Kunwar K. Rampal
*The Ohio State University*
*Columbus, OH 43210*

# Least Squares Collocation in Photogrammetry

## Least squares collocation is capable not only of determining random errors (noise) but also of discriminating systematic errors (signal).

### Introduction

The method of least squares interpolation, known sometimes as collocation, has been extensively applied to the solution of problems related to geodesy. Recently this powerful method has been applied to photogrammetry.[2,3,9]. The backbone of the method is the so-called covariance function, which takes into account the effect of correlation between observations. In the classical least squares solution the absence of knowledge of this correlation allows the use of diagonal elements only in the weight matrix. As a result, the residual systematic errors (not modeled mathematically) remain unaccounted for in the solution of photogrammetric problems such as the transformation of comparator to photo coordinates, and single-photo resection and multi-photo intersection (aerial triangulation) based on the projective equations. This can also be considered as modeling error even though such systematic errors as film deformation, atmospheric refraction, and lens distortion (radial and tangential) have already been accounted for. These corrections are applied to the photo coordinates or modeled in the mathematical projective equations of collinearity.

Abstract: *Generalized least squares, often called collocation, has been applied to the resection and transformation problems of photogrammetry. It has been shown that, in the contrast to the least squares theory, the results of the adjustment can be improved by use of the proper covariance function for correlated observations. The generalized least squares theory is capable of extracting both the signal and the noise portions of the observations whereas the classical least squares deals only with the random (noise) portion.*

Consider in general a function F which is dependent upon a set of parameters. The dependency may be linear or non-linear. Let us assume we have a set of observed values $\boldsymbol{x}$ so that we can write an observation equation in the linearized form

$$\boldsymbol{x} = F(\mathbf{X}) = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{A}$ is the coefficient matrix, also called the design matrix; $\mathbf{X}$ is the correction matrix, i.e., the corrections to the assumed values of the parameters; and $\boldsymbol{n}$ is the matrix of observation errors.

Equation 1 is the well-known equation for the least squares solution. However, the inadequacy of the model to describe the observed physical phenomenon can be compensated by including in Equation 1 an additional term $\boldsymbol{s}$ called *signal* so that the mathematical curve $\boldsymbol{AX}$, shown by the solid line in Figure 1, is replaced by the so called *trend* curve. We have, now, a mathematical equation

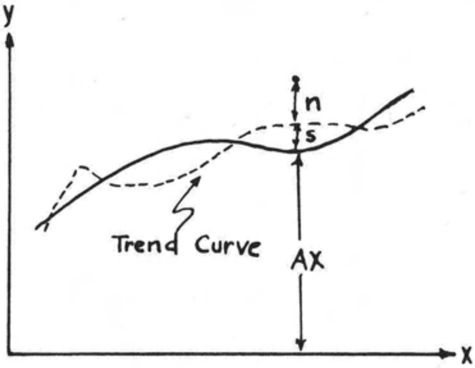


FIG. 1. Trend curve.

$$x = AX + n + s \tag{2}$$

The solution of Equation 1 by least squares is $X = (A'PA)^{-1} \; A'Px$ (3)

where **A** is the matrix of partial derivatives of the function F with respect to its parameters, **A'** is the transpose of the **A** matrix, **x** is the matrix of observations, and **P** is the weight matrix of observations. Since the correlation between observations is now known in a least squares solution, **P** is a diagonal matrix with elements proportional to $1/\sigma^2_i$ where $\sigma_i$ is the standard error of the $i$th observation. Equation 2 is the generalized observation equation and its solution, as given by Moritz[1], is

$$X = (A'C^{-1}A) \; A'C^{-1}x \tag{4}$$

where $C$ is the covariance matrix and takes into account the correlation between observations. In addition, the theory of collocation[1] gives $s$ as

$$s = C_s C^{-1}(x - AX) \tag{5}$$

where $C_s$ is the covariance matrix for the correlation between the points of prediction (where signal is to be predicted) and the observation points. It is determined in a manner similar to that of $C$. The second term of Equation 5, that is, $AX$, is the effect of correlation which was neglected in Equation 3. If there is no correlation between the signal and the noise, as is generally the case, then we can write $C = C_{ss} + C_{nn}$ where $C_{nn}$ is the covariance matrix due to the noise part and $C_{ss}$ is the covariance matrix of the signal at known points or data points. If we take $C_{ss}$ as a null matrix, the collocation solution given by Equation 4 reduces to that of least squares as given by Equation 3. Hence the knowledge of $C_{ss}$ or the covariance matrix of observations is the crux of the whole problem.

