John P. Snyder
Madison, NJ 07940

The Space Oblique Mercator Projection

A mathematical development for the Space Oblique Mercator Projection, which provides continuous mapping of satellite imagery true to scale along the groundtrack, is presented.

Introduction

In 1974 Alden P. Colvocoresses of the U. S. Geological Survey announced a new map projection which would permit continuous mapping of satellite imagery, especially Landsat.[1] He named it the "Space Oblique Mercator (SOM) Projection." The editor of this journal suggested the name "Colvo's Projection," after the originator's nickname. Until then no map projection had been devised which shows the satellite groundtrack continuously true to scale for a revolving satellite combined with a rotating Earth. It was preferred that the areas scanned by the satellite be mapped conformally with a minimum of scale error; thus, the relationship to the oblique Mercator.

The mapping problem was simplified by the relatively narrow swath covered by the Landsat (formerly ERTS) satellite at any given time, officially 185 km, but the mathematical formulas had not been developed when Dr. Colvocoresses presented the projection. He made numerous appeals for formulas, but they were not developed until 1977, when John L. Junkins of the University of Virginia and the writer independently and almost simultaneously developed equations. Both men were made fully aware of the progress of the other's work through Dr. Colvocoresses, who provided much inspiration, guidance, and intensive intercommunication with others who were interested.

Abstract: *The Space Oblique Mercator projection, a concept that was originated by Colvocoresses in 1974, has been mathematically implemented as the first map projection to provide continuous mapping of satellite imagery true to scale along the groundtrack, and within a few millionths of accurate conformal projection. Specifically designed for Landsat (formerly ERTS) imagery, it is also suitable for other satellites with broader scans. Formulas are given for both sphere and ellipsoid. A unique feature is the need for a curved groundtrack and skewed scan lines on the SOM projection, although they would be straight and parallel, respectively, on a normal oblique cylindrical projection.*

Dr. Junkins took a more general, theoretical approach. His formulas are much more complex, but are more universal, permitting application to non-circular orbits and other general cases. Indicated scale errors are somewhat greater than those for the following formulas.[2] This writer began with simpler approaches, including a programmable hand calculator and application to the sphere, eventually employing a mixture of rigorous theory and empirical approaches to arrive at apparently satisfactory formulas for the sphere and the ellipsoid (or spheroid), based on a circular satellite orbit.

The formulas in this article are not copyrighted and have been given to the Geological Survey for its and the public's use without restriction, except that credit in publications should be given. It should be stressed that the formulas do not give an exactly conformal projection, but scale factors are within a few millionths of correct values for a conformal projection, and are thus well within mapping accuracy.

SOM for the Sphere

For the stationary sphere, the regular Mercator projection formulas are frequently written:

$$x = R\lambda \tag{1}$$
$$y = R \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) \tag{2}$$

where ϕ and λ are geodetic (or geographic) latitude and longitude, respectively, and R is the radius of the globe. Radians are used throughout this paper, except where the degree symbol (°) or word is used.

To convert to the oblique Mercator, these formulas may be rewritten with ϕ' and λ' in place of ϕ and λ, respectively, followed by one of several forms of transformation equations. The following form seems most suitable for subsequent derivations:

$$\tan \lambda' = \cos i \tan \lambda + \sin i \tan \phi / \cos \lambda \tag{3}$$

$$\sin \phi' = \cos i \sin \phi - \sin i \cos \phi \sin \lambda \tag{4}$$

where ϕ and λ are as before, and i is the angle of inclination between the Earth's equator and the transformed equator of the oblique Mercator projection. Geodetic longitude λ is measured eastward from the intersection of the two equators. The transformed longitude λ' is the angular distance *along* the transformed equator, measured north from the same intersection, while the transformed latitude ϕ' is the angular distance *from* the transformed equator.

If a satellite is following a uniform circular orbit inclined counterclockwise i (99.092° for Landsat) to the Earth's equator, its groundtrack will follow the transformed equator, and λ' will be directly proportional to the time elapsed since the satellite crossed the plane of the Earth's equator in a northerly direction (the ascending node). (This crossing for Landsat occurs on the dark side of the Earth; mapping occurs as the satellite is proceeding south on the opposite face.) The angular distance along the scan line will be ϕ' measured from the groundtrack.

For a stationary, spherical Earth with a revolving satellite, these formulas are exact. Let us think, however, of the satellite orbit as fixed in space with the satellite revolving and the Earth rotating with respect to the orbit. At the time the satellite has reached λ', starting from zero, the parallels will not have changed latitude, but the meridians will have rotated so that, from the viewpoint of the satellite in space, the actual longitude λ will appear along the scanning line at the point where some other longitude λ_t would have appeared if the Earth had remained stationary. The "satellite-apparent" longitude λ_t is found from the equation

$$\lambda_t = \lambda + (P_2/P_1)\lambda', \tag{5}$$

where P_2 is the time required for the revolution of the satellite (103.267 min. for Landsat) and P_1 is the length of the Earth's rotation with respect to the precessed ascending node of the orbit. For Landsat, the satellite orbit is actually sun-synchronous—that is, it is always the same with respect to the sun—equating P_1 to the solar day (1440 min.). (The Landsat satellite is orbited to complete exactly 251 revolutions in 18 days.)

Letting λ_t take the place of λ in Equations 3 and 4:

$$\tan \lambda' = \cos i \tan \lambda_t + \sin i \tan \phi / \cos \lambda_t \tag{6}$$

$$\sin \phi' = \cos i \sin \phi - \sin i \cos \phi \sin \lambda_t. \tag{7}$$

Since λ_t is a function of λ', finding λ' for a given ϕ and λ involves trial and error (or iteration), but Equations 5, 6, and 7 are exact for the transformation in the case of the rotating sphere. (For calculations, see note 1.) Equations 1 and 2, with ϕ' and λ' substituted for ϕ and λ, respectively, are inappropriate, however. They would show the satellite groundtrack as a straight line and the scan lines as a series of parallel lines perpendicular to the groundtrack.

