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An Orientation and Calibration Method for Non-Topographic Applications*

A least-squares adjustment procedure, suitable for both metric and non-metric cameras, is derived.

INTRODUCTION

IN NON-TOPOGRAPHIC applications of photogrammetry, especially in the fields of close-range and industrial photogrammetry, metric cameras are not always available. They are either too expensive or a suitable metric camera is not on the market. For those cases several institutions have developed methods which allow the use of non-metric cameras. Such a method, the Direct Linear Transformation (DLT), was published by Abdel-Aziz and Karara (1971, 1974) and Marzan and Karara (1975). The DLT establishes a direct linear relationship between coordinates of image points and the corresponding object space coordinates. This linear approach for the calibration of a camera does not require fiducial marks on the photographs. Since the DLT is a linear solution for a non-linear problem, it implies some approximations. Based on the general linear transformation, an exact solution for the orientation and calibration is derived, which leads to a least-squares adjustment with linear fractional observation equations and non-linear constraints, treated as additional observation equations with zero variances. In contrast to the DLT, this solution can also be applied to cases where the known interior orientation should be enforced.

ABSTRACT: *From the basic relations of a linear transformation an exact solution for the orientation and calibration of cameras is derived. This solution leads to a least-squares adjustment with linear fractional observation equations and non-linear additional constraints. The transformation parameters are computed iteratively using three-dimensional object space control. With these parameters the object space coordinates of all image points, contained in at least two photographs, can be determined in a second step. The efficiency of the method is demonstrated with data of some control surveys of wide spanning surface structures.*

MATHEMATICAL FORMULATION OF THE PROBLEM

BACKGROUND PRINCIPLES

The general linear transformation of points between system $\{x_i\}$ and system $\{X_i\}$ $(i = 1, 2, 3)$ in the three-dimensional space can be expressed by

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$$x_i = \frac{a_{i1}X_1 + a_{i2}X_2 + a_{i3}X_3 + a_{i4}}{a_{41}X_1 + a_{42}X_2 + a_{43}X_3 + a_{44}} \quad (i = 1, 2, 3) \tag{1}$$

The introduction of homogeneous coordinates defined by

$$x_i = \frac{u_i}{t} \text{ and } X_i = \frac{U_i}{T} (i = 1, 2, 3)$$

leads, according to Klein (1926), to the four linear equations

$$u_i = a_{i1}U_1 + a_{i2}U_2 + a_{i3}U_3 + a_{i4}T \quad (i = 1, 2, 3)$$
$$t = a_{41}U_1 + a_{42}U_2 + a_{43}U_3 + a_{44}T$$

The general collineation (Equation 1) can be written in matrix notation

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ t \end{bmatrix} = \underset{4,4}{\boldsymbol{A}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ T \end{bmatrix} \tag{2}$$

The mathematical model used in photogrammetry is a singular collineation where the rank of $\boldsymbol{A}$ is three. After a suitable choice of the coordinate system, Equations 2 may be reduced to

$$\begin{bmatrix} u \\ v \\ t \end{bmatrix} = \underset{3,4}{\boldsymbol{C}} \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} \tag{3}$$

with the element $c_{34} \neq 0$.

These equations contain 12 parameters, c_{ik}, which are dependent, because the uncalibrated "perfect camera" can be represented by nine independent parameters. Using these parameters, we factor the matrix, $\boldsymbol{C}$, in a suitable way (Bender, 1972). Thus, the dependency between our transformation and the independent parameters becomes obvious.