## The Covariance Function

The information about the correlation between the observations forming $C_{ss}$ may be obtained empirically based on previous observations or through analytical procedure. Whatever the function describing the correlation, it has to be a function of the distance between the points. Based on our experience, points close together have a greater probability of correlation than those which are further apart. If F($s$) is a continuous function describing the signal, then the covariance function describing correlation between two points $P(x_1,y_1)$ and $Q(x_2, y_2)$, separated by a distance $R$ and averaged over a total separation of 2a, is given by

$$C(R) = \frac{1}{2(a-R)} \int_{R-a}^{a-R} F(s)\, F(s+u)\, du \; \cdot \tag{6}$$

One such function, which is a function of distance between two points and is positive, is given by

$$C(R) = C_o \exp(-a^2R^2) \tag{6a}$$

where $C_o$ and $a$ are constants—determined, known experimentally, or otherwise—and $R$ is the distance between $P$ and $Q$. These values are very often obtained by fitting the exponential curve to the observed values of $x$ at the data points. The expression for $\mathbf{C}$ must represent a positive definite function because all admissible covariance functions are positive definite.[1].

## Covariance Function for Photogrammetric Case

In order to develop the covariance function for photogrammetric distortions, we note that the distortion in the $x$ or $y$ direction at any point of the photographic plate may be represented as a continuous function of the coordinates $x_p$ and $y_p$ of the image points referred to the fiducial system. Let $s$ be the total signal (distortion) at a point $P$ and $s'$ at a point $P'$ such that the distance $PP' = R$. To determine the covariance $\mathbf{C}(R)$ we get the product $ss'$ and then take the average of the product $ss'$ for all pairs of points situated in the area under consideration which are at a constant distance $R$ apart. Let us consider a photographic plate of dimensions $2a$ by $2b$, with $O$ as the fiducial center and the $x$ and $y$ directions of axes as shown in Figure 2. In order to form the average we consider a circle drawn with $P$ as center and radius $R$ so that $P$ moves along the circle in the first step. In the second step we let $P$ move parallel to side $2b$ and in the third step we let it go at a distance $R$ apart from side $2a$. Let $\alpha$ be the angle as shown in Figure 2 so that if $x_p$ and $y_p$ are the rectangular coordinates of point $P$, those of $P'$ are $x_p + R\cos\alpha$ and $y_p + R\sin\alpha$. Then we have

$$\mathbf{C}(R) = \frac{1}{8\pi(a-R)(b-R)} \int_{R-a}^{a-R} \int_{R-b}^{b-R} \int_{0}^{2\pi} \mathrm{F}(x_p, y_p)\,\mathrm{F}(x_p + R\cos\alpha, y_p + R\sin\alpha)\,dx\,dy\,d\alpha \tag{7}$$

At this stage we would like to get a mathematical expression for F to carry out the integration in Equation 7. For this we make use of the fact that most of the photogrammetric distortions such as refraction, film distortion, and lens distortion (except the asymmetrical component of tangential distortion) are a continuous function of $OP$, the radial distance. Let $s_x$ and $s_y$ be the components of the signal $s$ ( the distortion) at a point $P$ so that $PP = s$, and $OP = r$. Since the numerical values of $s$ are known to be small compared to $r$, the radial distance, we have from Figure 3

$$s_x/s = x/r \tag{8}$$

$$s_y/s = y/r \tag{9}$$

$$s = (s^2_x + s^2_y)^{1/2} \tag{10}$$

$$r = (x^2 + y^2)^{1/2} \tag{11}$$

Taking the partial derivatives of Equations 8 and 9 with respect to $x$ and $y$, it can be shown[8] that

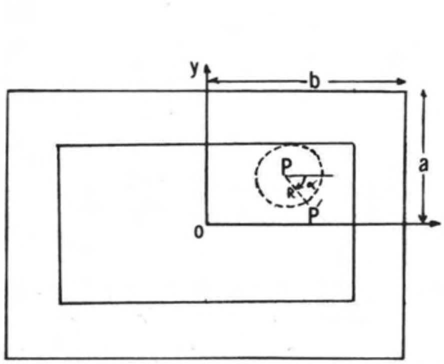

FIG. 2. Photo axes.