Actually, the groundtrack should not be plotted perpendicularly to the scan lines on the map projection because of the rotation of the Earth. At the Earth's equator, the direction of the Landsat groundtrack on the Earth is about 86° from that of the scan lines, although it is perpendicular in space. This angle becomes 90° when the satellite comes closest to the poles (at 180° minus 99.092° or about 81° N or S latitude). Derivation of the proper projected path for the groundtrack, if the scan lines remain parallel to each other, leads to the equations:

$$x = R\,H\lambda' \tag{8}$$

$$y = R\left[(P_2/P_1)\sin i \sin\lambda' + \ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi')\right] \tag{9}$$

where

$$H = 1 - (P_2/P_1)\cos i \tag{10}$$

and ϕ' and λ' are found from Equations 5, 6, and 7. The groundtrack is plotted by letting $\phi' = 0$ and determining x and y for a series of λ''s. It is a sinusoidal curve which is true to scale and conformal, inclined about 4° to the X-axis at the crossings of the equator. Time is proportional to x, but is not quite proportional to distance along the satellite path, and should not be because of the effect of Earth rotation.

This change from a straight to a curved transformed equator (or satellite groundtrack) is not enough to place scale errors within the satellite swath at less than the desired limit of about one part in 10,000 for mapping accuracy. The normal scale factor for a tangent, conformal cylindrical projection, whether regular, oblique, or transverse Mercator, is sec ϕ', or 1.000152 at 1° from the transformed equator, somewhat farther than the 0.83° limit of the Landsat scan. Dividing out this factor leaves remaining errors of over one part in 1000 at the polar approaches when $\phi' = \pm 1°$: the scale is too large poleward and too small on the opposite side of the groundtrack. At the equator, this discrepancy practically disappears.

Therefore, it was deduced that the sinusoidal groundtrack should be bent more sharply on the projection in the polar areas, but not equatorial areas, while the scan lines should continue to intersect the track at the same angles as before to prevent distortion along the track. The scan lines would thus become skew with respect to the Y-axis.

By observing the magnitude of this residual scale error at various values of λ', it was decided to double (approximately) the angle of the slope of the groundtrack in Equations 8 and 9 and to rotate the scan lines to the same extent. After determining new scale residuals and readjusting the angle of rotation, the following final formulas for the sphere were derived:

$$\frac{x}{R} = \int_0^{\lambda'} \frac{H - S^2}{\sqrt{1 + S^2}}\,d\lambda' - \frac{S}{\sqrt{1 + S^2}}\ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') \tag{11}$$

$$\frac{y}{R} = (H + 1)\int_0^{\lambda'} \frac{S}{\sqrt{1 + S^2}}\,d\lambda' + \frac{1}{\sqrt{1 + S^2}}\ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') \tag{12}$$

where $S = (P_2/P_1)\sin i \cos\lambda'$ (13)

and H, ϕ', and λ' are found from Equations 5, 6, 7, and 10. The resulting scale factors, after dividing by sec ϕ', are within six parts in one million of correct, in a zone 1° on either side of the satellite groundtrack. This is still not a perfectly conformal projection, but the discrepancy within the required scanning range is negligible, *for the sphere*. The X-axis is not now directly proportional to time, but λ' remains so. The poles are near to, but not on, the new X-axis. The scan lines vary between ±4.05° from vertical, at counterclockwise angle arc tan S, and ±4.01° (at clockwise angle arc tan (S/H)) from normal to the satellite groundtrack, which now is inclined 8.06° to the X-axis at the equator. Figure 1 shows a 30° graticule extended to most of the globe for one-and-one-half orbits. The progressive rotation of meridians may be observed. Table 1 lists scale factors at various points of the projection for Landsat imagery.

Numerical integration is now required, but this can be reduced to a one-time calculation of constants for a Fourier series for any given satellite orbit, as Dr. Junkins suggested to the writer. The integrals may be rewritten:

$$x:\ \int_0^{\lambda'} \frac{H - S^2}{\sqrt{1 + S^2}}\,d\lambda' = B\lambda' + A_2\sin 2\lambda' + A_4\sin 4\lambda' + \ldots \tag{14}$$

$$\text{where } A_n = \frac{1}{\pi n}\int_0^{2\pi}\left[\frac{H - S^2}{\sqrt{1 + S^2}} - B\right]\cos n\lambda'\,d\lambda' \tag{15}$$

$$\text{and } B = \frac{2}{\pi}\int_0^{\pi/2} \frac{H - S^2}{\sqrt{1 + S^2}}\,d\lambda'. \tag{16}$$

