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A General Sequential Algorithm for Photogrammetric On-Line Processing

In a computer assisted comparator operation, errors may be detected and corrected as they occur.

Abstract: *Recently, powerful minicomputers have been interfaced to a variety of photogrammetric equipment for on-line processing support. The resulting capability to measure and immediately process data greatly affects the conventional procedures and mathematical models for analytical triangulation. This implies that observations may be processed one at a time rather than in groups as is currently done. This paper addresses such a processing scheme and presents a general algorithm for sequential processing of data from computer-assisted photogrammetric systems.*

Introduction

Precision stereocomparators have been interfaced to digital computers in an attempt to reduce the time required for and to improve the accuracy of photogrammetric tasks such as analytical triangulation and map compilation. The concept of all these devices is similar although there may be some hardware and software differences which provide for more efficient operation in the functional assignments with which each type is primarily concerned.

In the past, a completely comprehensive mathematical development for statistical estimation problems associated with computer-assisted comparators was not possible, due mainly to the limitations of the computer which was interfaced to the systems. With the introduction of contemporary mini-computers in the measuring system, even complex computations and data cataloging functions are possible. Therefore, it is now reasonable to address not only advantages to be gained from the implementation of a rigorous mathematical model, but also the tailoring of the mathematical model to the data flow associated with the on-line measurement process.

This paper presents background information related to on-line data processing and a derivation of a sequential algorithm for the analytical triangulation of a stereopair.

Background

An understanding of conventional data reduction procedures associated with analytical triangulation is helpful in demonstrating the advantages to be gained from a computer-assisted comparator. Conventional off-line processing is usually divided into two phases: first, data acquisition which includes primarily the measurement of image coordinates of selected ground points on all required photographs; and second, data reduction which addresses (1) the correction of measurements for known sources of error, (2) relative orientation solutions for model-by-model editing purposes, and (3) the grand analytical triangulation solution. These phases are sequential: that is, measurements are completed and then at a later date processing is initiated. If erroneous measurements are discovered in the data processing, it is necessary either to delete them and continue the processing with a reduced data set, or to repeat the measurements. The latter is a very time consuming process which can significantly disrupt the production cycle.

The conventional processing scheme is modified with a computer-assisted comparator since the measurement and some portion of the processing phases may be combined. With present-day equipment, processing up to and including the model-by-model editing may be accomplished. It is possible to emulate the conventional off-line procedure for a reduced data set with the computer-assisted comparator. In this case, the operator may perform all measurements and then wait for a short period of time while the on-line computer performs the corrections for systematic errors, the analytical triangulation solution, and the editing of image measurements. Any re-measurements which are required may be completed immediately before the stereopair is removed from the comparator. The operator is confident when this procedure is completed that re-measurement at a later date will be minimized.

Although one would expect increased output by employing the system as described above, the procedure does have some disadvantages. First, the operator must revisit any image point which was determined to be in error. This takes more time than if he were signaled that the measurement was not satisfactory just after it was obtained. Second, measurements added or deleted may require that the computations for the analytical solution be performed again. Third, the system is not being utilized while the computations are being performed. These disadvantages could be overcome if a data processing algorithm were available with which each measurement's contribution to the analytical solution was computed immediately. This implies that the contribution of one measurement may be added or deleted without repeating the calculations for all other measurements. The following sections will demonstrate that such an algorithm can be developed and will look into the feasibility of its application.

Mathematical Derivation

For computer-assisted comparators, the analytical photogrammetric model need be capable only of processing data from a small number of photographs, usually no more than two or three. This solution is accomplished by employing least squares estimation to the non-linear photogrammetric model. With the model defined, the solution is mechanical: that is, the normal equation matrix and constant vector are formed, the corrections to the parameters are computed and added to *a priori* approximations. This procedure is repeated until the corrections are insignificant.

The photogrammetric model presented by Brown[2] is a sufficient starting point for deriving an algorithm for the on-line processing associated with computer-assisted comparators. The structure of this model will be exploited to derive the sequential algorithm for the simultaneous solution of a stereopair, given that *a priori* standard deviations for all parameters are available. If required, the sequential algorithm may easily be extended to a solution of more than two photographs. In the following presentation, all the background mathematics are not given. The mentioned reference may be reviewed as required.

It can be shown that the normal equation matrix and constant column vector for the simultaneous solution of a stereopair can be used to compute the corrections to the parameters according to the matrix equation,

$$\boldsymbol{\delta} = \boldsymbol{N}^{-1}\boldsymbol{c} \tag{1}$$

where $\boldsymbol{\delta}$ is the vector of corrections to the parameters,
$\boldsymbol{N}$ is the normal equation coefficient matrix, and
$\boldsymbol{c}$ is the constant column vector.