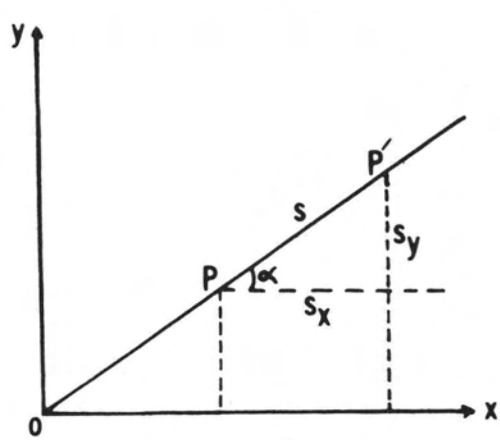

FIG. 3. Distortion components.

$$\frac{\partial^2 s_x}{\partial x^2} + \frac{\partial^2 s_x}{\partial y^2} = 0 \cdot \tag{12}$$

Equation 12 is the well-known Laplace differential equation of 2nd order. The solution is given by

$$s_x = \sum_{n=o}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)\, (r/K)^n \tag{13}$$

where $A_n$, $B_n$, and $k$ are arbitrary constants to be determined from boundary conditions. Similarly for $s_y$ we have

$$s_y = \sum_{n=o}^{\infty} (A'_n \cos n\theta + B'_n \sin n\, \theta) (\frac{r}{K})^n \tag{14}$$

where $\cos\theta = x/r$ and $\sin\theta = y/r \cdot$

We go back to Equation 7 to evaluate the covariance function using Equations 13 and 14 for $x$ and $y$ respectively:

$$C(R) = \frac{1}{8\pi (a-R)(b-R)} \int_{x=R-a}^{a-R} \int_{y=R-b}^{b-R} \int_{\alpha=o}^{2\pi} \left\{ \sum_{n=o}^{\infty} (A_n \cos n\, \theta + B_n \sin n\, \theta) \, (\frac{r^n}{k}) \right\}$$
$$\left\{ \sum_{n=o}^{\infty} A_n \cos n\theta' + B_n \sin n\theta' \right) (\frac{r'^n}{K}) \right\} \, dx dy d\alpha \tag{15}$$

The integration in Equation 15 can be performed term by term, retaining as many terms of the infinite series as contribute significantly to the covariance function.

It can be shown that the contribution up to nine terms for the case $a = b$ is given by

$$C(R) = A_o^2 + 1/3\, (A_1^2 + B_1^2) (\frac{a-R}{K})^2 + \frac{1}{45} \left[8A_2^2 + 10B_2^2\right] (\frac{a-R}{K})^4$$
$$+ \frac{1}{70} \left[ 49\, A_3^2 + 48\, B_3^2 \right] (\frac{a-R}{K})^6 + \frac{1}{1575} \left[944\, A_2^4 + 384\, B_2^4\right] (\frac{a-R}{K})^8 \tag{16}$$

Equation 16 is, therefore, a required covariance function for the $x$ coordinate of the photogrammetric case. We notice that

- The covariance function is positive as all terms are even powers;
- It is isotropic, i.e., the function depends only upon the distance between the points;
- Its numerical value decreases as the distance $R$ between the two correlated points increases; and
- when we reach the edge of the photo, i.e., when $a = R$, the covariance function reaches its minimum value and becomes a constant.

## Numerical Determination of $C$(R)

The first step is to determine the $A$ and $B$ parameters. This can be done in exactly the same way as in physical geodesy where the covariance function is determined from a knowledge of the values of the potential harmonic coefficients solved from a set of simultaneous equations formed by observation of gravity or gravity anomalies over a large number of well-distributed points over the globe. An exact parallelism exists in the photogrammetric case. The procedure is as follows:

(a) Select a photo which has the required density (say 10 to 15 points per photo) of well distributed control points;

(b) Perform a single photo resection and determine the residuals $e_x$ and $e_y$ for the photo coordinates by least square adjustments;
(c) Take these residuals as the best estimates for the determination of $A$'s and $B$'s by the following set of equations:

$$(e_x)_n = \sum_n (A_n \cos n\theta) + B_n \sin n\theta \quad (\frac{R}{K})^n \tag{17}$$

$$(e_y)_n = \sum_n (A'_n \cos n\theta + B'_n \sin n\theta) \quad (\frac{R}{K})^n \tag{18}$$

where the other symbols have the meaning as explained earlier. If necessary two or more photos can be used to determine the $A$'s and $B$'s (if the necessary control exists) and a weighted mean of the $A$'s and $B$'s taken from the least squares adjustment.

It might be argued that, instead of determining the $A$'s and $B$'s in this fashion, why not include them as parameters in the original projective equations and solve for them along with the other parameters like $X_o$, $Y_o$, $Z_o$, $\kappa$, $\varphi$ and $\omega$. The objections to this procedure are (1) in the first instance it increases the number of unknowns considerably, and (2), if the control available pertains to flat or nearly flat terrain, the solution will be come indeterminant. Here the matrix $\boldsymbol{N} = \boldsymbol{A'PA}$ becomes ill conditioned and hence almost impossible to invert. The only way is to use either control data of mountainous terrain, or methods where the ill conditioning can be avoided, for example, by using the method suggested by Uotila[7]. This involves adding a term $\mathbf{A'ZA}$ to the ill conditioned matrix so that the solution now is given by