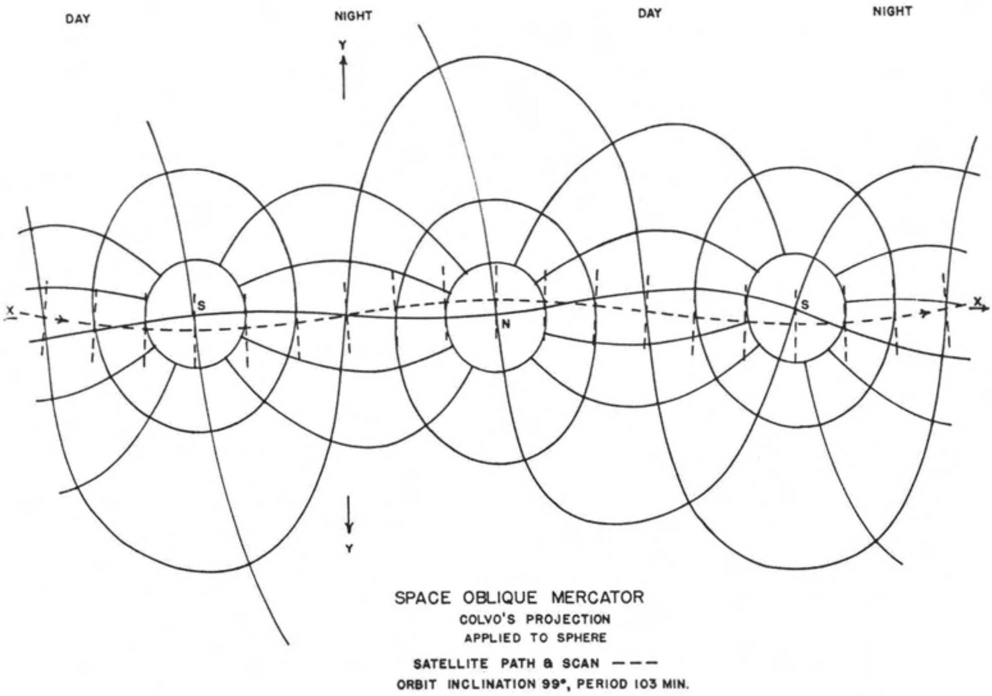

FIG. 1. A 30° graticule extended to most of the globe for one-and-one-half orbits.

$$y\text{: } (H+1)\int_0^{\lambda'} \frac{S}{\sqrt{1+S^2}}\,d\lambda' = C_1 \sin\lambda' + C_3 \sin 3\lambda' + \ldots \tag{17}$$

$$\text{where } C_n = \frac{H+1}{\pi n}\int_0^{2\pi} \frac{S}{\sqrt{1+S^2}} \cos n\lambda'\, d\lambda' . \tag{18}$$

If P_2 = 103.267 min., P_1 = 1440 min., and i = 99.092°,

B = 0.0175853340 for λ' in degrees
A_2 = −0.0018820
A_4 = 0.0000007
C_1 = 0.1421598
C_3 = −0.0000296

Additional terms are unnecessary for up to 7-place accuracy for the sphere.

INVERSE EQUATIONS FOR THE SPHERE

For ϕ' and λ' in terms of x and y: Equation 19 is to be used for finding λ' by iteration.

$$\frac{x + y\,S}{R} = \int_0^{\lambda'} \frac{H - S^2}{\sqrt{1+S^2}}\,d\lambda' + S\,(H+1)\int_0^{\lambda'} \frac{S}{\sqrt{1+S^2}}\,d\lambda' \tag{19}$$

The repeated integration may be eliminated by use of Equations 14 and 17, combining and transposing as follows, although trial and error remains (see sentence following Equation 72 for iteration procedure):

$$B\lambda' = (x/R) + (y/R)\,D\cos\lambda' - E_2 \sin 2\lambda' - E_4 \sin 4\lambda', \tag{19a}$$

where

$D = (P_2/P_1)\sin i = 0.0708122$
$E_2 = A_2 + \tfrac{1}{2}D\,(C_1 + C_3) = 0.0031503$
$E_4 = A_4 + \tfrac{1}{2}DC_3 = -0.0000003$.

TABLE 1. RELATIVE SCALE FACTORS

Calculated along sides and diagonals of quadrilaterals 0.01° of λ'' and ϕ'' (λ' and ϕ' for the sphere), respectively, on each side, using inverse formulas. Calculated factors would vary for ellipsoid up to 0.000006 if direct formulas were used. For sphere, inverse formulas are exactly equivalent to direct formulas. Scale factors for each quadrant of the projection are identical except for reversals due to symmetry, etc. Scale factors are given *relative to* sec ϕ'' to indicate "conformal trueness," and must be multiplied by sec ϕ'' for true scale factors.

	**	Sphere: ϕ'': 1°	0°	−1°	Ellipsoid: 1°	0°	−1°
λ''	1)	1.000152	1.000000	1.000152	1.000152	1.000000	1.000152
0°	2)	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
	3)	0.999999	1.000000	0.999999	0.999999	1.000000	0.999999
	4)	0.999995	1.000000	0.999995	0.999995	1.000000	0.999995
	5)	1.000005	1.000000	1.000005	1.000005	1.000000	1.000005
15°	2)	1.000000	1.000000	1.000000	1.000004	0.999999	0.999994
	3)	0.999997	0.999999*	1.000001	1.000000	1.000000	0.999999
	4)	0.999994	1.000000	0.999995	0.999984	0.999999	1.000006
	5)	1.000004	1.000000	1.000006	1.000022	0.999999	0.999986
30°	2)	1.000000	1.000000	1.000000	1.000006	0.999996	0.999988
	3)	0.999997	1.000000	1.000002	1.000000	1.000000	1.000000
	4)	0.999995	1.000000	0.999997	0.999976	0.999998	1.000014
	5)	1.000003	1.000000	1.000006	1.000034	0.999998	0.999971
45°	2)	1.000000	1.000000	1.000000	1.000010	0.999994	0.999984
	3)	0.999998	1.000000	1.000002	1.000001	1.000000	0.999999
	4)	0.999995	1.000000	0.999998	0.999976	0.999997	1.000016
	5)	1.000002	1.000000	1.000005	1.000038	0.999997	0.999965
60°	2)	1.000000	0.999999*	1.000000	1.000015	0.999996	0.999983
	3)	0.999999	1.000000	1.000001	1.000002	1.000000	0.999999
	4)	0.999997	1.000000	0.999998	0.999983	0.999998	1.000014
	5)	1.000003	1.000000	1.000003	1.000036	0.999998	0.999967
75°	2)	0.999999*	1.000000	1.000000	1.000020	0.999998	0.999985
	3)	0.999999	1.000000	1.000001	1.000004	1.000000	0.999999
	4)	0.999998	1.000000	0.999999	0.999998	0.999999	1.000006
	5)	1.000001	1.000000	1.000002	1.000026	0.999999	0.999977
90°	2)	1.000000	1.000000	1.000000	1.000018	1.000001	0.999981
	3)	1.000000	1.000000	1.000000	1.000005	1.000000	0.999999
	4)	1.000000	1.000000	1.000000	1.000011	1.000000	0.999990
	5)	1.000000	1.000000	1.000000	1.000012	1.000000	0.999990

* Should be 1.000000, but reported as calculated for consistency with other data, all obtained from 10-place programmable hand calculator.

