karara/Abdel Aziz predictor σXYZ and the simulation predictor $rXYZ$ (Figure 11), and subsequently between the Karara/Abdel Aziz predictor σXYZ and experimental accuracies (we proved (Figure 13) the agreement between the simulation predictor and experiments) at a 20 per cent level for the ratio range 0.1 to 1.0.
- However, the simulation predictor $rXYZ$ seems to be systematically superior to the accuracy σXYZ at the central point, and the difference eclipses the difference between the normal and the central case for ratios between 0.1 − 1.0. In addition, the random variations of $rXYZ$ (Figure 11) also appear to be much more important than the slight difference between the normal and the central case.

Other remarks can be derived from a comparison between the values of the Karara/Abdel Aziz predictor, which gives the accuracy at the central point, and the experimental curves of the RMS residuals (Figures 8, 9, and 10).

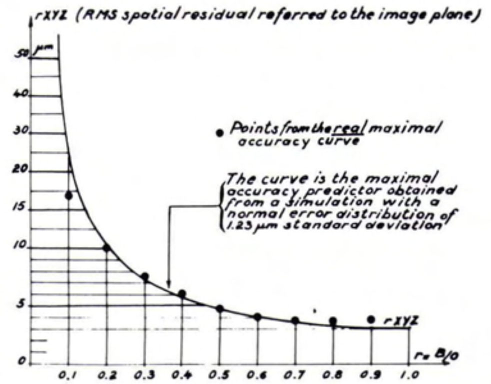

FIG. 13. Adjustment of the simulation predictor and of the experiments.

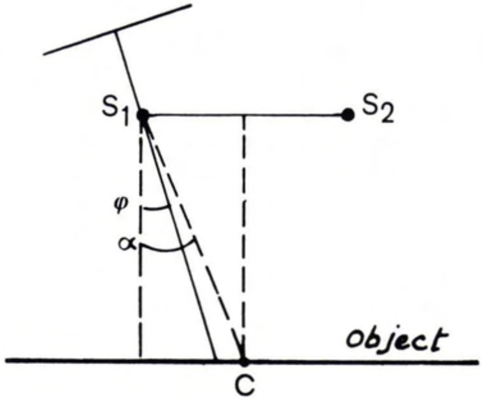

FIG. 14. Definition of the central point C.

TABLE 4. COMPARISONS OF NORMAL CASE AND CENTRAL CASE AND COMPARISON OF THE KARARA/ABDEL AZIZ PREDICTOR σ_{XYZ} AND THE SIMULATION PREDICTOR $rXYZ$ (FIGURE 11) (σ_{XYZ}; UNIT μm; ONLY NORMAL ANGULAR FIELD IS CONSIDERED).

ratio base/object distance r	Normal case σ_{XYZ}	Central case φ (grads)	Central case σ_{XYZ}	RMS residual (simulation predictor) $rXYZ$
0.1	28.4 μm	3.18	28.4 μm	35.0 μm
0.2	14.3	6.31	14.3	15.0
0.3	9.6	9.5	9.7	12.0
0.4	7.3	12.6	7.5	9.0
0.5	6.0	15.6	6.2	7.4
0.6	5.1	18.5	5.3	6.6
0.7	4.5	21.4	4.8	6.0
0.8		24.2	4.4	5.3
0.9		26.9	4.1	5.2
1.0		29.5	3.8	5.0
1.5		41.0	3.3	5.6
2.0		45.0	3.3	5.0

The predictor gives for the X component (normal case) $\sigma X = ct$, but experimental and simulated results show that the RMS residual in X rapidly decreases when the ratio goes from 0.1 to 0.5. The same remark holds for the Z component. In addition, the constant value of σX for $\sigma = 2$ μm is 1.4 μm when the value of rX is $\simeq 2$ μm for $r > 0.6$.

It can be concluded that (at least for $0.1 < r < 1.0$) the accuracy at the central point cannot represent in an ideal way the average accuracy over the whole object plane, particularly for small values of the base-to-object-distance ratio. (Besides, it is obvious that, for small values of r the central point accuracy overestimates the average accuracy. The central point XY accuracy is of course constant, but it is not the same for corner points, and this is probably the explanation for the fact that we found the RMS residual $rXYZ$ systematically superior to the central accuracy σXYZ.)

NON-METRIC CAMERAS

The source for non-metric cameras will be chiefly the American authors Karara and Abdel Aziz[1,2] who have studied the accuracy of the photopair (symmetrical case) for five different non-metric cameras.

Karara/Abdel Aziz trials. The characteristics of the five cameras (and their prices) are listed below:

Camera and lens	Focal length (mm)	Image format (mm)	Approx. Price ($)
Kodak Instamatic 154	43	12 × 12	15
Crown Graphic-Graphex *f*/4.7	135	120 × 100	300
Honeywell Pentax Super Takumar *f*/1.4	50	36 × 24	500
Hasselblad 500-Planar *f*/2.8	80	55 × 55	550
Hasselblad MK70-Biogon *f*/5.6	60	55 × 55	4500

The geometry of the picture taking (Figure 15) was the same for all the cameras (targets in two planes, determined with an accuracy better than 0.1 mm). For each camera, five photographs (camera hand-held) were taken at both stations (ten plates for each camera).

At first the authors studied the means to reduce film deformation and lens distortion by using different mathematical models (accounting for linear deformations and radial distortion with polynomials of different degrees, and eventually for asymetrical lens distortion due to non-linear film deformation).

The conclusion is very neat. A sophisticated model is not required, a model with linear film deformations and radial distortion represented by $\Delta r = k\, r^3$ (r = radial distance), considering the number of supplementary unknowns to be introduced in the computations, being the most efficient and economical. Others refinements seem to be unnecessary.

With this model, the average RMS plate residuals σ_o estimated for each camera from ten resections in space are given in Table 5.