Since in the collineation (Equation 3) only the ratio between the 12 parameters is of importance (Klein, 1926), we can define a matrix, $\boldsymbol{B}$, such that

$$\begin{aligned} \boldsymbol{B} &= \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & 1 \end{bmatrix} = \frac{1}{c_{34}} \cdot \boldsymbol{C} \\ &= \frac{1}{c_{34}} \begin{bmatrix} c & 0 & -x_p \\ 0 & c & -y_p \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -X_o \\ 0 & 1 & 0 & -Y_o \\ 0 & 0 & 1 & -Z_o \end{bmatrix} \\ &= \frac{1}{\lambda} \begin{bmatrix} cm_{11}-x_pm_{31} & cm_{12}-x_pm_{32} & cm_{13}-x_pm_{33} & \alpha \\ cm_{21}-y_pm_{31} & cm_{22}-y_pm_{32} & cm_{23}-y_pm_{33} & \beta \\ -m_{31} & -m_{32} & -m_{33} & \lambda \end{bmatrix} \end{aligned} \tag{4}$$

where

$$\alpha = (x_pm_{31}-cm_{11})X_o + (x_pm_{32}-cm_{12})Y_o + (x_pm_{33}-cm_{13})Z_o$$
$$\beta = (y_pm_{31}-cm_{21})X_o + (y_pm_{32}-cm_{22})Y_o + (y_pm_{33}-cm_{23})Z_o$$
$$\lambda = m_{31}X_o + m_{32}Y_o + m_{33}Z_o$$

and

c is the principal distance of the camera,
x_p, y_p are the coordinates of the principal point,
m_{ij} are the elements of an orthogonal matrix, $\boldsymbol{M}$, and

X_0, Y_0, Z_0 are the object space coordinates of the perspective center.

The conversion from homogeneous to Cartesian coordinates leads to the collinearity equations represented by the linear fractional functions

$$x = \frac{b_{11}X + b_{12}Y + b_{13}Z + b_{14}}{b_{31}X + b_{32}Y + b_{33}Z + 1}$$

$$y = \frac{b_{21}X + b_{22}Y + b_{23}Z + b_{24}}{b_{31}X + b_{32}Y + b_{33}Z + 1} \tag{5}$$

where X,Y,Z are the object space coordinates,

x,y are the corresponding measured image coordinates of a point, and

b_{ik} are 11 transformation parameters.

The dependency between these parameters and the nine parameters necessary was demonstrated in Equation 4.

Equations 5 are the basic equations of the DLT method, where the 11 parameters are considered as being independent, which is one of the aforementioned approximations. In order to obtain an exact solution for the calibration it is necessary either to eliminate two of these 11 parameters or to establish two constraints between them, which must be enforced. The elimination would lead to the well known nonlinear collinearity equations which are normally used. Here we follow the second way and derive the two necessary constraints.

NECESSARY CONSTRAINTS

From Equations 4 we obtain

- using $m_{31}^2 + m_{32}^2 + m_{33}^2 = 1$ (6)

  $$b_{31}^2 + b_{32}^2 + b_{33}^2 = \lambda^{-2} \tag{7}$$

- using $m_{11}m_{31} + m_{12}m_{32} + m_{13}m_{33} = 0$ (8)

  and Equation 6,

  $$x_p = (b_{11}b_{31} + b_{12}b_{32} + b_{13}b_{33})\lambda^2 \tag{9}$$

- using $m_{21}m_{31} + m_{22}m_{32} + m_{23}m_{33} = 0$ (10)

  and Equation 6,

  $$y_p = (b_{21}b_{31} + b_{22}b_{32} + b_{23}b_{33})\lambda^2 \tag{11}$$

- using $m_{11}^2 + m_{12}^2 + m_{13}^2 = 1$ (12)

  and Equations 6 and 8,

  $$c^2 = (b_{11}^2 + b_{12}^2 + b_{13}^2)\lambda^2 - x_p^2 \tag{13}$$

- using $m_{21}^2 + m_{22}^2 + m_{23}^2 = 1$ (14)

  and Equations 6 and 10,

  $$c^2 = (b_{21}^2 + b_{22}^2 + b_{23}^2)\lambda^2 - y_p^2 \tag{15}$$

In Equations 13 and 15 the principal distance, c, is a non-linear function of different transformation parameters. Equating Equations 13 and 15 and making use of Equations 7, 9, and 11, we obtain the first constraint between the parameters

$$(b_{11}^2+b_{12}^2+b_{13}^2) - (b_{21}^2+b_{22}^2+b_{23}^2) + \left[(b_{21}b_{31}+b_{22}b_{32}+b_{23}b_{33})^2 - (b_{11}b_{31}+b_{12}b_{32}+b_{13}b_{33})^2\right](b_{31}^2+b_{32}^2+b_{33}^2)^{-1} = 0 \tag{16}$$