** (1) sec ϕ'', or normal scale factor in all directions for a truly conformal cylindrical projection.
(2) scale factor divided by sec ϕ'' along scan line of SOM (constant λ'').
(3) scale factor divided by sec ϕ'' in direction parallel to (or along) satellite groundtrack (constant ϕ'').
(4) scale factor divided by sec ϕ'' along diagonal of quadrilateral, ϕ'' and λ'' both becoming more positive.
(5) scale factor divided by sec ϕ'' along diagonal of quadrilateral, ϕ'' becoming more negative and λ'' becoming more positive.

Equation 20, for finding ϕ', requires finding λ' from Equation 19 or 19a:

$$\ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') = \sqrt{1 + S^2}\left[\frac{y}{R} - (H + 1)\int_0^{\lambda'} \frac{S}{\sqrt{1 + S^2}}\, d\lambda\right] \qquad (20)$$

With series, it may be rewritten:

$$\ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi') = (y/R)\sqrt{1 + D^2\cos^2\lambda'} - L_1 \sin \lambda' - L_3 \sin 3\lambda' \qquad (20a)$$

where

$$L_1 = C_1G_0 - \tfrac{1}{2}G_2\,(C_1 - C_3) = 0.1422488$$
$$L_3 = C_3G_0 + \tfrac{1}{2}C_1\,(G_2 - G_4) = 0.0000594$$
$$G_0 = 1 + D^2/4 - 3D^4/64 = 1.0012524$$
$$G_2 = D^2/4 - D^4/16 = 0.0012520$$
$$G_4 = -D^4/64 = -0.0000004\ .$$

For ϕ and λ in terms of ϕ' and λ' (for calculations, see note 1):

$$\lambda = \lambda_t - (P_2/P_1)\lambda' \tag{21}$$

$$\tan \lambda_t = \cos i \tan \lambda' - \sin i \tan \phi'/\cos \lambda' \tag{22}$$

$$\sin \phi = \cos i \sin \phi' + \sin i \cos \phi' \sin \lambda'. \tag{23}$$

SOM for the Ellipsoid

Accurate though these formulas may be for the sphere, there are errors of over a half percent in using these (or any accurate spherical formulas) in place of formulas based on the ellipsoid, especially in tropical latitudes. In maps of very large areas, errors fundamental to plane projection far outweigh the effect of the ellipsoid but, for topographic mapping of small areas or strips, the use of the ellipsoid is essential for high-quality mapping. Therefore, the development of SOM formulas for the sphere served largely to avoid more complex derivations and calculations in determining the feasibility of various concepts.

At first, the writer assumed that Hotine's classic work on the ellipsoidal oblique Mercator would be the logical link in changing from sphere to ellipsoid.[3] Indeed, John B. Rowland of the Geological Survey had applied it in five stationary zones to approximate each north-to-south pass of the satellite, with consequent discontinuities.[4] It was soon evident that Hotine's work was less applicable to a continuous projection than the basic geometry of the ellipsoid, assisted by Thomas.[5]

For ellipsoidal formulas, the path of the satellite groundtrack can be derived rigorously without excessive work, but the positioning of points along the scan lines seems to require certain simplifying assumptions to prevent staggering derivations and overly cumbersome formulas. The chief assumption made is that the radius of curvature of the ellipsoid in the direction of the scan lines does not vary from its value at the groundtrack in the same direction. This leads to errors of about 2 parts in 100,000 in scale, relative to sec ϕ', at $\phi' = \pm 1°$. It was found possible to compensate for this effect in the direction of the scan lines with a fairly simple empirical formula, but this causes other errors to increase in patterns which are not readily correctable, so this adjustment was omitted, since residual errors remain well within the desired range. Scanning is assumed to be instantaneous. Two assumptions also affect the groundtrack: (1) a mass-centered perfect ellipsoid, and (2) a uniform circular orbit for the satellite (see note 8 below).

The ellipsoidal formulas are presented here without describing the derivation in any further detail. It is hoped that the derivation will be available at a later date in a Geological Survey publication, which will also include any refinement of the formulas resulting from further testing. Figure 2 shows an enlargement of the second quadrant of the projection after the ascending node. This graticule is 10°, calculated for the ellipsoid, but almost imperceptibly different from the sphere at this scale.

For x and y in terms of ϕ'' and λ'':

$$\frac{x}{a} = \int_0^{\lambda''} \frac{HJ - S^2}{\sqrt{J^2 + S^2}} d\lambda'' - \frac{S}{F\sqrt{J^2 + S^2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') \tag{24}$$

$$\frac{y}{a} = \int_0^{\lambda''} \frac{S(H + J)}{\sqrt{J^2 + S^2}} d\lambda'' + \frac{J}{F\sqrt{J^2 + S^2}} \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') \tag{25}$$

where

$$S = (P_2/P_1) \sin i \cos \lambda'' \sqrt{\frac{1 + T \sin^2 \lambda''}{(1 + W \sin^2 \lambda'')(1 + Q \sin^2 \lambda'')}} \tag{26}$$

$$H = \sqrt{\frac{1 + Q \sin^2 \lambda''}{1 + W \sin^2 \lambda''}} \left[\frac{1 + W \sin^2 \lambda''}{(1 + Q \sin^2 \lambda'')^2} - (P_2/P_1) \cos i\right] \tag{27}$$

$$F = \sqrt{\frac{1 + Q \sin^2 \lambda''}{1 + T \sin^2 \lambda''}} \left[1 + \frac{U (1 + Q \sin^2 \lambda'')^2}{(1 + W \sin^2 \lambda'')(1 + T \sin^2 \lambda'')}\right] \tag{28}$$