Furthermore, in Table 6, the estimated values for the mean RMS spatial residuals $rXYZ$, rX, rY, and rZ (in micrometers referred to the image plane) in the planes 1 and 2 are given.

Comparison of experimental and predicted accuracies. We have two predictors at our disposal.

(1) The predictor obtained from simulations, and which concerns the RMS residuals (Figures 11 and 12).

For $r = 0.73$ (case of the envisaged geometry plane 1), we read on the prediction curves that, for $\sigma_o = 2$ μm, we can anticipate $rXYZ = 5.6$ μm, $rX = 1.8$ μm, $rY = 4.5$ μm, and $rZ = 2.5$ μm.

For any other value σ_o, we multiply these quantities by $\sigma_o/2$, so that

$$rXYZ = 2.80\,\sigma_o;\; rX = 0.9\,\sigma_o;\; rY = 2.25\,\sigma_o;\; rZ = 1.25\,\sigma_o \tag{22}$$

(2) The Karara/Abdel Aziz predictor which concerns the central point accuracy:

$$\sigma XYZ = \sqrt{\sigma X^2 + \sigma Y^2 + \sigma Z^2}$$

$$\sigma X = \frac{1}{\sqrt{2}}\,\frac{1 + \tan\alpha\tan\varphi}{1 - \tan(\alpha - \varphi)\tan\varphi}\sigma_o;\; \sigma Y = \frac{2}{r}\,\sigma X;$$

$$\sigma Z = \frac{1}{\sqrt{2}}\,\frac{1/\cos\varphi}{1 - \tan(\alpha - \varphi)\tan\varphi}\sigma_o;$$

in which $\varphi = 15°$ and $\tan\alpha = \frac{r}{2} = 0.365$.

Then we obtain for the envisaged geometry (plane 1)

$$\sigma XYZ = 2.44\,\sigma_o;\; \sigma X = 0.79\,\sigma_o;\; \sigma Y = 2.18\,\sigma_o;\; \sigma Z = 0.75\,\sigma_o \tag{23}$$

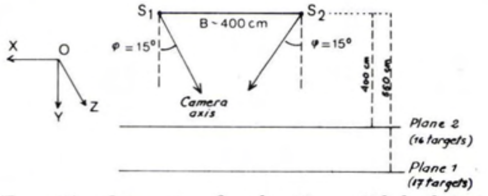

FIG. 15. Geometry for the Karara/Abd el Aziz studies.

TABLE 5. EVALUATION OF ERROR MEASUREMENT FROM RESECTIONS IN SPACE (COMPARE WITH METRIC CAMERAS: THE RMS PLATE RESIDUAL WAS FOUND TO BE s_o = 2.8 μm).

Camera	RMS plate residual (μm) σ_o
Kodak	15.2
Crowngraphic	11.5
Honeywell Pentax	3.9
Hasselbled 500 C	6.1
Hasselbled MK70	5.0

TABLE 6. NON-METRIC CAMERAS: COMPARISON BETWEEN EXPERIMENTAL AND PREDICTED ACCURACIES IN MICROMETERS (A = EXPERIMENTAL VALUES; B = SIMULATION PREDICTOR; C= KARARA/ABDEL-AZIZ PREDICTOR)

		Plane 1				Plane 2			
		rXYZ	*rX*	*rY*	*rZ*	*rXYZ*	*rX*	*rY*	*rZ*
Kodak	A	24.1	10.1	19.4	10.1	19.8	6.5	16.0	9.7
	B	42.6	13.7	34.2	19.6	38.0	13.6	28.9	18.2
	C	37.1	12.0	33.1	11.4	31.2	12.9	25.8	11.7
Crown	A	34.3	10.1	31.7	8.3				
	B	32.2	10.3	25.9	14.4				
	C	28.1	9.1	25.1	8.6				
Pentax	A	7.4	2.2	6.7	2.2	9.6	3.6	8.6	2.4
	B	10.9	3.5	8.8	4.9	9.7	3.5	7.4	4.7
	C	9.5	3.1	8.5	2.9	8.0	3.3	6.6	3.0
Hass.	A	17.9	5.3	16.7	3.8	17.0	5.4	15.2	5.4
500	B	17.1	5.4	13.7	7.6	15.2	5.5	11.6	7.3
	C	14.9	4.8	13.3	4.6	12.5	5.2	10.3	4.7
Hass.	A	13.8	4.9	12.4	3.4	12.0	2.4	11.1	3.7
MK70	B	14.0	4.5	11.2	6.2	12.5	4.5	9.5	6.0
	C	12.2	3.9	10.9	3.7	10.2	4.2	8.5	3.8

Table 6 gives the experimental and predicted accuracies computed from the values of σ_o (Table 5) and Equations 22 and 23 (Analog formulas were established for the plane 2).

It can be observed that, for the corresponding value of r (base-to-object distance), the two predictors give nearly the same results, at least for $rXYZ$, rX, and rY. However, there is some disagreement for rZ (vertical). The value σ_o for the Kodak camera (Table 5) is probably over-estimated. The value to use in the predictors is 10 μm rather than 15.6 μm.

It is also interesting to compare the accuracy of non-metric and metric cameras. If we use 2.5 μm for σ_o, we find (Figures 11 and 12) for minimum accuracy of metric cameras ($r = 0.73$), $rXYZ = 7.0$ μm, $rX = 2.25$ μm, $rY = 5.6$ μm, and $rZ = 3.1$ μm.

From all of the results obtained by Karara/Abdel Aziz, only the ones for the Pentax camera give for the same measurement effort the same accuracy as metric cameras. All the other cameras give a lower accuracy.

Maximum accuracy with non-metric cameras. All the Karara/Abdel Aziz trials correspond to minimum accuracy conditions, that is, to minimum measurement redundance (one setting per image point, one target per object point, one frame per station).