Between the elements of an orthogonal (3, 3)-matrix there exist six independent relations. Five of them were used up to now. The remaining relation

$$m_{11}m_{21} + m_{12}m_{22} + m_{13}m_{23} = 0$$

applied to Equation 4 yields

$$x_p y_p = (b_{11}b_{21} + b_{12}b_{22} + b_{13}b_{23})\lambda^2$$

Thus, we obtain the second constraint

$$(b_{11}b_{21}+b_{12}b_{22}+b_{13}b_{23}) - (b_{31}^2+b_{32}^2+b_{33}^2)^{-1}(b_{11}b_{31}+b_{12}b_{32}+b_{13}b_{33}) \cdot (b_{21}b_{31}+b_{22}b_{32}+b_{23}b_{33}) = 0 \tag{17}$$

In an exact solution of a calibration, Equations 16 and 17 must be fulfilled by the 11 transformation parameters. Before we discuss how these constraints can be incorporated in a least-squares adjustment, we will regard the situation where the interior orientation of a camera is known. In this case the collinearity equations, i.e., Equation 5, contain only six independent parameters, i.e., we have to enforce $11-6 = 5$ constraints. In addition to Equations 16 and 17, the Equations 9, 11, and 13 or 15 must also be fulfilled by the transformation parameters.

DETERMINATION OF THE TRANSFORMATION PARAMETERS

The transformation parameters in Equation 5 can be determined by a least-squares adjustment, if the measured coordinates, x,y, and object space coordinates, $X,Y,Z,$ of at least six control points are known. In this first study we consider the object space coordinates to be free of errors, an assumption which is common in industrial photogrammetry where control points are normally provided with sufficiently high accuracy. Therefore, Equation 5 leads to the observation equations with regard to point P_i

$$
\begin{aligned}
x_i + v_{xi} &= \frac{b_{11}X_i + b_{12}Y_i + b_{13}Z_i + b_{14}}{b_{31}X_i + b_{32}Y_i + b_{33}Z_i + 1} \\
y_i + v_{yi} &= \frac{b_{21}X_i + b_{22}Y_i + b_{23}Z_i + b_{24}}{b_{31}X_i + b_{32}Y_i + b_{33}Z_i + 1}
\end{aligned}
\qquad (18)
$$

There are two ways to consider the two sets of constraints (calibration or orientation) which must be fulfilled in the solution:

(1) the constraints are incorporated in a least-squares adjustment with parameters and additional constraints between these parameters.

In matrix notation we have the observation equations

$$\boldsymbol{v}_1 = \mathbf{A}_1\boldsymbol{u} - \mathbf{l} \text{ with the weight matrix, } \boldsymbol{P}_1,$$
$$\text{and } \boldsymbol{O} = \mathbf{A}_2\boldsymbol{u} - \boldsymbol{d} \text{ as additional constraints.}$$

The least-squares solution leads to

$$\boldsymbol{u} = (\mathbf{A}_1^T\boldsymbol{P}_1\mathbf{A}_1)^{-1}\,(\mathbf{A}_1^T\boldsymbol{P}_1\mathbf{l} + \mathbf{A}_2^T\boldsymbol{k})$$
$$\text{with } \boldsymbol{k} = \left[\mathbf{A}_2(\mathbf{A}_1^T\boldsymbol{P}_1\mathbf{A}_1)^{-1}\mathbf{A}_2^T\right]^{-1}\left[\boldsymbol{d} - \mathbf{A}_2(\mathbf{A}_1^T\boldsymbol{P}_1\mathbf{A}_1)^{-1}\mathbf{A}_1^T\boldsymbol{P}_1\mathbf{l}\right]$$

(2) the constraints are interpreted as additional observation equations with zero variances.