$$J = (1 - e^2)^3 \tag{29}$$

$$W = \left[(1 - e^2 \cos^2 i)^2/(1 - e^2)^2\right] - 1 \tag{30}$$

$$Q = e^2 \sin^2 i/(1 - e^2) \tag{31}$$

$$T = e^2 \sin^2 i\, (2 - e^2)/(1 - e^2)^2 \tag{32}$$

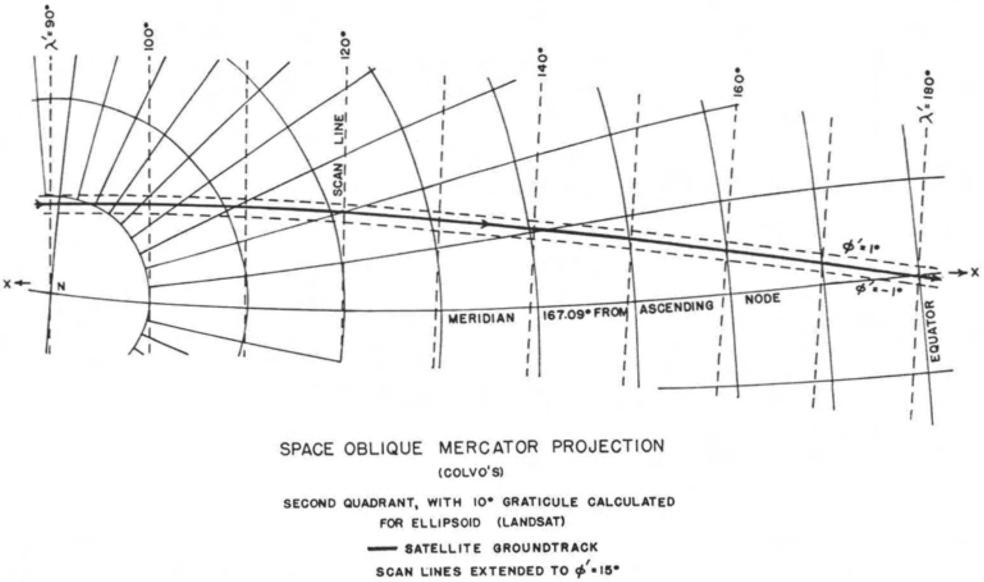

FIG. 2. The second quadrant of the Space Oblique Mercator Projection after the ascending node with a 10° graticule.

$U = e^2 \cos^2 i/(1 - e^2)$ (33)

λ'' = pseudotransformed longitude relative to a geocentric satellite groundtrack. Because the satellite scans vertically, this is slightly different from λ' (see Equation 40).

ϕ'' = pseudotransformed latitude relative to a geocentric satellite groundtrack (see Equation 39).

a = major semi-diameter of ellipsoid (6378206.4 meters for the 1866 Clarke ellipsoid).

e^2 = square of eccentricity of ellipsoid (square is 0.00676866 for the 1866 Clarke ellipsoid).

For Fourier equivalents, see Equations 58 and 66.

For ϕ and λ in terms of ϕ'' and λ'':

$$\sin \phi = K/\sqrt{1 + e^2 K^2} \tag{34}$$

where

$$K = \frac{1}{1 - e^2}\left[\sin i \sin \lambda''\left(\frac{1}{\sqrt{1 + Q \sin^2 \lambda''}} - \frac{1 - \cos \phi''}{F}\right) + \frac{\cos \theta \sin \phi''}{F}\right] \tag{35}$$

$$\tan \theta = \tan i \,(1 - e^2 \cos^2 \lambda'')/(1 - e^2) \tag{36}$$

$$\lambda = \lambda_t - (P_2/P_1)\lambda'' \tag{37}$$

where

$$\tan \lambda_t = \cos i \tan \lambda'' - \frac{\sin i \tan \phi''}{F \cos \lambda'' \sqrt{1 - e^2 \sin^2 \phi}} \tag{38}$$

Note: θ should be in the same quadrant as i. For other quadrant adjustments, see note 1.

For ϕ'' and λ'' in terms of ϕ' and λ':

$$\phi'' = \phi' + j_1 \sin \lambda' + j_3 \sin 3\lambda' \tag{39}$$

$$\lambda'' = \lambda' + m_2 \sin 2\lambda' + m_4 \sin 4\lambda' \tag{40}$$

Fourier forms are given here because of the unwieldiness of closed forms of the equations. Constants are determined as follows:

$$j_n = \frac{1}{\pi}\int_0^{2\pi} \phi'' \sin n\lambda' \, d\lambda' \tag{41}$$

$$m_n = \frac{1}{\pi}\int_0^{2\pi} (\lambda'' - \lambda') \sin n\lambda' \, d\lambda' \tag{42}$$

where ϕ'' and λ'' are determined for the groundtrack as functions of λ', from Equations 55, 56, 46, 47, and 48. If $P_2/P_1 = 18/251$, $i = 99.092°$, $R_0 = 7294.69$ km, and the 1866 Clarke ellipsoid is used, these are the only significant terms:

$j_1 = 0.008556$ for ϕ'' and ϕ' in degrees
$j_3 = 0.000818$ for ϕ'' and ϕ' in degrees
$m_2 = -0.023840$ for λ'' and λ' in degrees
$m_4 = 0.000105$ for λ'' and λ' in degrees

Note: ϕ' and λ' have the same definitions as they have for the spherical form of the projection; thus λ' is proportional to time along the true groundtrack.

For ϕ' and λ' in terms of ϕ'' and λ'':

$$\phi' = \phi'' - j_1 \sin \lambda'' - j_3 \sin 3\lambda'' \tag{43}$$

$$\lambda' = \lambda'' - m_2 \sin 2\lambda'' - m_4 \sin 4\lambda'' \tag{44}$$

These are within 0.000003° and 0.000009°, respectively, of the true inverses of Equations 39 and 40.