Expanding the normal equation matrix $\boldsymbol{N}$ and constant vector $\boldsymbol{c}$ to full partitioned form we get:

$$\boldsymbol{N} = \begin{bmatrix} \dot{\boldsymbol{N}} & \overline{\boldsymbol{N}} \\ \overline{\boldsymbol{N}}^T & \ddot{\boldsymbol{N}} \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} \dot{\boldsymbol{c}} \\ \ddot{\boldsymbol{c}} \end{bmatrix} \tag{2}$$

In this equation, $\dot{\boldsymbol{N}}$ and $\ddot{\boldsymbol{N}}$ implicitly contain the weight matrices $\dot{\boldsymbol{W}}$ and $\ddot{\boldsymbol{W}}$ of the parameters, and $\dot{\boldsymbol{c}}$ and $\ddot{\boldsymbol{c}}$ implicitly contain the term $\dot{\boldsymbol{W}}\dot{\boldsymbol{\epsilon}}$ and $\ddot{\boldsymbol{W}}\ddot{\boldsymbol{\epsilon}}$ respectively.

By definition,

$$\boldsymbol{N}^{-1} = \boldsymbol{M} = \begin{bmatrix} \dot{\boldsymbol{M}} & \overline{\boldsymbol{M}} \\ \overline{\boldsymbol{M}}^T & \ddot{\boldsymbol{M}} \end{bmatrix} \tag{3}$$

and

$$\boldsymbol{NM} = \boldsymbol{I} \tag{4}$$

where $\boldsymbol{I}$ is the identity matrix of corresponding dimensions.

Multiplication of equation 4 yields four simultaneous matrix equations which when solved and substituted into equation 1 gives

$$\dot{\boldsymbol{\delta}} = \dot{\boldsymbol{M}}\,(\dot{c} - \overline{\boldsymbol{N}}\,\ddot{\boldsymbol{N}}^{-1}\,\ddot{c}) = \dot{\boldsymbol{M}}\dot{\boldsymbol{T}} \qquad (5a)$$

$$\ddot{\boldsymbol{\delta}} = \ddot{\boldsymbol{N}}^{-1}\,(\ddot{c} - \overline{\boldsymbol{N}}^{T}\,\dot{\boldsymbol{\delta}}) = \ddot{\boldsymbol{N}}^{-1}\,\ddot{\boldsymbol{T}} \qquad (5b)$$

Equations 5a and 5b indicate the following. First, the solutions can be performed in two steps. In the first the corrections to the elements of exterior orientation ($\dot{\boldsymbol{\delta}}$) are computed; in the second the corrections to each ground point ($\ddot{\boldsymbol{\delta}}$) are computed. The second step will depend on the results of the first. Second, the largest matrix which must be inverted is of the order of $\dot{\boldsymbol{M}}$. Third, the corrections $\ddot{\boldsymbol{\delta}}$ can be computed on a point-by-point basis since $\ddot{\boldsymbol{N}}^{-1}$ is block diagonal in form.

In order to accomplish the first step of the solution, the inverse corresponding to the $\dot{\boldsymbol{N}}$ portion of the normal equation matrix is required. The $\dot{\boldsymbol{N}}$ portion is of the order 12, 12 for the simultaneous stereopair solution. The formation of the $\dot{\boldsymbol{N}}$ matrix is a summation process for each photo given by

$$\underset{6,6}{\dot{\boldsymbol{N}}_i} = \sum_{j=1}^{m} \underset{6,2}{\dot{\boldsymbol{B}}^T}\,\underset{2,2}{\boldsymbol{W}}\,\underset{2,6}{\dot{\boldsymbol{B}}} \qquad (6)$$

where $\dot{\boldsymbol{B}}$ is a matrix composed of partial derivatives of the collinearity equations with respect to the exterior orientation elements,
$\boldsymbol{W}$ is the weight matrix for each image measurement,
m is the number of points measured on the i^{th} photo, and
i is either photo 1 or 2 of the stereopair.

An $\dot{\boldsymbol{N}}_i$ is computed for each photo in the solution and, therefore, the expanded form is written

$$\underset{12,12}{\dot{\boldsymbol{N}}} = \begin{bmatrix} \dot{\boldsymbol{N}}_1 & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \dot{\boldsymbol{N}}_2 \end{bmatrix} \qquad (7)$$

where $\boldsymbol{\phi}$ is the null matrix of appropriate dimensions.