$$\mathbf{X} = [\mathbf{A}'(\Sigma_{L_b} + \mathbf{AZA}')^{-1} \mathbf{A}\ ]^{-1} \mathbf{A}'\ (\Sigma_{L_b} + \mathbf{AZA}')\, \boldsymbol{x} \tag{19}$$

where $\mathbf{Z}$ is an arbitrary matrix but satisfying the condition that the rank of $(\Sigma_{L_b} + \mathbf{AZA}')$ matrix is equal to the rank of the augmented matrix $(\Sigma_{L_b}^{\cdot} 1\mathbf{A}) \cdot \Sigma_{L_b}$ being the variance covariance matrix of observations. The procedure adopted here is that used for Equations 17 and 18 stated above because of the simplicity of operation.

## Error Estimations

An important result of least square interpolation/collocation is that the solution is optimal in the sense that it gives the most accurate results obtainable on the basis of the available data. This can be seen from the expressions for the standard errors and error covariances of the quantities $X$ and $s$, i.e., the parameters and the signal. It has been shown by Moritz[1] that,

Table 1. Transformation of Coordinates Using Collocation

| No. | Photo No. | Point Name | $x$ Coordinate | $y$ Coordinate | $s_x$ (mm) | $s_y$ (mm) |
|---|---|---|---|---|---|---|
| 1 | 81 | AC-45 | 76.07873 mm | −108.85883 mm | 0.0 | 0.00003 |
| 2 | 81 | AD-45 | 21.36250 | −108.26322 | −0.00132 | −0.00010 |
| 3 | 81 | AE-45 | −31.73751 | −106.96785 | −0.00040 | −0.00032 |
| 4 | 81 | AF-45 | −85.34009 | −106.65100 | −0.00001 | −0.00011 |
| 5 | 81 | AC-46 | 75.61224 | − 55.44642 | 0.0 | 0.00002 |
| 6 | 81 | AD-46 | 22.21169 | − 56.06745 | 0.00042 | 0.00002 |
| 7 | 81 | AE-46 | −30.19707 | − 55.88943 | 0.00022 | −0.00015 |
| 8 | 81 | AC-47 | 70.49319 | − 3.86392 | 0.00039 | −0.00013 |
| 9 | 81 | AD-47 | 25.31277 | − 0.28166 | 0.00002 | −0.00011 |
| 10 | 81 | AE-47 | −27.26767 | 1.27908 | 0.00076 | −0.00012 |
| 11 | 81 | AF-47 | −82.71080 | 0.63158 | −0.00013 | −0.00095 |
| 12 | 81 | AC-48 | 78.90378 | 47.00334 | 0.00036 | 0.00054 |
| 13 | 81 | AD-48 | 25.86640 | 50.15817 | −0.00035 | 0.00008 |
| 14 | 81 | AE-48 | −30.62685 | 51.52508 | 0.00182 | −0.00037 |
| 15 | 81 | AC-49 | 76.49361 | 99.00907 | 0.00020 | −0.00040 |
| 16 | 81 | AD-49 | 26.09323 | 98.46003 | −0.00019 | 0.00084 |
| 17 | 81 | AE-49 | −26.82053 | 99.81833 | −0.00030 | −0.00028 |
| 18 | 81 | AF-49 | −78.58911 | 101.04509 | 0.00007 | 0.00071 |

if the quantities $X$ and $s$ are first obtained by an arbitrary linear unbiased estimation and then extended to get for the least squares collocation, the following equations result:

$$\boldsymbol{E}_{XX} = (\boldsymbol{A}'\,\boldsymbol{C}^{-1}\,\boldsymbol{A})^{-1} \tag{20}$$

$$\boldsymbol{E}_{SS} = \boldsymbol{C}_{SS} - \boldsymbol{C}_{SX}\,\boldsymbol{C}^{-1}\,\boldsymbol{C}_{XS} + \boldsymbol{HAE}_{XX}\,\boldsymbol{A}'\,\boldsymbol{H}' \tag{21}$$

$$\boldsymbol{E}_{XS} = \boldsymbol{E}_{XX}\,\boldsymbol{A}'\boldsymbol{H}' \text{ where } \boldsymbol{H} \text{ is given by:} \tag{22}$$