For ϕ'' and λ'' in terms of ϕ and λ:

$$\tan \lambda'' = \cos i \tan \lambda_t + (1 - e^2) \sin i \tan \phi / \cos \lambda_t . \tag{46}$$

Equation 46, like 6, requires trial and error. For techniques, see note 1.

$$\sin \phi'' = \frac{F\left[(1 - e^2)\cos i \sin \phi - \sin i \cos \phi \sin \lambda_t\right]}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{47}$$

$$\lambda_t = \lambda + (P_2/P_1)\lambda'' \tag{48}$$

Equations 46 to 48 are not exact inverses of Equations 34 to 38, except when $\phi'' = 0$. Within the band $\phi'' = \pm 1°$, however, ϕ'' and λ'' vary less than 0.000005° between the two sets of equations.

For ϕ'' and λ'' in terms of x and y:

Equation 49, for λ'', requires numerical integration as well as iteration, but Equation 70 with its Fourier series should be substituted to eliminate repeated numerical integration:

$$\frac{x + (S/J)\, y}{a} = \int_0^{\lambda''} \frac{HJ - S^2}{\sqrt{J^2 + S^2}} d\lambda'' + \frac{S}{J}\int_0^{\lambda''} \frac{S(H + J)}{\sqrt{J^2 + S^2}} d\lambda'' \tag{49}$$

Equation 50, for ϕ'', requires the calculation of λ'' from Equation 49 or 70, and preferably the substitution of Fourier equivalent 73 for Equation 50:

$$\ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') = \frac{F\sqrt{J^2 + S^2}}{J}\left[\frac{y}{a} - \int_0^{\lambda''} \frac{S(H + J)}{\sqrt{J^2 + S^2}} d\lambda''\right]. \tag{50}$$

Equations for the Satellite Groundtrack

If $\phi' = 0$, equations to find λ' from ϕ_0 or λ_0, latitude or longitude of any point along the true (vertical) satellite groundtrack, may be written as follows (all these formulas are thoretically exact for the ellipsoid and circular orbit):

$$\sin \lambda' = \sin \phi_g / \sin i \tag{51}$$

where
$$\phi_g = \phi_o - \arcsin\left[\frac{a\, e^2 \sin 2\phi_o}{2\, R_o \sqrt{1 - e^2 \sin^2 \phi_o}}\right] \quad (52)$$

= geocentric latitude along the geocentric satellite groundtrack.

R_o = radius of circular satellite orbit (in same units as a) (7294.69 km for Landsat).

Also
$$\tan \lambda' = \tan \lambda_t / \cos i, \text{ requiring iteration,} \quad (53)$$

since
$$\lambda_t = \lambda_o + (P_2/P_1)\lambda'. \quad (5)$$

(If tracking were geocentric, $\tan \phi_g = (1 - e^2) \tan \phi_o$.) (54)
Conversely, for ϕ_o or λ_o from λ',

$$\phi_o = \arcsin(\sin i \sin \lambda') + \arcsin \frac{a\, e^2 \sin 2\phi_o}{2R_o \sqrt{1 - e^2 \sin^2 \phi_o}} \quad (55)$$

requiring iteration; convergence is quite rapid.
Also
$$\lambda_o = \arctan(\cos i \tan \lambda') - (P_2/P_1)\lambda'. \quad (56)$$
(If tracking were geocentric, $\tan \phi_o = \tan \phi_g/(1 - e^2)$, where $\sin \phi_g = \sin i \sin \lambda'$.) (57)

The ϕ_o corresponding to a given λ_o, or vice versa, may also be found from Equations 53, 5, and 55 for ϕ_o, and Equations 51, 52, and 56 for λ_o. Table 2 lists these values for the first two quadrants for both sphere and ellipsoid, and also shows the extent of errors using the sphere instead of the ellipsoid.

Notes

(1) The programming of Equations 6, 46, and 53 requires care to avoid quadrant problems and discontinuities near λ'' (or λ') of 90°, 270°, etc., in the calculation of λ'' as an arc tangent determined for a given ϕ and/or λ. These trial and error equations converge rapidly if, after selecting the desired ϕ and λ, the λ'' of the nearest polar approach, λ'_p, is used as the first trial λ'' on the right-hand side, solving for the corresponding λ'' on the left-hand side, adding a factor (see below), using that as the next trial λ'', etc. This value may be calculated as $\lambda'_p = 90° \times (4N + 2 \pm 1)$, where N is the number of orbits completed at the last ascending node before the satellite passes the nearest pole, and the ± takes minus in the northern hemisphere and plus in the southern (either for the equator). Since a computer normally calculates the arc tangent as an angle between −90° and 90°, it is necessary to add the proper

Table 2. Geodetic Coordinates Along Satellite Groundtrack

λ'	Ellipsoid *or* sphere Longitude	Sphere Latitude	Ellipsoid Latitude	Error using sphere Shape*	Error using sphere Displacement**
0°	0.00000°	0.00000°	0.00000°	0.68%	0.00 km
15	−3.50023	14.80720	14.89143	0.64	−9.32
30	−7.36423	29.58525	29.73140	0.52	−16.20
45	−12.20673	44.28458	44.45437	0.35	−18.87
60	−19.60972	58.77571	58.92599	0.18	−16.74
75	−35.90801	72.51389	72.61090	0.06	−10.83
90	−96.45418	80.90800	80.96079	0.02	−5.89
105	−157.00035	72.51389	72.61090	0.06	−10.83
120	−173.29865	58.77571	58.92599	0.18	−16.74
135	179.29837	44.28458	44.45437	0.35	−18.87
150	174.45586	29.58525	29.73140	0.52	−16.20
165	170.59187	14.80720	14.89143	0.64	−9.32
180	167.09163	0.00000	0.00000	0.68	0.00

Constants: $P_2/P_1 = 18/251$; $i = 99.092°$; Clarke 1866 ellipsoid.
Radius of satellite orbit: 7294.69 km.

* A shape error calculated as the ratio of the scale along a meridian of the spherical projection to that of the ellipsoidal after scales along the parallel are equalized, or $(1 - e^2 \sin^2 \phi)/(1 - e^2)$, minus 1. It is only one of the useful correlation factors.