Adding the contribution of a new image measurement from one ground point appearing on both photographs of the stereopair to the $\dot{\boldsymbol{N}}$ matrix may be done as follows:

$$\underset{12,12}{\dot{\boldsymbol{N}}_l} = \underset{12,12}{\dot{\boldsymbol{N}}_k} + \underset{12,4}{\dot{\boldsymbol{B}}^T_l}\,\underset{4,4}{\boldsymbol{W}_l}\,\underset{4,12}{\dot{\boldsymbol{B}}_l} \qquad (8)$$

where $\dot{\boldsymbol{N}}_l$ is the $\dot{\boldsymbol{N}}$ portion of the partitioned normal equation matrix after the contribution of the l-th ground point is included,
$\dot{\boldsymbol{N}}_k$ is the $\dot{\boldsymbol{N}}$ matrix before the l-th ground point is included, and

$$\underset{4,12}{\dot{\boldsymbol{B}}_l} = \begin{bmatrix} \dot{\boldsymbol{B}}_{1_l} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \dot{\boldsymbol{B}}_{2_l} \end{bmatrix} \qquad (9a)$$

$$\underset{4,4}{\dot{\boldsymbol{W}}_l} \; \begin{bmatrix} \dot{\boldsymbol{W}}_{1_l} & \boldsymbol{\phi} \\ \boldsymbol{\phi} & \dot{\boldsymbol{W}}_{2_l} \end{bmatrix} \qquad (9b)$$

It is assumed that no correlation exists between the measurements of the images of l-th ground point on each photo. Also, subscripts 1_l and 2_l refer to the matrix associated with the first and second photo respectively.

For future reference the computation of $\overline{\boldsymbol{N}}$ and $\ddot{\boldsymbol{N}}$ is defined by

$$\underset{12,3}{\overline{\boldsymbol{N}}} = \underset{12,4}{\dot{\boldsymbol{B}}^T}\,\underset{4,4}{\boldsymbol{W}}\,\underset{4,3}{\ddot{\boldsymbol{B}}} \qquad (10a)$$

$$\underset{3,3}{\ddot{\boldsymbol{N}}} = \underset{3,4}{\ddot{\boldsymbol{B}}^T}\,\underset{4,4}{\boldsymbol{W}}\,\underset{4,3}{\ddot{\boldsymbol{B}}} \qquad (10b)$$

where $\ddot{\boldsymbol{B}}$ represents the partial derivatives of the collinearity equations with respect to the object space coordinates.

For one additional ground point measured on both photos of the stereopair;

$$\ddot{\boldsymbol{B}}^T{}_l = \begin{bmatrix} \ddot{\boldsymbol{B}}^T_{1l} & \ddot{\boldsymbol{B}}^T_{2l} \end{bmatrix} . \qquad (11)$$

It can also be shown that the inversion of the portion of the normal equation matrix corresponding to $\dot{\boldsymbol{N}}$ may be computed by the formula

$$\dot{\boldsymbol{M}}_l = (\dot{\boldsymbol{N}}_l - \overline{\boldsymbol{N}}_l\,\ddot{\boldsymbol{N}}_l^{-1}\,\overline{\boldsymbol{N}}_l^T)^{-1} . \qquad (12)$$

From the above equation let us define

$$\widetilde{\boldsymbol{N}}_l = \dot{\boldsymbol{N}}_l - \overline{\boldsymbol{N}}_l\,\ddot{\boldsymbol{N}}_l^{-1}\,\overline{\boldsymbol{N}}_l^T . \qquad (13)$$

Substituting Equation 10a into Equation 13 gives

$$\widetilde{\boldsymbol{N}}_l = \dot{\boldsymbol{N}}_l - \dot{\boldsymbol{B}}_l^T\,\boldsymbol{W}_l\,\ddot{\boldsymbol{B}}_l\,\ddot{\boldsymbol{N}}_l^{-1}\,\overline{\boldsymbol{N}}_l^T . \qquad (14)$$

Further substitution for $\dot{\boldsymbol{N}}_l$ in Equation 14 using Equation 8 yields

$$\widetilde{\boldsymbol{N}}_l = \dot{\boldsymbol{N}}_k + \dot{\boldsymbol{B}}_l^T\,\boldsymbol{W}_l\,\dot{\boldsymbol{B}}_l - \dot{\boldsymbol{B}}_l^T\,\boldsymbol{W}_l\,\ddot{\boldsymbol{B}}_l\,\ddot{\boldsymbol{N}}_l^{-1}\,\overline{\boldsymbol{N}}^T . \qquad (15)$$