Added to these results are two trials performed at the IGN. The conditions of the picture taking and measuring process are summarized as follows:

Camera	Lens	Focal length	r	φ(gr)	Measuring process
Hasselblad 500	Planar	100 mm	0.29	12.5	3 frames/station and
Hasselblad 500	Distagon	40 mm	0.44	18	3 setting/image point

After averaging the settings for each frame, the three frames of each station were superposed by the means of a homography. The computation conditions were analogous to those of Karara and Abdel Aziz (particularly for the Distagon camera where an image-refinement method was practiced, i.e., plate-residual filtering and correction). The number of control points was 27.

It can be seen (Table 7) that the maximum accuracy obtained from the best non-metric cameras is nearly the same as that obtained for metric cameras.

Note that the RMS values of plate residuals were 3.05 μm for the Planar and 2.98 μm for the Distagon. Thus, in contrast to minimum accuracy conditions, it is impossible to predict real accuracy from such values. The maximum accuracy is obtained for $\sigma = 1.23$ μm (metric cameras) and the use of the value 3 μm would lead one to underestimate the accuracy. This is not an isolated observation, and is also valid for metric cameras. In other words, the value of σ to use for accuracy prediction can be derived only from experimental accuracy studies.

NON-SYMMETRICAL CASE OF THE STEREOPAIR AND MULTI-STATION GEOMETRY

Studies of the non-symmetrical case of the stereopair are currently (August 1975) underway at the University of Illinois. As for multi-station geometry, it is currently used by

TABLE 7. MAXIMUM ACCURACY AND NON-METRIC CAMERAS.

	Experimental values				Predicted maximum accuracy for metric cameras (Figures 8, 9, and 10)				
	rXYZ	*rX*	*rY*	*rZ*	*rXYZ*	*rX*	*rY*	*rZ*	unit:μm
Planar	8.7	1.9	8.2	1.9	7.2	1.5	6.6	2.4	
Distagon	6.1	2.1	5.5	1.9	5.0	1.5	4.4	1.8	

DBA Systems[8] in a very original way. Long focal lengths (1000 mm), associated with small angular fields (10° by 10°) and multi-station geometry, seem to eliminate the need for control nets. The use of long focal lengths provides relative insensitivity to emulsion unflatness and principal point error. Kenefick[8] quotes other assorted advantages of the narrow angle, long focal length cameras, e.g., constant resolution and no problems of variations of lens distortion with object distance. The use of more than two bundles with high convergence seems to allow precise reconstruction of the object despite the narrowness of the bundles (which would be impossible with only two bundles if there were no control net). The accuracy obtained seems to be the same as in the symmetrical case.

CONCLUDING REMARKS

The results presented here are obviously incomplete and not definitive. I had only two relatively abundant sources at my disposal for presentation in this report, and in my opinion these results and other propositions are worth more verification and thorough investigations, at last for theoretical aims. It is particularly true for the presented predictors, which should require some refinements. It also would be interesting to estimate the measurement RMS bias (in fact maximum accuracy) in photogrammetric systems other than the one studied at the IGN (metric camera plus Gevapan 30 emulsion plus Zeiss Asco-Record comparator). Finally, the study of accuracy with multi-station geometry is almost a virgin area.

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APPENDIX A

ESTIMATION OF THE RMS SPATIAL RESIDUAL FROM CONTROL POINTS (AND NOT FROM CHECK POINTS)

The estimation $RXYZ$ of the "true" RMS residual $\dot{R}XYZ$ has been defined as

$$RXYZ = \sqrt{\frac{1}{n}\sum_{1}^{n}\left[(X_{iPH} - X_{iT})^2 + (Y_{iPH} - Y_{iT})^2 + (Z_{iPH} - Z_{iT})^2 \right.} \qquad \text{(A-1)}$$

where X_{iPH}, Y_{iPH}, . . . , are the photogrammetrically computed spatial coordinates of n check points well acounting for the object volume. It should be recalled that a check point is a point whose true coordinates are not used in the computations.

If control points are used instead of check points Equation A-1 gives a biased estimation $R'XYZ$ of $\dot{R}XYZ$ with (in a statistical way) $R'XYZ < RXYZ$. The accuracy estimated from control points is, then, statistically overestimated.

However, if only control points are available, it can be seen that in many computational procedures it is possible to compute a corrective coefficient K so that $E(K \times R'XYZ) = E(RXYZ)$.

FIRST CASE: RESECTIONS IN SPACE OF THE TWO BUNDLES WITH THE RIGOROUS LEAST-SQUARES METHOD, AND INTERSECTION OF HOMOLOGOUS RAYS.

Let us call m_{io} the observed image point corresponding to the control point M_i. The compensated image point, M_{iC}, is the intersection of SM_i with the plane of the picture (S determined by compensation) (Figure A-1).

The residuals of the observation equations are the two components v_{ix} and v_{iy} of the vector $\overrightarrow{m_{io}\, m_{iC}}$. As a consequence of the least-squares theory, we have

$$E(v_{ix}) = E(v_{iy}) = 0$$

$$s^2 = \frac{1}{2\,n - r} \sum_{1}^{n} (v_{ix}{}^2 + v_{ig}{}^2) \tag{A-2}$$

where s^2 is the unbiased estimation of the variance σ^2 of the comparator measurements and r is the number of unknown parameters.

The following assumption is considered verified in a practical sense though not theoretically correct, i.e., all the residuals v_{ix} and v_{iy} $(i = 1, n)$ obey the same law, with a standard deviation σ_v, and are independent. Under these conditions, an unbiased estimate of $\sigma_v{}^2$ is

$$s_v{}^2 = \frac{1}{2n} \sum_{1}^{n} (v_{ix}{}^2 + v_{iy}{}^2) = \frac{2n - r}{2n} s^2. \tag{A-3}$$

Then it is possible to define the corrective coefficient K. If $RXYZ$ is the true RMS spatial residual, we know that $RXYZ = q\,\sigma$ where σ is the RMS error of comparator measurements and q depends only on the object volume. It is, as a rule, the value obtained from a sufficient check-point population.