$$\boldsymbol{H} = \boldsymbol{C}_{SX}\,\boldsymbol{C}^{-1} \tag{23}$$

Equation 20 is the variance covariance matrix for the estimated parameters $X$ and is very similar to the one used in the least squares solution except that $\boldsymbol{C}^{-1}$ is replaced by $\boldsymbol{P}$, the weight matrix. Thus $\boldsymbol{C}^{-1}$, which is essentially the sum of the covariances matrix due to the noise part and that of signal part (assuming no correlation between noise and signal), acts like a weight matrix in the collocation technique. Equation 21 is the covariance matrix for the predicted signal. The first term $\boldsymbol{C}_{SS}$, is the covariance function like the one given in Equation 16 above. The second term $\boldsymbol{C}_{SX}\,\boldsymbol{C}^{-1}\,\boldsymbol{C}_{XS}$, is the effect of the improvement as a result of adjustment (least squares collocation) and is similar to the second term of the variance covariance term of the adjusted observations. The last term, $\boldsymbol{HAE}_{XX}\boldsymbol{A}'\boldsymbol{H}'$, represents the effect of the inaccurate estimation of the parameters, $\boldsymbol{X}$. If there were no parameters, this term would drop to zero. Equation 22 represents the variance covariance matrix or simply the cross covariance between the signal and the parameter. Equations 4, 5, 6, 20, 21, and 22 constitute the basic computational formulas for the least squares collocation, giving the estimates together with their accuracy.

## Application of Collocation to Photogrammetry

The application of collocation to photogrammetric problems is considered here for two cases: (a) transformation of coordinates and (b) single photo resection. The transformation of coordinates in the classical least squares is based on the following set of equations in matrix form:


Fig. 4. Contours for $x$ signals. Contour interval, 1 $\mu$m. Photo no. 81.

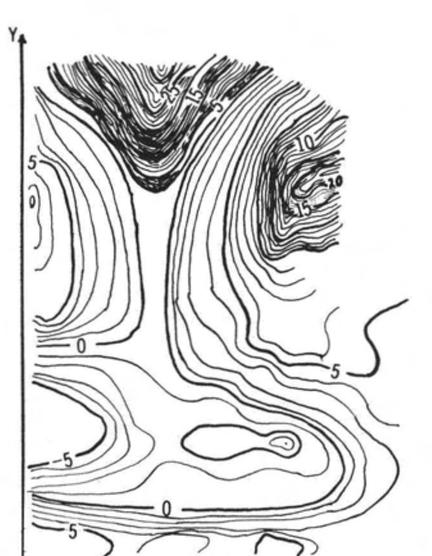


Fig. 5. Contours for $y$ signals. Contour interval, 1 $\mu$m. Photo no. 81.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x_p & y_p & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_p & y_p & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \tag{24}$$

which in abbreviated form can be written as $\boldsymbol{B} = \boldsymbol{AX}$ ·

In Equation 24 $x'$ and $y'$ are the transformed coordinates (comparator), $x_p$ and $y_p$ are the observed comparator coordinates, and $a$, $b$, $c$, $d$, $e$, and $f$ are the parameters. The least squares solution for the parameters, $\boldsymbol{X}$, is given by

$$\boldsymbol{X} = (\boldsymbol{A}'\boldsymbol{PA})^{-1}\,\boldsymbol{A}'\,\boldsymbol{Px} \tag{25}$$

Here $\boldsymbol{P}$ is the weight matrix of observation matrix, $\boldsymbol{x}$, having diagonal elements only. The solution by collocation is given by

$$\boldsymbol{X} = (\boldsymbol{A}'\boldsymbol{C}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{C}^{-1}\boldsymbol{x} \tag{26}$$

The signals, which in this case are non-linear film deformations, can be obtained from the collocation relation

$$\boldsymbol{s}_p = \boldsymbol{C}_p\,\boldsymbol{C}^{-1}\boldsymbol{Z} \tag{27}$$