This equation may be rearranged in the following manner:

$$\widetilde{\boldsymbol{N}}_l = \dot{\boldsymbol{N}}_k + \dot{\boldsymbol{B}}_l^T\,\boldsymbol{W}_l\,(\dot{\boldsymbol{B}}_l - \ddot{\boldsymbol{B}}_l\,\ddot{\boldsymbol{N}}_l^{-1}\,\overline{\boldsymbol{N}}_l^T) \qquad (16)$$

or

$$\underset{12,12}{\widetilde{\boldsymbol{N}}_l} = \underset{12,12}{\dot{\boldsymbol{N}}_k} + \underset{12,4}{\dot{\boldsymbol{B}}_l^T}\,\underset{4,4}{\boldsymbol{W}_l}\,\underset{4,12}{\overline{\overline{\boldsymbol{B}}}_l} \qquad (17)$$

where

$$\overline{\overline{\boldsymbol{B}}}_l = (\dot{\boldsymbol{B}}_l - \ddot{\boldsymbol{B}}_l\,\dot{\boldsymbol{N}}_l^{-1}\,\overline{\boldsymbol{N}}_l^T) . \qquad (18)$$

Equation 17 shows that the augmented normal equation matrix with the contribution of a new ground point included ($\widetilde{N}_l$) can be written as a function of the augmented normal equation matrix without the new measurement in $\dot{N}_k$. The inverse of $\widetilde{N}_l$ will be $\dot{M}_l$ which is required for the solution of Equation 5a. At this point we have derived a sequential algorithm. By using any numerical inversion computation for Equation 17 the contribution of a new point may be added to the solution.

Rather than accept this sequential procedure, a different algorithm will be developed which gives the inverse directly. In order to obtain the inverse of $\widetilde{N}_l$ a matrix identity equation will be employed which states that for a matrix equation of the form

$$\underset{n,n}{A} = \underset{n,n}{B} \pm \underset{n,r}{U}\,\underset{r,r}{W}\,\underset{r,n}{V} \tag{19}$$

The inverse of A may be computed directly by the formula

$$A^{-1} = B^{-1} \mp B^{-1} U \,(W^{-1} \pm VB^{-1} U)^{-1}\, VB^{-1} \tag{20}$$

provided the inverses in this equation exist. The correspondence between Equation 17 and Equation 19 is apparent and therefore the inverse of $\widetilde{N}_l$ can be written directly as

$$\dot{M}_l = \dot{M}_k \left[I - \dot{B}_l^T \,(W_l^{-1} + \bar{\bar{B}}_l \dot{M}_k \dot{B}_l^T)^{-1} \,\bar{\bar{B}}_l \dot{M}_k\right] \tag{21}$$

where

$$\dot{M}_k = \dot{N}_k^{-1} \ .$$

Equation 21 gives the desired expression for sequential processing. The inverse including one new ground point ($\dot{M}_l$) has been written as a function of the inverse before the point was added ($\dot{M}_k$). It should be noted that if the signs of Equation 21 are changed in accordance with those in Equation 20, the process is reversed; that is, a point, the contribution of which is already included in the solution, may be removed. This is a valuable characteristic of the algorithm which may be employed for deleting erroneous data from the solution.

In order to determine the corrections to the elements of exterior orientation as shown in Equation 5a, it is necessary to compute a constant vector.

$$\dot{T} = \dot{c} - \overline{N}\, \ddot{N}^{-1}\, \ddot{c} \ . \tag{22}$$

This constant vector must also be modified as the measurements associated with a new ground point are added to the solution. Including the image measurements from both photos of the stereopair changes the constant vector in the following manner.

Considering first the $\dot{c}$ term of Equation 22, it can be shown that

$$\underset{12,1}{\dot{c}_l} = \underset{12,1}{\dot{c}_k} + \underset{12,4}{\dot{B}_l^T}\,\underset{4,4}{W_l}\,\underset{4,1}{\epsilon_l} \tag{23}$$

where $\dot{c}_l$ is the constant vector after the measurements from a new ground point are added,

$\dot{c}_k$ is the constant column before the measurements are added, and

ϵ is the discrepancy vector.

$$\epsilon_l^T = \left[\epsilon_{1l}{}^T \quad \epsilon_{2l}{}^T\right] \ .$$

For completeness the $\ddot{c}$ portion of Equation 22 is computed by

$$\ddot{c}_l = \ddot{B}_l^T\, W_l\, \epsilon_l \ . \tag{24}$$

Substituting Equation 10a into Equation 22 gives

$$\dot{T}_l = \dot{c}_l - \dot{B}_l^T\, W_l\, \ddot{B}_l\, \ddot{N}_l^{-1}\, \ddot{c}_l \ . \tag{25}$$