If we consider an n control-point population, we will have

$$R'^2XYZ = q'^2\, s_v{}^2.$$

(For a given control point, M_i, the photogrammetric determination, M_{iPH}, is obtained as an intersection of two rays, $S_1\, m_{io}{}^1$ and $S_2\, m_{io}{}^2$, and subsequently the spatial residual M_iM_{iPH} is a function only of the left and right residuals v_{ix} and v_{iy}).

q' is a stochastic variable depending only on the choice of the n control points and is independent of s_v such that $E\, q^{2'} = q^2$.

Then, from Equation A-3,

$$E(R'^2XYZ) = Eq^{2'}\, Es_v{}^2 = q^2\, \frac{2n - r}{2n}\, \sigma^2$$

$$E(R'^2XYZ) = \frac{2n - r}{2n}\, R^2XYZ.$$

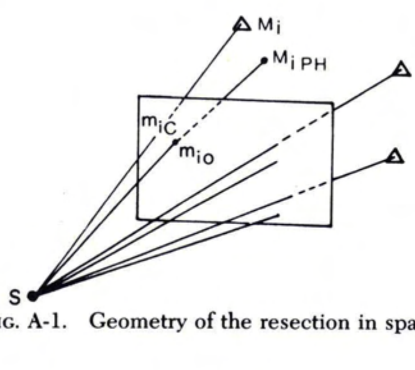

FIG. A-1. Geometry of the resection in space.

And, finally, if $RXYZ$ is the RMS spatial residual estimated from the check points,

$$E(RXYZ) = K \times E(R'XYZ) \quad \text{(A-4)}$$

$$\text{where } K = \sqrt{\frac{2n}{2n - r}}\ .$$

Results for several pairs, for each of the three values 6,7, and 10 of n, are listed in Table A-1. In these cases, Equation A-4 becomes

$$K = \sqrt{\frac{2n}{2n - 9}}\ .$$

SECOND CASE: RESECTIONS IN SPACE OF THE TWO BUNDLES WITH THE DIRECT LINEAR TRANSFORMATION METHOD FOLLOWED BY INTERSECTION OF HOMOLOGOUS RAYS.

The same justification as in the first case can be applied. The residuals w_i of the observation equations are no longer the components of the vector $\overrightarrow{m_{io}\ m_{ic}}$ (Figure A-1) but are functions of $\overrightarrow{m_{io}\ m_{iC}}$, i.e., $w_i = f(v_i)$.

However, in the general case, if the volume depth is not too great, it can be shown that w_i is sensibly proportional to v_i and that

$$\sum w_i^2 \text{ minimum} \Leftrightarrow \sum v_i^2 \text{ minimum.}$$

In other words, the two solutions are practically identical. As a consequence:

$$E\,(R'XYZ) = \sqrt{\frac{2n}{2n - 11}}\ E\,(RXYZ).$$

The results of the tests are given in Table A-2.

THIRD CASE: RELATIVE ORIENTATION AND LEAST-SQUARE ADJUSTMENT OF THE MODEL TO N CONTROL POINTS.

The same assumptions are made in the third case as were made in the first and second cases. The following equation was employed to obtain the results listed in Table A-3:

$$E\,(R'XYZ) \simeq \sqrt{\frac{3n}{3n - 7}}\ E\,(RXYZ) \quad \text{(A-5)}$$

TABLE A-1. PROCEDURE WITH RESECTIONS IN SPACE (RIGOROUS SOLUTION). TEST OF THE FORMULA: $(\overline{r}/r') \simeq \sqrt{2n/(2n-9)}$

Pair reference	n	$r'XYZ$ control	$rXYZ$ check	$\frac{r}{r'}$	$\sqrt{\frac{2n}{2n-9}}$
Ren.	6	7.8	13.3	1.71	
Ren.	6	4.2	11.3	2.69	
Ren.	6	6.4	10.8	1.69	
				2.03	2.00
101	7	5.5	8.5	1.55	
102	7	5.3	7.1	1.34	
106	7	5.6	11.8	2.11	
107	7	13.0	16.4	1.26	
11	7	29.0	37.3	1.29	
12	7	19.2	37.1	1.93	
				1.58	1.67
101	10	4.6	9.8	2.13	
102	10	5.0	6.9	1.38	
107	10	15.2	14.3	.94	
11	10	38.2	30.4	.80	
12	10	22.7	33.6	1.48	
				1.35	1.35

TABLE A-2. PROCEDURE WITH RESECTIONS IN SPACE (DIRECT LINEAR METHOD). TEST OF THE FORMULA: $(\overline{r}/r') \simeq \sqrt{2n/(2n-11)}$

Pair reference	n	$r'XYZ$ control	$rXYZ$ check	$\frac{r}{r'}$	$\sqrt{\frac{2n}{2n-11}}$
101	7	5.3	7.57	1.43	
102	7	4.9	7.4	1.51	
106	7	3.8	10.9	2.87	
107	7	8.7	18.0	2.08	
				1.97	2.16
101	10	4.6	7.7	1.66	
102	10	4.9	7.1	1.46	
106	10	5.1	9.5	1.87	
107	10	12.7	15.3	1.21	
11	10	31.9	32.9	1.03	
12	10	22.4	34.6	1.55	
				1.46	1.49
101	15	5.2	6.3	1.22	
102	15	5.9	7.3	1.24	
106	15	5.5	9.0	1.65	
107	15	13.7	15.3	1.11	
11	15	29.3	29.7	1.02	
12	15	25.2	33.2	1.32	
				1.26	1.26
101	28	5.5	5.7	1.03	
102	28	5.8	6.9	1.20	
11	28	26.8	28.9	1.08	
12	28	25.0	35.3	1.40	
				1.18	1.12