** A location error in plotting the latitude along the groundtrack at a given time, calculated as the spherical latitude minus the ellipsoidal latitude, converted to distance along the ellipsoidal meridian.

factor. Each λ'' (or λ') given on the left-hand side of the computer must be increased by λ'_p minus the following factor: 90° times sin λ'_p times ±1 (taking the sign of cos λ_{tp}, where $\lambda_{tp} = \lambda + (P_2/P_1)\lambda'_p$). If cos λ_{tp} is zero, the final λ'' is λ'_p. Thus λ'_p is 90°, 90°, 270°, 270°, 450°, etc., and the adder to arc tan is 0°, 180°, 180°, 360°, 360°, etc., for each successive quadrant beginning at the origin (ϕ, λ, ϕ'', and $\lambda'' = 0$). These quadrants automatically change along the equator, rather than along the scan line which crosses the equator at the node.

Different corrections are required in inverse Equations 22, 38, and 56. Subtract from the λ initially calculated in these equations this factor: 90° times (1 ± 1, taking the opposite sign of cos λ'') times ± 1, taking the sign of the initial λ (assume − if $\lambda = 0$). If cos $\lambda'' = 0$, add 10^{-8} degree to λ'' for the calculation. Thus the subtrahend is 0, ±180°, ±180°, 0, 0, etc., in successive quadrants of the orbit, depending on the sign of the initial λ.

For Equation 36, add 180° to the θ initially calculated as arc tan. In other equations, no correction is needed in arc function calculations.

(2) In order to reduce computer time substantially, it is recommended that Fourier series be used as follows, especially to replace repeated numerical integration, since the constants need to be computed only once for a given satellite:

For Equation 24, use:

$$x/a = B\lambda'' + A_2 \sin 2\lambda'' + A_4 \sin 4\lambda'' - \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') \cdot (b_1 \cos \lambda'' + b_3 \cos 3\lambda'' + b_5 \cos 5\lambda''), \quad (58)$$

where

$$B = \frac{2}{\pi} \int_0^{\pi/2} \frac{HJ - S^2}{\sqrt{J^2 + S^2}} \, d\lambda'' \quad (59)$$

$$A_n = \frac{1}{\pi n} \int_0^{2\pi} \left[\frac{HJ - S^2}{\sqrt{J^2 + S^2}} - B\right] \cos n\lambda'' \, d\lambda'' \quad (60)$$

$$b_1 = \tfrac{1}{2} (D_1 g_0 + D_1 g_2 + D_3 g_2) \quad (61)$$

$$b_3 = \tfrac{1}{2} (D_3 g_0 + D_1 g_2 + D_1 g_4) \quad (62)$$

$$b_5 = \tfrac{1}{2} (D_5 g_0 + D_3 g_2 + D_1 g_4) \quad (63)$$

$$D_n = \frac{1}{J\pi} \int_0^{2\pi} S \cos n\lambda'' \, d\lambda'' \quad (64)$$

$$g_n = \frac{1}{\pi} \int_0^{2\pi} \frac{J}{F\sqrt{J^2 + S^2}} \cos n\lambda'' \, d\lambda'' \quad (65)$$

If P_2/P_1 = 18 days/251 orbits, or 103.2669 . . . /1440, i = 99.092°, and the ellipsoid is the Clarke 1866,

B = 0.0175544891 for λ'' in degrees
A_2 = −0.00109792
A_4 = −0.00000129
b_1 = 0.07211679
b_3 = −0.00004718
b_5 = −0.00000013

For Equation 25,

$$y/a = C_1 \sin\lambda'' + C_3 \sin 3\lambda'' + \ln \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') \cdot (\tfrac{1}{2}g_0 + g_2 \cos 2\lambda'' + g_4 \cos 4\lambda''), \quad (66)$$

where g_n is as in Equation 65, and

$$C_n = \frac{1}{\pi n} \int_0^{2\pi} \frac{S(H + J)}{\sqrt{J^2 + S^2}} \cos n\lambda'' \, d\lambda'' \,. \quad (67)$$

With the above constants,

C_1 = 0.14344099
C_3 = 0.00002851
g_0 = 2.00038442
g_2 = −0.00295993
g_4 = −0.00000324 .

In Equations 35, 38, and 47, the calculation for F may be replaced, with some benefit, with

$$F = \tfrac{1}{2}d_0 + d_2 \cos 2\lambda'' + d_4 \cos 4\lambda'' \tag{68}$$

where

$$d_n = \frac{1}{\pi}\int_0^{2\pi} F \cos n\lambda''\, d\lambda''. \tag{69}$$

$$d_0 = 1.99702155$$
$$d_2 = 0.00165458$$
$$d_4 = 0.00000480$$

For Equation 49, consolidating the series and transposing leads to the following important substitution:

$$B\lambda'' = (x/a) + (y/a)(D_1 \cos \lambda'' + D_3 \cos 3\lambda'' + D_5 \cos 5\lambda'') - E_2 \sin 2\lambda'' - E_4 \sin 4\lambda''. \tag{70}$$

where B and D_n are defined in Equations 59 and 64, and

$$E_2 = A_2 + \tfrac{1}{2}(C_1D_1 + C_3D_1 - C_1D_3) \tag{71}$$

$$E_4 = A_4 + \tfrac{1}{2}(C_3D_1 + C_1D_3) \tag{72}$$

With the above constants,

$$D_1 = 0.07220987$$
$$D_3 = 0.00005979$$
$$D_5 = 0.00000007$$
$$E_2 = 0.00407775$$
$$E_4 = 0.00000402\ .$$

While Equation 70 is still trial and error, it converges rapidly by trying almost any λ'' in the right-hand side, solving for λ'' on the left side and using the latter λ'' for the next trial, etc., until there is no significant change between successive trial λ'''s.