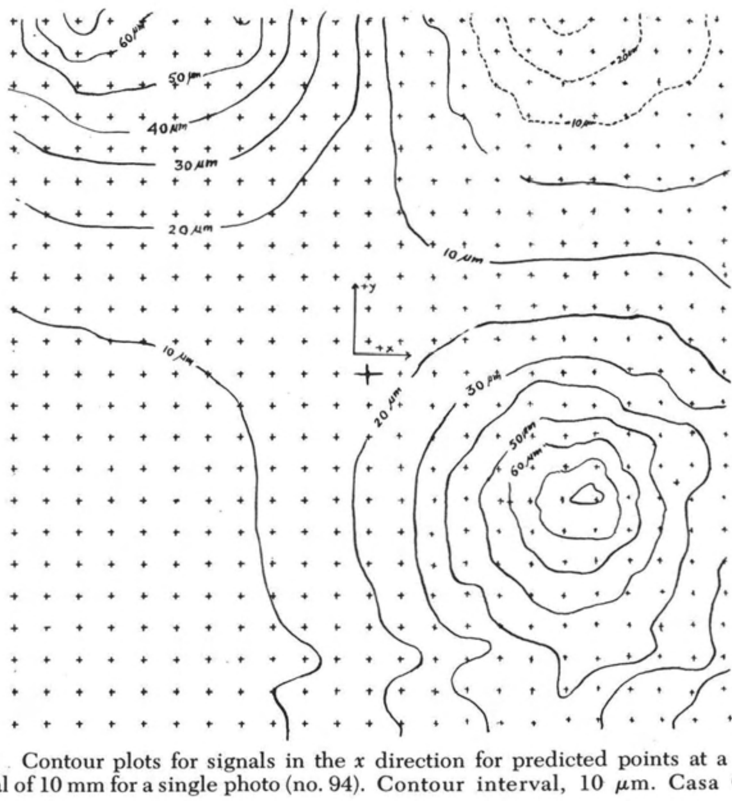


FIG. 6. Contour plots for signals in the $x$ direction for predicted points at a regular interval of 10 mm for a single photo (no. 94). Contour interval, 10 $\mu$m. Casa Grande Range.

where $\boldsymbol{Z} = \boldsymbol{x} - \boldsymbol{AX}$ is the vector of distortions (non-linear film deformation) at known points, as obtained at the reseaux corners of reseau photography. The photography used for the present investigation was taken over Casa-Grande Range with a Zeiss RMK-AR 15/23 camera with calibrated focal length of 152.01 mm at an approximate scale of 1:30.000. Table 1 gives the computed photo coordinates as obtained from collocation. The parameters of transformation, such as $a, b, \ldots, f$, were computed using Equation 20 with $\boldsymbol{C}^{-1}$ as outlined earlier. The values $s_x$ and $s_y$ are the signals in the $x$ and $y$ coordinates. It will be seen that the maximum value of the signals does not exceed 1.82 micrometers. This means that, using bracketing reseau, very little systematic film distortion remains unaccounted for in our usual transformation Equation 24.

The residual systematic distortions, such as those resulting from causes other than film deformation and not modeled in our corrections for lens distortion and refraction, etc., can be studied using the case of single photo resection. The mathematical model, in this case, is the set of well known projective equations of the type

$$x_p - x_o = c\,\frac{\left[a_{11}\,(X_p - X_o) + a_{12}\,(Y_p - Y_o) + a_{13}\,(Z_p - Z_o)\right]}{\left[a_{31}\,(X_p - X_o) + a_{32}\,(Y_p - Y_o) + a_{33}\,(Z_p - Z_o)\right]}$$

$$y_p - y_o = c\,\frac{\left[a_{21}(X_p - Xo) + a_{22}(Y_p - Y_o) + a_{23}(Z_p - Z_o)\right]}{\left[a_{31}(X_p - X_o\,) + a_{32}(Y_p - Y_o) + a_{33}(Z_p - Z_o)\right]}\;. \tag{28}$$

The solution by least squares is given by

$$\boldsymbol{X} = -\boldsymbol{N}^{-1}\boldsymbol{U} \tag{29}$$

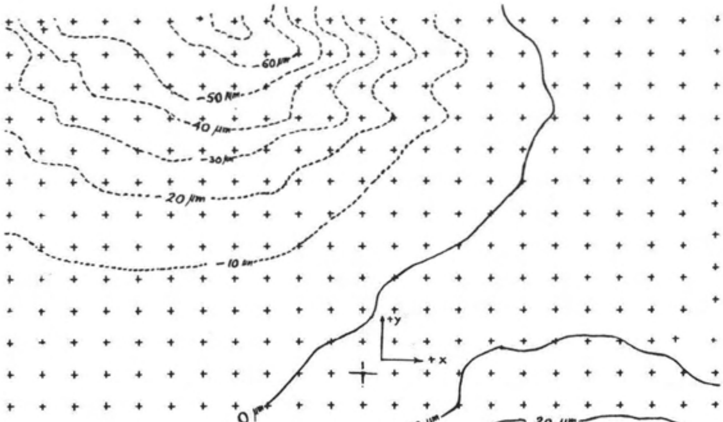


FIG. 7. Contour plots for signals in the $y$ direction for predicted points at a regular interval of 10 mm for a single photo (no. 94). Contour interval, 10 $\mu$m. Casa Grande Range.