TABLE A-3. PROCEDURE WITH RELATIVE ORIENTATION TEST OF THE FORMULA: $(\overline{r}/r') \simeq \sqrt{3n/(3n-7)}$

Pair reference	n	$r'XYZ$ control	$rXYZ$ check	$\frac{r}{r'}$	$\sqrt{\frac{3n}{3n-7}}$
101	4	10.1	9.3	.92	
102	4	7.1	8.8	1.24	
106	4	3.6	10.4	2.89	
107	4	27.1	19.6	.27	
11	4	15.9	36.2	2.28	
12	4	18.7	39.8	2.13	
				1.70	1.55
101	4	8.8	8.2	.93	
102	4	5.7	8.6	1.40	
106	4	7.7	11.8	1.53	
107	4	13.0	19.6	1.51	
11	4	18.9	38.4	2.03	
Ren. 1	4	10.4	14.7	1.41	
Ren. 2	4	9.3	9.8	1.05	
				1.41	1.28
Tu. 1	8	3.6	3.6	1.00	
Tu. 2	8	4.6	6.0	1.30	
Tu. 3	8	8.7	10.8	1.24	
				1.18	1.18
101	15	8.0	8.1	1.01	
102	15	8.4	8.0	.95	
106	15	9.3	10.0	1.08	
107	15	17.3	18.3	1.06	
11	15	33.7	31.5	.93	
12	15	34.9	35.0	1.00	
				1.01	1.09

APPENDIX B

EVOLUTION OF ACCURACY WITH THE NUMBER OF CONTROL POINTS

One particular computational method will be considered: Resections in space of the two bundles with direct linear transformation (collinearity equations and eleven unknown parameters per bundle) and then intersection of homologous rays. The resections are computed from n control points whose coordinates we assume to be exactly known. It is an experimental fact that the final accuracy of the object points depends on the number and relative disposition of the n control points. Here we try to determine if there is a simple law accounting for the corresponding accuracy variation.

The theoretical derivation is performed as follows: For each bundle the eleven unknown parameters $p_1, p_2, \ldots, p_{11}$ are determined from the least-square solutions of the observation equations

$$x_i = \frac{p_1 X_i + p_2 Y_i + p_3 Z_i + p_4}{p_9 X_i + p_{10} Y_i + p_{11} Z_i + 1}, \quad y_i = \frac{p_5 X_i + p_6 Y_i + p_7 Z_i + p_8}{p_9 X_i + p_{10} Y_i + p_{11} Z_i + 1}$$

where X_i, Y_i, and Z_i are the spatial coordinates of the i^{th} control point and x_i and y_i are the plate coordinates.

The precision of the estimations of the eleven parameters is the variance (matrix)

$$\Gamma_P \text{ with } P = \begin{Vmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_{11} \end{Vmatrix}$$

But this is not the accuracy. Then how is the accuracy estimated?

In the case of a one-dimensioned quantity estimated from n independent measurements, the mean-square-error $\sqrt{Ee^2}$ of the mean of n measurements can be written as

$$Ee^2 = E\,\beta^2 + \sigma^2/n,$$

σ being the standard deviation of replicated measurements and $\sqrt{E\,\beta^2}$ being the RMS bias of the measuring system, which includes the estimated quantity. If systematic errors are well corrected, it can be assumed* that

$$\sqrt{E\,\beta^2} = \sigma$$

and subsequently

$$Ee^2 \simeq (1 + \tfrac{1}{n})\,\sigma^2 = (1 + \tfrac{n}{1})\,(\tfrac{\sigma^2}{n}). \qquad \text{(B-1)}$$

In the case of an r-dimensioned quantity P, estimated from n $> r$ measurements, we can generalize Equation B-1 as follows:

$$E\,e_P\,e_P{}^T \simeq (1 + \tfrac{n}{r})\,\Gamma_P \simeq (1 + \tfrac{r}{n})\,\Gamma_o \qquad \text{(B-2)}$$

where $e_p = P - \dot{P}(\dot{P} =$ true value of $P)$, $\Gamma_P =$ variance of P, and $\Gamma_o = n/r\ \Gamma_P$, the mean variance matrix of all elementary determinations of P (r measurements instead of n and control nets with neighboring geometrical characteristics). Γ_P is of course proportional to σ^2, σ being the RMS error of comparator measurements. Thus we can rewrite Equation B-2 as

$$E\,e_P\,e_P{}^T = \Gamma'_o\left[(1 + \tfrac{r}{n}\right]\sigma^2 \qquad \text{(B-3)}$$

with Γ'_o independent of σ. This can be adopted as the accuracy of the resection. Thus, one can conclude that the simplest way to account for the bias due to the resections operations is to substitute for all accuracy questions the value $\sqrt{1 + \frac{r}{n}}\,\sigma$ for the value σ.

As a consequence, the RMS spatial residual R_nXYZ (resections with n control points), which can be computed from check measurements after intersection of homologous rays and which we know to be proportional to the RMS error of the comparator measurements, is proportional to $\sqrt{1 + \frac{r}{n}}\,\sigma$, i.e.,

$$R_nXYZ = k\,\sqrt{1 + \tfrac{r}{n}}\,\sigma$$
$$R_nXYZ = \sqrt{1 + \tfrac{r}{n}}\,R_o. \qquad \text{(B-4)}$$

THE TESTS

The raw results of the tests are recorded in Tables B-1, B-2, and B-3. Six different photo-pairs and three different base-to-object distance ratios (0.86 for 101 and 102, 0.33 for 106 and 107, 0.14 for 11 and 12) were employed. Table B-1 includes results for the direct linear transformation method, the principle of which was given earlier. Table B-2 presents results for the method of relative orientation followed by a least-squares adjustment to a control net. Table B-3 concerns the exclusive problem of relative orientation (number and definition of image points). The accuracy criterion is the RMS spatial residual $rXYZ$ (μm) referred to the image plane in Tables B-1 and B-2, and the RMS residual parallax (μm) for Table B-3, computed from more than 100 distinct check targets. For each value of n (number of control points for