Equation 50 may be replaced with

$$\ln \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'') = (y/a)\cdot(\tfrac{1}{2}G_0 + G_2 \cos 2\lambda'' + G_4 \cos 4\lambda'' + G_6 \cos 6\lambda'') - L_1 \sin \lambda'' - L_3 \sin 3\lambda'' - L_5 \sin 5\lambda''. \tag{73}$$

where

$$G_n = \frac{1}{\pi}\int_0^{2\pi} \frac{F\sqrt{J^2 + S^2}}{J} \cos n\lambda''\, d\lambda'' \tag{74}$$

$$L_1 = \tfrac{1}{2}(C_1G_0 - C_1G_2 + C_3G_2) \tag{75}$$
$$L_3 = \tfrac{1}{2}(C_3G_0 + C_1G_2 - C_1G_4) \tag{76}$$
$$L_5 = \tfrac{1}{2}(C_1G_4 + C_3G_2) \tag{77}$$
$$G_0 = 1.99962441$$
$$G_2 = 0.00295883$$
$$G_4 = 0.00000761$$
$$G_6 = 0.00000002$$
$$L_1 = 0.14320189$$
$$L_3 = 0.00024017$$
$$L_5 = 0.00000059$$

While all constants are calculated specifically for Landsat, the formulas permit calculating constants for other orbits.

(3) Scan lines are inclined with respect to vertical at an angle of arc tan $(-S/J)$, thus varying from 4.13° counterclockwise from vertical at the ascending node to 4.13° clockwise at the descending node. The linear distance on the ellipsoid along a given scan line from the true vertical groundtrack is $a\phi'/F$.

(4) For a given differential time interval along the geocentric satellite groundtrack, the distance is proportional to $\sqrt{H^2 + S^2}$. The true groundtrack is for all practical purposes (within 10^{-7}) at the same true scale as the geocentric. It is at the most 0.008° displaced.

(5) The maximum deviation of scale from the normal sec ϕ' in any direction is 0.004 percent at $\phi' = \pm 1°$, 0.03 percent at $\phi' = \pm 5°$, 0.08 percent at ±10°, and 0.15 percent at ±15°. The effect of sec ϕ' itself is much greater (0.015 percent, 0.38 percent, 1.5 percent, and 3.5 percent, respectively). The projection is thus suitable for weather and other satellites with a scan broader than that of Landsat. Table 1 lists various scale factors affecting Landsat imagery.

(6) If $e = 0$, the preceding formulas for the sphere result. Any other ellipsoid or circular orbit may be used by substitution of the appropriate a, e, i, R_o, and P_2/P_1 in the ellipsoidal formulas.

(7) If $P_2 = 0$, the formulas give the ordinary oblique Mercator projection (transverse, if $i = 90°$) for sphere or ellipsoid. The ellipsoidal formulas are limited to a narrow band.

(8) For a non-circular orbit, λ' is not proportional to time, but if the orbit is elliptical it may be calculated from the mean longitude in orbit of the satellite by Keplerian formulas. The factor $(P_2/P_1)\lambda'$ is now $(P_2/P_1)\cdot(L + \omega)$, and this change necessitates rederivation of most earlier formulas. The writer has not attempted this, except as follows, although Dr. Junkins includes elliptical and even non-elliptical orbits in his analysis.

These formulas permit use of an elliptical orbit in plotting the groundtrack only on the ellipsoid: Use Equations 51 through 57, and 5, except that Equation 5 becomes

$$\lambda_t = \lambda_o + (P_2/P_1)\,(L + \omega), \tag{78}$$

Equation 56 becomes

$$\lambda_o = \text{arc tan}\,(\cos i \tan \lambda') - (P_2/P_1)\,(L + \omega), \tag{79}$$

λ' is found from Equation 80:

$$\tan \tfrac{1}{2}\,(\lambda' - \omega) = (\tan \tfrac{1}{2} E')\,\sqrt{(1 + e')/(1 - e')}, \tag{80}$$

and R_o from Equation 81:

$$R_o = a'\,(1 - e' \cos E') \tag{81}$$

where

$$E' - e' \sin E' = L \tag{82}$$

L = mean anomaly, or mean longitude of satellite from perigee, directly proportional to time

ω = geocentric longitude in orbit of satellite perigee from ascending node of orbit

e' = eccentricity of satellite orbit

a' = mean distance of satellite from center of earth, same units as "a"

E' = "eccentric anomaly" of satellite orbit

λ' = geocentric longitude of satellite at a given time, relative to ascending node, as in earlier formulas.

Acknowledgments

In addition to thanking Drs. Colvocoresses and Junkins, I wish to thank Paul H. Salamonowicz, Tod E. Huffman, and Mr. Rowland of the USGS Topographic and Computer Center Divisions for their helpful comments in computer-testing of these formulas. Furthermore, without the initial encouragement of Prof. Waldo R. Tobler, and the technology available in hand calculators, I would not have undertaken this derivation.

References

1. Colvocoresses, Alden P., "Space Oblique Mercator, A new map projection of the Earth . . .", *Photogrammetric Engineering*, 40:8, pp. 921-926, 1974.
2. Junkins, John L., and James D. Turner, *Formulation of a Space Oblique Mercator Map Projection.* Final Report No. UVA/525023/ESS77/105, submitted to Computer Science Laboratory, U. S. Army Engineer Topographic Laboratories, Nov. 1977.
3. Hotine, M., "The Orthomorphic Projection of the Spheroid," *Empire Survey Review*, 8, pp. 300-311, 1946; 9, pp. 25-35, 52-70, 112-123, 157-166, 1947.
4. Rowland, John B., *The Hotine Oblique Mercator Projection Applied to Landsat Mapping,* U. S. Geological Survey Memorandum (EC-50-Landsat), Apr. 5, 1977.
5. Thomas, Paul D., *Conformal Projections in Geodesy and Cartography,* U. S. Coast and Geodetic Survey Sp. Pub. No. 251, Government Printing Office, Washington, D. C., 1952.