In matrix notation the observation equations are

$$\begin{bmatrix}\boldsymbol{v}_1\\ \boldsymbol{v}_2\end{bmatrix} = \begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix}\boldsymbol{u} - \begin{bmatrix}\mathbf{l}\\ \boldsymbol{d}\end{bmatrix} \text{ with the weight matrix } \begin{bmatrix}\boldsymbol{P}_1 & \mathbf{0}\\ \mathbf{0} & a\boldsymbol{E}\end{bmatrix}$$

The constant, a, is a suitable high weight which enforces the zero variances. We obtain the least-squares solution

$$\boldsymbol{u} = \left[\mathbf{A}_1^T\boldsymbol{P}_1\mathbf{A}_1 + a\mathbf{A}_2^T\mathbf{A}_2\right]^{-1}\left[\mathbf{A}_1^T\boldsymbol{P}_1\mathbf{l} + a\mathbf{A}_2^T\boldsymbol{d}\right]$$

As it is more complicated to adapt the first way into a computer program and as it is less easy to handle, we prefer the second way which is described by Koch and Pope (1969).

Equations 18, as well as the constraints, are non-linear functions involving the parameters. That is the reason for an iterative least-squares solution. Due to the simplicity of our observation equations, being linear fractional functions, initial values for the parameters can easily be determined if there are at least six control points imaged in a photo. We obtain these initial values by solving the following system of 11 linear equations derived from Equation 5:

$$
\begin{aligned}
X_ib_{11} + Y_ib_{12} + Z_ib_{13} + b_{14} - x_iX_ib_{31} - x_iY_ib_{32} - x_iZ_ib_{33} - x_i &= 0 \; (i=1 \ldots 6) \\
X_ib_{21} + Y_ib_{22} + Z_ib_{23} + b_{24} - y_iX_ib_{31} - y_iY_ib_{32} - y_iZ_ib_{33} - y_i &= 0 \; (i=1 \ldots 5)
\end{aligned}
\qquad (19)
$$

If there is only the minimum number of control points given (orientation, 3; calibration, 5) the initial values can be determined in a way similar to that used in the conventional non-linear approach. Because in non-topographic applications enough object space control usually is available, the suggested way seems to be the most efficient one.

DETERMINATION OF THE INTERIOR AND EXTERIOR ORIENTATION WITH THE TRANSFORMATION PARAMETERS

With the computed values of the transformation parameters we obtain the interior orientation from Equations 9, 11, and 13 or 15.

The exterior orientation can also easily be determined (Thompson, 1971). From Equation 4 we have three linear equations for the coordinates, X_o, Y_o, Z_o, of the camera station

$$b_{11}X_o + b_{12}Y_o + b_{13}Z_o + b_{14} = 0$$
$$b_{21}X_o + b_{22}Y_o + b_{23}Z_o + b_{24} = 0$$
$$b_{31}X_o + b_{32}Y_o + b_{33}Z_o + 1 = 0$$

The elements of the orthogonal matrix, $\boldsymbol{M}$, result from Equations 4 and 7, too, i.e.,

$$m_{31} = -\lambda b_{31}$$
$$m_{32} = -\lambda b_{32}$$
$$m_{33} = -\lambda b_{33}$$
$$m_{11} = (\lambda b_{11}+x_p m_{31})/c \qquad m_{21} = (\lambda b_{21}+y_p m_{31})/c$$
$$m_{12} = (\lambda b_{12}+x_p m_{32})/c \qquad m_{22} = (\lambda b_{22}+y_p m_{32})/c$$
$$m_{13} = (\lambda b_{13}+x_p m_{33})/c \qquad m_{23} = (\lambda b_{23}+y_p m_{33})/c$$

DETERMINATION OF OBJECT SPACE COORDINATES

Now the calibration or orientation problem is solved, the transformation parameters are determined, and the interior and exterior orientation are known. In non-topographic applications these parameters are usually not of great interest, as we want to compute object space coordinates of points.