* This can be justified as follows: It can be thought that the heart of the measuring process is the realization of a coincidence between two "spots", one of the measuring device and one of the measured quantity. In fact, a spot never appears to the observer as a point (mathematical concept) but as a brilliance gaussian distribution with a standard deviation σ_c; and two different spots are known as distinct only when the sum of the two curves present a central minimum, that is, if the distance between the two spots is greater than 2-3σ_c (Figure B-1). In other words, the definition of the spot of the measured quantity is defined with an RMS bias of σ_c. In addition, for the same reasons, the observation error for the coincidence also has a standard deviation σ_c, which we can assume for a well elaborated, precise, and calibrated measuring device to be equal to σ (standard deviation of the elementary replicated observations). Finally the RMS bias is equal to σ.

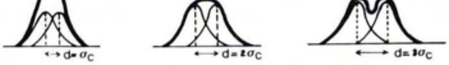

FIG. B-1. Gaussian distribution of a measurement.

TABLE B-1. RESECTIONS IN SPACE FOLLOWED BY INTERSECTION OF HOMOLOGOUS RAYS. RMS SPATIAL RESIDUAL *rXYZ* (REFERRED TO THE IMAGE PLANE) WHEN VARYING THE NUMBER AND DEFINITION OF CONTROL POINTS.

		rXYZ (unit:μm)					
		UMK pairs				TMK pairs	
n	p	101	102	106	107	11	12
7	1	7.6	7.5	10.9	18.0	57.1	52.5
	2	7.1	7.3	11.5	16.7	38.6	60.3
	3	7.1	7.1	11.3	17.4	44.4	58.4
	4	7.1	7.5	11.2	17.0	45.4	49.1
10	1	7.7	7.2	9.5	15.4	32.9	34.5
	2	6.8	7.3	8.8	14.4	30.1	34.0
	3	7.0	7.1	9.0	14.4	30.2	33.4
	4	6.9	7.0	9.4	12.56	30.2	35.5
15	1	6.3	7.4	9.0	15.3	29.8	33.2
	2	6.2	6.6	8.9	13.6	30.0	33.4
	3	6.0	6.7	9.1	13.7	29.8	33.0
	4	5.6	6.6	10.0	11.2	28.6	34.1
28	1	5.5	6.9	*9.1	*15.0	28.9	35.3
	2	5.8	6.3	8.7	13.6	29.5	33.1
	3	5.7	6.5	9.3	14.9	29.1	33.8
	4	5.4	6.6	9.3	13.6	28.6	33.0

* (23 control points instead of 28)
n: number of control points
p: number of neighbouring targets per control point

TABLE B-2. RELATIVE ORIENTATION FOLLOWED BY LEAST-SQUARE ADJUSTMENT WITH *N* CONTROL POINTS. RMS SPATIAL RESIDUAL (REFERRED TO THE IMAGE PLANE) WHEN VARYING NUMBER AND DEFINITION OF CONTROL-POINTS.

		rXYZ (unit:μm)					
		UMK pairs				TMK pairs	
n	p	101	102	106	107	11	12
4	1	9.3	8.8	10.4	19.6	36.2	39.8
	2	8.9	10.2	12.9	19.7	34.3	41.6
	3	9.0	9.4	11.8	19.1	33.4	40.9
	4	8.9	9.2	11.8	17.6	33.9	40.3
6	1	8.2	8.0	11.8	19.6	38.4	53.8
	2	7.7	7.8	11.3	29.0	42.7	44.1
	3	7.7	8.0	11.5	22.8	45.4	43.1
	4	7.6	8.1	12.0	24.3	43.3	43.5
10	1	8.6	8.5	10.2	18.9	32.4	37.3
	2	8.5	8.9	10.1	17.0	31.0	36.5
	3	8.5	8.8	9.6	15.2	31.1	36.1
	4	8.4	8.7	9.3	13.8	30.3	36.5
15	1	8.1	8.0	10.0	18.3	31.5	35.0
	2	7.6	7.9	9.8	16.9	32.1	35.3
	3	7.7	7.8	9.2	14.8	32.8	36.2
	4	7.3	7.9	9.2	13.0	31.7	36.5
28	1	7.6	7.8	*10.1	*18.7	31.6	36.5
	2	7.6	7.5	10.0	18.3	30.5	33.3
	3	7.7	7.7	9.7	16.7	31.0	33.1
	4	7.5	7.9	10.3	19.2	30.9	33.1

n, *p*, * see Table B-1.

tests B-1 and B-2 and number of image points for test B-3), four trials have been made with 1, 2, 3, and 4 neighbouring targets per object point, respectively.

One can make the following remarks:

- There is no significant change when each control point (test B-3, image point) is defined with several neighboring targets. In other words, there is no significant improvement in the accuracy when demultiplying each ray (Figure B-2). This is not surprising if we note that the measuring system is in fact invariant. We are in the situation of an observer comparing the means of different measurement series: The random errors are eliminated and the bias is the same for each series.
- There is a significant improvement in the accuracy when increasing the number of control points (tests B-1, and B-2) or relative orientation points (test B-3), on the condition of an equi-repartition in the whole object-volume of the supplementary points.

Now the law of accuracy improvement we should test is the law given by Equation B-4, i.e.,

$$r_n XYZ = \sqrt{1 + \frac{r}{mn}}\, r_o \qquad \text{(B-5)}$$

where r is the number of unknown parameters, n is the number of control (or image) points, and m is the number of equations per point.

(1) Test B-2 (Resections in space followed by intersection of homologous rays).