TABLE 2. LEAST SQUARES SOLUTION FOR PARAMETERS OF SINGLE PHOTO RESECTION. CASA GRANDE RANGE. CASE OF FIVE DATA POINTS PER PHOTO.

| | Parameters (m) | | | | | | Parameters (radians) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Photo No. | $X_o$ | $Y_o$ | $Z_o$ | $\sigma_{X_0}$ | $\sigma_{Y_0}$ | $\sigma_{Z_0}$ | $\kappa$ | $\varphi$ | $\omega$ | $\sigma_\kappa$ | $\sigma_\varphi$ | $\sigma_\omega$ |
| 65 | 425430.6444 | 3628421.8254 | 5158.0184 | 6.8673 | 3.0033 | 3.6112 | −0.05229 | −0.02336 | 0.01341 | 0.00026 | 0.00147 | 0.00052 |
| 66 | 428096.5291 | 3628449.0663 | 5164.4595 | 2.3145 | 1.3767 | 0.8637 | −0.04724 | −0.03781 | 0.01752 | 0.00010 | 0.00045 | 0.00022 |
| 67 | 430729.5509 | 3628841.4345 | 5177.6959 | 0.9487 | 0.6305 | 0.7433 | −0.03938 | −0.3392 | 0.02342 | 0.00006 | 0.00021 | 0.00009 |
| 80 | 432584.9300 | 3633272.0640 | 5140.6001 | 0.9438 | 1.2337 | 0.2548 | −0.00764 | 0.02261 | −0.01027 | 0.00005 | 0.00019 | 0.00020 |
| 81 | 429963.0227 | 3633249.3609 | 5141.6330 | 2.0713 | 2.0083 | 0.8090 | 0.01531 | 0.00761 | −0.03713 | 0.00017 | 0.00037 | 0.00033 |
| 82 | 427261.6925 | 3633067.8116 | 5133.2370 | 4.3245 | 5.0660 | 2.4936 | 0.12209 | 0.00265 | −0.01954 | 0.00024 | 0.00083 | 0.00105 |
| 93 | 423133.3394 | 3635741.0078 | 5099.8418 | 3.3392 | 1.5631 | 1.4095 | −0.01243 | −0.00874 | 0.02876 | 0.00011 | 0.00071 | 0.00030 |
| 94 | 426013.9032 | 3635854.8603 | 5124.7374 | 4.3760 | 2.0810 | 1.3100 | −0.02470 | −0.03774 | 0.02730 | 0.00014 | 0.00091 | 0.00035 |
| 95 | 428747.2534 | 3635976.5357 | 5148.1555 | 2.5136 | 3.1597 | 0.9467 | −0.01502 | −0.02330 | 0.02135 | 0.00017 | 0.00042 | 0.00048 |

TABLE 3. COLLOCATION SOLUTION FOR PARAMETERS OF SINGLE PHOTO RESECTION. CASA GRANDE RANGE. CASE OF FIVE DATA POINTS PER PHOTO.

| | Parameters (m) | | | | | | Parameters (radians) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Photo No. | $X_o$ | $Y_o$ | $Z_o$ | $\sigma_{X_0}$ | $\sigma_{Y_0}$ | $\sigma_{Z_0}$ | $\kappa$ | $\varphi$ | $\omega$ | $\sigma_\kappa$ | $\sigma_\varphi$ | $\sigma_\omega$ |
| 65 | 425430.6246 | 3628421.8216 | 5158.0286 | 1.3562 | 0.6263 | 0.6831 | −0.05229 | −0.02336 | 0.01341 | 0.00004 | 0.00029 | 0.00011 |
| 66 | 428096.3058 | 3628448.9178 | 5164.3944 | 0.5442 | 0.3990 | 0.2004 | −0.04725 | −0.03785 | 0.01755 | 0.00002 | 0.00011 | 0.00007 |
| 67 | 430729.5554 | 3628481.4323 | 5177.6987 | 0.1587 | 0.1278 | 0.1230 | −0.03938 | −0.03392 | 0.02342 | 0.00007 | 0.00003 | 0.00002 |
| 80 | 432584.9145 | 3633272.1314 | 5140.5976 | 0.1858 | 0.3437 | 0.0581 | −0.00763 | 0.02261 | −0.01028 | 0.00003 | 0.00003 | 0.00005 |
| 81 | 429961.9246 | 3633258.9150 | 5143.3058 | 0.5288 | 0.8976 | 0.2592 | 0.01521 | 0.00736 | −0.03880 | 0.00001 | 0.00009 | 0.00015 |
| 82 | 427261.6981 | 3633067.8109 | 5133.2338 | 0.6270 | 0.8309 | 0.2941 | 0.12209 | 0.00265 | −0.01954 | 0.00005 | 0.00010 | 0.00016 |
| 93 | 423133.3299 | 3635740.9874 | 5099.8488 | 0.6668 | 0.4059 | 0.2928 | −0.01243 | −0.00874 | 0.02877 | 0.00002 | 0.00014 | 0.00008 |
| 94 | 426013.8964 | 3635854.8615 | 5124.7391 | 0.7535 | 0.3975 | 0.2325 | −0.02470 | −0.03774 | 0.02730 | 0.00002 | 0.00016 | 0.00007 |
| 95 | 428746.9515 | 3635986.3325 | 5148.9094 | 0.5370 | 1.2491 | 0.7910 | −0.01510 | −0.02342 | 0.01988 | 0.00004 | 0.00009 | 0.00019 |