Including the formula for $\dot{c}_l$ from Equation 23 yields the final form of the modified constant vector.

$$\dot{T}_l = \dot{c}_k + \dot{B}_l^T\, W_l\, \epsilon_l - \dot{B}_l^T\, W_l\, \ddot{B}_l\, (\ddot{N}_l^{-1})\, \ddot{c}_l \tag{26}$$

or

$$\dot{T}_l = \dot{c}_k + \dot{B}_l^T\, W_l \left[\epsilon_l - \ddot{B}_l^T\, \ddot{N}_l^{-1}\, \ddot{c}_l\right] \ . \tag{27}$$

This completes the derivation of the sequential algorithm for the solution of a stereopair. The modified inverted normal equation matrix is computed by Equation 21 and the constant vector by Equation 27. Multiplication according to Equation 5a gives the correction vector $\dot{\delta}$ for the exterior orientation parameters.

Discussion

One might ask why such a complex sequential algorithm as given by Equation 21 should be employed when Equation 17 can be formed and inverted in a seemingly more straight-forward manner. The reasons are twofold. First, to utilize Equation 17 may require as much as two times more computer storage. For this equation to compete with Equation 21 in execution time, the normal equation matrix must be saved after each point is added to the solution and it cannot therefore be inverted in place as is commonly done in practice. Second, execution time can be significantly more when utilizing Equation 17. Review of Equation 17 shows that the inverse which must be computed is of the order of six times the number of photos being reduced in the on-line solution. Equation 21 on the other hand, requires the inversion of a matrix of order two times the number of photos on which a new ground point has been measured. For the stereopair derivation this amounts to a 4,4 matrix. Trade-offs exist in performing the large inversion of Equation 17 or the small inversion plus the remaining matrix calculations in Equation 21. The exact savings realized with either approach are totally dependent on the on-line procedures and computer hardware and software available for a particular application. Based on present experience, either equation may be employed for the stereopair without seriously affecting response time or computer memory allocations.

Application Considerations

In applying the sequential algorithm several items should be discussed: the initial priming of the solution, the effect of non-linear photogrammetric models, and finally computer execution time estimates.

In order to start or prime the sequential algorithm of Equation 21 it is necessary to have an initial estimate of $\dot{\boldsymbol{M}}_k$. When no measurements have been performed, this estimate is provided by the *a priori* covariance matrix of the exterior orientation parameters. This is in one sense a drawback for on-line operation; however, the flexibility provided by a general mathematical approach will certainly be rewarding in the long run. By employing *a priori* constraints one may accomplish varying degrees of solution from a strict relative orientation to the condition where all parameters in the solution are exactly known. For a strict relative orientation solution for example, seven of twelve exterior orientation parameters would be considered known and a covariance matrix which reflects this condition would be generated. Measurements would then be added until the best estimates, in the least squares sense, were computed.

This leads to the consideration of non-linear photogrammetric mathematical models. As previously mentioned, non-linear models such as the one utilized above for the stereopair, require that the solution be re-linearized some number of times depending on the closeness of the *a priori* approximations of the parameters. It must be emphasized that the sequential algorithm is exact only for linear solutions and therefore data sets must not be expanded inside a re-linearization if agreement with conventional non-linear least squares estimation is to be had.[7]

Finally, the estimated execution time for the sequential algorithm of Equation 21 and 27 has been determined. These equations have been programmed in FORTRAN for the UNIVAC 1108 computer and MODCOMP II/45 minicomputer. Execution times of approximately 1.0 to 1.5 seconds were experienced for adding the contribution of one new ground point to the solution. This means that the quality of a set of measurements could be established in less than two seconds. These times could be reduced with some machine language programming; however, it is felt that the execution times quoted above are reasonable for sequential on-line processing.

Summary

This paper has derived a general sequential algorithm for on-line processing. Simulation tests have shown that the algorithm can be implemented and that the execution time is reasonable for on-line response. Discussion of the practical aspects of implementing such an algorithm are dependent on the scope of the on-line problem and the supporting computer hardware and software. Significant computational efficiency can be realized depending on the number of photos in the solution. If properly utilized, the algorithm will provide adjusted parameters equivalent to conventional least squares solutions. The general nature of the approach enables it to be used with equal facility for other functions such as on-line editing. Anyone considering implementation of an on-line system should seriously evaluate the sequential processing mode before software development is initiated.

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