In Test B-1, $r = 22$ (11 parameters per bundle) and $m = 4$, and Equation B-5 becomes

$$r_n XYZ = \sqrt{1 + \frac{11}{2n}}\, r_o. \qquad \text{(B-6)}$$

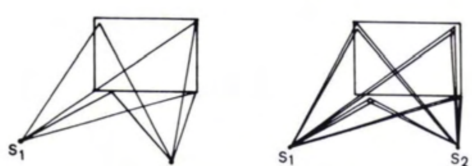

FIG. B-2. No significant improvement in accuracy when demultiplying each ray.

TABLE B-3. RELATIVE ORIENTATION FOLLOWED BY LEAST-SQUARE ADJUSTMENT WITH N CONTROL POINTS. RMS RESIDUAL PARALLAX WHEN VARYING NUMBER AND DEFINITION OF IMAGE-POINTS.

		RMS residual parallax (unit: μm)					
		UMK pairs				RMK pairs	
n	p	101	102	106	107	11	12
	1	8.0	8.3	11.5	6.8	16.0	13.3
4	2	5.9	6.9	11.5	6.5	14.1	13.5
	3	5.4	7.0	12.0	11.6	13.6	
	4	5.4	6.7	12.5	6.5	11.7	13.1
6	1	5.6	7.3	6.1	5.8	13.3	13.0
	2	5.5	7.4	6.5	5.7	14.5	13.7
	3	5.5	—	6.3	5.8	12.6	13.2
	4	5.3	6.7	6.0	5.7	13.4	12.7
10	1	5.4	7.3	6.1	5.8	15.4	14.0
	2	5.4	7.4	6.5	5.7	14.4	14.0
	3	5.4	6.9	6.4	5.5	12.6	13.2
	4	5.3	6.7	6.0	5.5	12.0	12.8
15	1	6.0	6.2	5.8	5.4	14.1	14.0
	2	5.5	6.2	6.1	5.4	13.0	12.9
	3	5.3	6.1	6.1	5.4	11.9	12.7
	4	5.3	6.1	5.9	5.4	11.7	12.6
28	1	5.5	6.1	*5.9	*5.3	13.7	14.2
	2	5.3	6.0	5.8	5.5	12.1	13.1
	3	5.2	5.8	5.7	5.4	11.5	12.9
	4	5.2	5.8	5.7	5.4	11.5	13.0

* (23 image-points instead of 28)
n: number of image-points
p: number of neighbouring targets per image-point

For each pair, and each value of n, the values of r_nXYZ (Table B-1) have been summarized with their mean. Then r_o is estimated from

$$r_o = \frac{1}{4}\left[\frac{r_7}{\sqrt{1+\frac{11}{2\times 7}}} + \frac{r_{10}}{\sqrt{1+\frac{11}{2\times 10}}} + \frac{r_{15}}{\sqrt{1+\frac{11}{2\times 15}}} + \frac{r_{28}}{\sqrt{1+\frac{11}{2\times 28}}}\right].$$

If the law (Equation B-6) is correct, one should have

$$\frac{r_nXYZ}{r_o} \simeq \sqrt{1+\frac{11}{2n}}$$

for the six photo-pairs. The results are given in Table B-4.

(2) Test B-2 (Relative orientation with 15 image points followed by least-square adjustment to n *control points)*

In Test B-2, $r = 7$ (translation, similitude) and $m = 3$, and Equation B-4 becomes

$$r_nXYZ = \sqrt{1+\frac{7}{3n}}\ \mathrm{r}_o \tag{B-7}$$

The results are given in Table B-5.

(3) Test B-3 (Relative orientation with n *image-points and adjustment to the control net with 15 control points.*

In Test B-3, $r = 5$ (parameters of the relative orientation) and $m = 1$, and Equation B-4 becomes

$$p_n = \sqrt{1+\frac{5}{n}}\ p_o \tag{B-8}$$

where p_n is the RMS residual parallax and p_o is the minimum parallax. The results are given in Table B-6.

It is interesting to note that, one photo-pair excepted, there is no significant improvement

TABLE B-4 (TEST B-1). TEST OF THE LAW (EQUATION B-6) (r = RMS SPATIAL RESIDUAL FOR n CONTROL POINTS; r_o = MAXIMUM ACCURACY)

Photopair reference		n (number of control points) 7	10	15	23	28	r_o
101	r	7.21 μm	7.10	6.14		5.75	5.38 μm
	r/r_o	1.34	1.32	1.14		1.07	
102	r	7.34 μm	7.15	6.83		6.57	5.78 μm
	r/r_o	1.27	1.24	1.18		1.14	
106		11.21	9.18	9.25	9.10		7.97
		1.41	1.15	1.16	1.14		
107		17.28	14.19	13.46	14.30		12.18
		1.42	1.17	1.11	1.17		
11			30.88	29.56		29.05	25.61
			1.21	1.15		1.13	
12			34.38	33.43		33.82	29.11
			1.18	1.15		1.16	
$\overline{(r/r_o)}$		1.36	1.21	1.15	1.15	1.13	
$\sqrt{1 + 11/2n}$		1.34	1.24	1.17	1.11	1.09	

TABLE B-5 (TEST B-2). TEST OF THE LAW (EQUATION B-7) (r = RMS SPATIAL RESIDUAL FOR N CONTROL POINTS; r_o = MAXIMAL ACCURACY)

Photopair reference		n (number of control points) 4	6	10	15	23	28	r_o
101	r	9.03 μm	7.80	8.50	7.68		7.60	7.18 μm
	r/r_o	1.26	1.09	1.18	1.07		1.06	
102	r	9.40 μm	7.98	8.73	7.90		7.73	7.38 μm
	r/r_o	1.27	1.08	1.18	1.07		1.05	
106		11.73	11.85	9.80	9.55	10.03		9.30
		1.26	1.25	1.05	1.03	1.08		
107		19.00	22.18	16.29	15.75	18.23		16.12
		1.18	1.38	1.01	0.98	1.13		
$\overline{r/r_o}$		1.21	1.20	1.11	1.04	1.10	1.05	
$\sqrt{1 + 7/3n}$		1.26	1.18	1.11	1.07	1.05	1.04	

TABLE B-6 (TEST B-3). TEST OF THE LAW (EQUATION B-8) (p = RMS RESIDUAL PARALLAX FOR n IMAGE-POINTS; p_o = MINIMAL RMS PARALLAX).