where $\boldsymbol{X}$ is the vector of corrections to the assumed values of exterior orientation elements $X_o$, $Y_o$, $Z_o$, $\kappa$, $\varphi$, and $\omega$; $\boldsymbol{U} = \boldsymbol{A'PL}$; and $\boldsymbol{N}$ is the normal matrix ($\boldsymbol{A'PA}$). $\boldsymbol{P}$ is a diagonal matrix consisting of the variances of observations (the photo coordinates). In the collocation solution $\boldsymbol{P}$ is replaced by $\boldsymbol{C}^{-1}$, which is now a full matrix and is computed with the help of Equation 16. Table 2 gives the values of the parameters of exterior orientation as obtained from least squares and Table 3 gives the collocation solution. Although the two solutions differ slightly, the important thing to note is the information on the nature of systematic signals, that is, the residual distortions obtained as a by-product of collocation and the improvement of standard errors $\sigma$ of the parameters. Figures 4 and 5 illustrate the plot of these signals for a single photo # 81 from a set of twelve discrete data points. Figures 6 and 7 show the plot for photo # 94. In both cases the systematic pattern can be noticed. Towards the edges the signals are large on the order of 60 to 70 micrometers. This could be due to a number of causes, the two main causes appearing to be due to

(a) Errors in identification of photo points. The signalized ground marks consisted of large cross arms. In many cases it was not possible to ascertain the exact center of the mark. A full account of the ground signals has been given by Byars[10].
(b) Errors in ground control, especially vertical, due to land subsidence between the time of ground control surveys and aerial photography. The fact that there is land subsidence in the Casa Grande Test Range has been reported by Byars.

## Conclusion

Several interesting features can be studied by using the collocation technique instead of the straightforward approach by least squares. In particular, the following can be be emphasized for its application in photogrammetry:

- The use of the covariance function allows residual systematic errors to be determined instead of being absorbed in the residuals in the least squares solution.
- The values of the parameters are improved. It can be proved mathematically that the standard error of estimation of parameters by collocation technique has the least possible value (see Moritz).[1]
- Since collocation is essentially an interpolation problem, the covariance function as expressed by Equation 16 may be used in interpolation problems, such as the digitization of terrain models, as alternatives to those discussed by Schut[6]. For details the reader is referred to Rampal.[9]
- The nature of the systematic portion of the residual errors can be predicted and, therefore, the photo coordinates to be used for subsequent work can be corrected in the same way as other systematic errors such as refraction, lens distortion, and film deformation.
- Since collocation is primarily suited to the evaluation of systematic residual errors, it is recommended that this method be used for land subsidence studies by taking photography over the area at intervals.

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## References

1. Moritz, Helmut, *Advanced Least Squares Methods,* Report No. 175, Dept. of Geodetic Science, Ohio State University, Columbus, Ohio (June, 1972).
2. Mikhail, E. M., K. Kraus, "Linear Least-Squares Interpolation," *Photogrammetric Engineering,* October 1972, pp. 1016-1029.
3. Kraus, K., "Correction of Film Deformation by Least Squares Interpolation," *Photogrammetric Engineering,* May 1972, pp. 487-493.
4. Moritz, H., *Interpolation and Prediction of Gravity and Their Accuracy,* Report No. 24, Dept. of Geodetic Science, The Ohio State University, Columbus, Ohio, 1962.
5. Uotila, U. A., *Least Squares Adjustment Using Matrices,* Lecture Notes, Dept. of Geodetic Science, The Ohio State University, Columbus, Ohio.
6. Schut, G. H., "Two Interpolation Methods," *Photogrammetric Engineering,* December 1974, pp. 1447-1453.

7. Uotila, U. A., "Generalized Inverse as Weight Matrix" *International Symposium On Redefinition of North American Geodetic Networks*, May 20-25, 1974, Fredericton, N. B., Canada.
8. Rampal, Kunwar K., "Analysis and Prediction of Photogrammetric Residual Errors", *Journal of the Surveying & Mapping Division*, ASCE, Vol. 100, No. SUI pp. 33-48, May 1974.
9. Rampal, Kunwar K., *Filtering, Prediction and Interpolation in Photogrammetry*, unpublished Ph.D Dissertation, Dept. of Geodetic Science, The Ohio State University, Columbus, Ohio.
10. Byars, David D., "The Casa Grande Photogrammetric Test Range," *Proceedings of the Fall Convention*, October 26-31, 1975, American Society of Photogrammetry, Phoenix, Az.
11. Kupfer, G., "Image Geometry as Obtained from Rheidt Test Area Photography", Special Publication # 8, OEEPE, Bonn, 1975.

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