Equations 5 are a simple relation between the measurements, x_i and y_i (in an arbitrary orthogonal system), the unknown object space coordinates, X_i, Y_i, Z_i, and the parameters, b_{ik}, which were determined above. They can be used for the least-squares computation of these unknowns, if the corresponding point, P_i, has been observed in at least two photographs.

Regarding the parameters, b^n_{jk}, as uncorrelated constants, we get for each point, P_i, in photograph n the observation equations

$$x_{i,n} + v_{x_{i,n}} = \frac{b^n_{11}X_i + b^n_{12}Y_i + b^n_{13}Z_i + b^n_{14}}{b^n_{31}X_i + b^n_{32}Y_i + b^n_{33}Z_i + 1}$$
$$y_{i,n} + v_{y_{i,n}} = \frac{b^n_{21}X_i + b^n_{22}Y_i + b^n_{23}Z_i + b^n_{24}}{b^n_{31}X_i + b^n_{32}Y_i + b^n_{33}Z_i + 1} \qquad (20)$$

where

$x_{i,n}$, $y_{i,n}$ are the measured coordinates of point P_i in photograph n,
$y_{x_{i,n}}$, $v_{y_{i,n}}$ are the corresponding corrections,
b^n_{jk} are the 11 transformation parameters of photograph n, and
X_i, Y_i, Z_i are the unknown object space coordinates of P_i.

Since Equations 20 are linear fractional functions, we have to solve the least-squares problem iteratively. The necessary initial values can easily be determined out of the (3,3) system derived from Equation 5, which is again linear. With the measured coordinates, x, y, $\bar{x}$, of point P in two photographs, we obtain

$$(xb_{31}-b_{11})X + (xb_{32}-b_{12})Y + (xb_{33}-b_{13})Z + x - b_{14} = 0$$
$$(yb_{31}-b_{21})X + (yb_{32}-b_{22})Y + (yb_{33}-b_{23})Z + y - b_{24} = 0$$
$$(\bar{x}\bar{b}_{31}-\bar{b}_{11})X + (\bar{x}\bar{b}_{32}-\bar{b}_{12})Y + (\bar{x}\bar{b}_{33}-\bar{b}_{13})Z + \bar{x} - \bar{b}_{14} = 0$$

Practical Experiences

In this first study we tested this method and its applicability to industrial photogrammetry with data of several photogrammetric control surveys using a metric camera. In these tests we regarded the coordinates of the control points as constant, an assumption which is common in these applications. As mentioned earlier, we considered the transformation

TABLE 1. COMPARISON OF COMPUTING TIME

Example	conventional collinearity method (seconds)	new method orientation (seconds)	new method calibration (seconds)
1 photo 28 control points	—	—	5.2
2 photos 13 control points in each photo 123 object points	16.8	7.6	7.7
4 photos 14 control points in each photo 178 object points	19.3	10.4	11.0
14 photos 12 control points in each photo 693 object points	37.2	31.5	31.9

parameters, b, to be uncorrelated and we did not yet incorporate any distortion parameters into the solution. These three facts will be taken into consideration in further studies.

We took four examples in order to compare the necessary computing time in system seconds (CP+IO) of the described method with that of the conventional collinearity solution (see Table 1). The computations were performed on the CYBER 174 computer of the University of Stuttgart.

All photographs were taken with the UMK 10/1318 of the Institut für Anwendungen der Geodäsie im Bauwesen of the University of Stuttgart. The plates were measured in the Zeiss PSK. With these data—without reduction of the coordinates of the principal point—we determined the object space coordinates, enforcing the known interior orientation. The same computer program was utilized for the calibration using the same data.

The data used in the last example were taken from the photogrammetric control survey of a cable net structure which forms the supporting construction of the shell of a large cooling tower (Bopp *et al.*, 1977).

The resulting object space coordinates in the case of the orientation were identical to those obtained with the conventional method. The differences between the results of the on-the-job calibration and those of the orientation were negligible. They can be explained through the fact that the configurations of the control points of the examples were not designed for a calibration.

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