Photopair reference		n (number of points for relative orientation) 6	9	10	15	23	28	p_o
101	p	6.18 μm	5.48	5.37	5.53		5.30	4.61 μm
	p/p_o	1.34	1.19	1.17	1.20		1.15	
102	p	7.23 μm	7.13	7.08	6.15		5.93	5.53 μm
	p/p_o	1.31	1.29	1.28	1.11		1.07	
106			6.23	6.25	5.98	5.78		
			1.21	1.22	1.16	1.12		
107		6.55	5.75	5.63	5.40	5.40		
		1.38	1.21	1.19	1.14	1.14		
11		13.35	13.20	13.60	12.68		12.08	10.74
		1.24	1.23	1.27	1.18		1.12	
12		13.38	13.15	13.50	13.05		13.30	11.01
		1.22	1.19	1.23	1.19		1.21	
$\overline{p/p_o}$		1.30	1.22	1.23	1.16	1.13	1.14	
$\sqrt{1 + 5/n}$		1.35	1.25	1.22	1.15	1.10	1.09	

in the accuracy of final determinations in space when increasing the number of image points. For the exception, however, the improvement was about 50 per cent.

I explain this fact as follows: the criterion of the RMS residual parallax is not an accuracy criterion on the model, because slight errors in focal length values or principal points positions do not always significantly change the RMS residual parallax, which can be small despite residual systematic effects. The improvement of the RMS parallax with the number of image points only indicates a better realization of the coplanarity of homologous rays, not necessarily a better realization of the model.

Engineering Reports

Business—Equipment—Literature

Gordon R. Heath

ALTEK Corporation Announces New Digital Scaling Device

No longer do you have to rely upon mechanical counters subject to wear and limited scaling ability, according to ALTEK Corp. Their Model AC 98 offers a six-digit and sign display with presetting and a five-digit scaler. With a rotary or linear pulse encoder, the AC 98 may be used to measure the amount of material produced in units such as inches, feet, yards, meters or square inches, square feet, etc. (i.e., length of steel rod, square feet of paper). The encoder is mounted to a pulley or belt assembly on a production line. The AC 98 will count pulses transmitted by the encoder and display the amount. Calibration is easily performed by adjusting the scaling switches. Once the system is calibrated, production rate and volume may be controlled by quick visual inspection. System reliability far exceeds mechanical type counters. As an expansion on the AC 98, the digital readout may be fed into a control system for automatic production measurement and control.

Another application involves precise measurement devices such as coordinatographs or stereoplotters. The AC 98 with encoder attachment may be used to display X, Y, or Z axis data scaled to real values directly from maps or drawings. The AC 98 costs $950.00 in single unit quantities. Lower priced readouts without scaling are also available.

For further information contact ALTEK Corporation, 2141 Industrial Parkway, Silver Spring, Maryland, 20904. (301) 622-3906

Engineers Joint Council Urges Adoption of Metrication System in Senate Commerce Committee Hearings

Engineers Joint Council has again called for the replacement of the present traditional forms of measure in use in the United States by a single metric system. The recommendation was made during the current round of hearings on metric legislation being held by the Senate Committee on Commerce. William G. McLean, Vice Chairman of the Council's Metric Commission, reiterated a resolution passed by the EJC Board on November 14, 1974 which endorsed "the adoption of a single system of measurement in the United States" and identified that system as "the International System of units commonly known as SI, as described by the resolutions of the General Conference of Weights and Measures."

Dept. of Interior Announces Three New Maps

(1) Land use maps and data for 65,000 square miles of the Ozarks region in Arkansas, Missouri, and Oklahoma have been completed by the U.S. Geological Survey, Department of the Interior, in cooperation with the Ozarks Regional Commission. The Ozarks was one of the first areas to have maps produced in a nationwide program by the Survey's recently established Land Information and Analysis (LIA) Office to complete land use mapping for all parts of the United States within six to seven years. Dr. James R. Anderson, Chief Geographer for the USGS, said land use maps are designed for use mainly by local, State, Federal, and private planners and decision makers to enable them to make better use of land and other resources while still protecting the environment.

(2) A space view of Louisiana from about 570 miles above the Earth has been prepared by the U.S. Geological Survey, Department of the Interior, from pictures recorded by a satellite. The black and white space mosaic of the Pelican State was made from parts of 11 images recorded by a multi-spectral line scanner on NASA's LANDSAT-1 Earth resources survey satellite in March, April, and May of 1973. Each of the 11 individual LANDSAT images covered 115-by-115 miles (13,225 square miles) of the Earth's surface.

(3) Land use maps and data for all of Louisiana are now available to State, regional, and local planners and decision makers as well as the general public, the U.S. Geological Survey, Department of the Interior, announced. Louisiana is also one of the first States to be completed in a six to seven year program by the USGS to produce land use maps covering the entire Nation. It was the first State to use such maps in an emergency situation when Louisiana officials last spring used the land use maps, along with satellite pictures of the State, to determine the number of acres of various types of land inundated by floods along the Mississippi, Red, Ouachita, Black, and Atchafalaya rivers.

Prospectus Offered of a Major Study in Computer Cartography

A comprehensive study, *Computer Cartography: Worldwide Technology and Markets,* is being offered by International Technology Marketing, Newton, Massachusetts. The study covers applications of computer-aided cartography by government agencies, utilities, industry, and university