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Selection of Additional Parameters for the Bundle Adjustment*

Although unknown systematic errors were corrected for by solving for linear functions of parameters of image coordinates, nearly as great an improvement was obtained by employing measurements of eight rather than four fiducial marks.

Introduction

At the National Research Council of Canada, a program for the adjustment of bundles has been in use since 1972 (Schut, 1972). Its mathematical formulation has the following main features in common with most of the programs used elsewhere:

- The condition equations used are those for collinearity of image point, projection center, and terrain point.
- Given coordinates of terrain points are treated in essentially the same way as unknown coordinates. Thus, each terrain point is essentially given three coordinate corrections.
- A direct solution of the total system of normal equations is used.

Other features, apparently not encountered in other programs, are—

KEY WORDS: **Bundle adjustment; Errors; Linear functions; Parameters; Photographic coordinates**
ABSTRACT: In the block adjustment of bundles, additional parameters may be introduced to correct for unknown systematic errors. For this purpose, correction to the measured photograph coordinates are expressed as suitable linear functions of such parameters, and the latter are solved for during the adjustment. It has been stated that this procedure greatly improves the accuracy of the adjustment.
This paper discusses the selection of such parameters and shows the effect of various combinations of them upon the adjustment of blocks of photographs taken over three test areas and with four cameras. It is shown that no single set of parameters can consistently produce the best results. The improvements obtained by using the most suitable set have varied from very moderate to, indeed, very considerable. However, a large part of this improvement can be achieved equally well by means of advance corrections based upon measurements of eight fiducial marks.
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- The formation and solution of the normal equations is carried out exclusively by operations on submatrices rather than on elements.
- The attitude of the photographs is defined by the four Rodriguez parameters of the orthogonal matrix.

Since early 1975, the program has been extensively modified in an effort to simplify

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the input requirements and to adapt the program for use on a computer with a medium-size fast-access storage supplemented by a disk or three tapes (Schut, 1978). The following features have been added to the mathematical formulation:

- Corrections for Earth curvature and refraction, if required, are now applied to the rays in object space rather than to the image coordinates.
- While the early version of the program already contained provision for the incorporation of additional parameters in the normal equations to correct for unknown systematic deformation of the photographs, a subroutine for the computation of such parameters has now been added to the program.

This paper reports on the theoretical considerations and on the experiments that have been performed to arrive at a sensible selection of such parameters.

Review of Parameters Used

In the bundle adjustment, the measured comparator coordinates, x and y, are first converted to photograph coordinates with origin at the principal point. Let these coordinates be denoted by u and v, and let the u-axis be chosen in the flight direction. In most of the publications that deal with the use of additional parameters, the errors Δu and Δv of the photograph coordinates caused by systematic deformation of the photographic image are approximated by polynomials in the photograph coordinates. The coefficients of these polynomials are called additional or, alternatively, added parameters. They are added to the regular parameters in the adjustment and are solved for simultaneously.

In these polynomials for Δu and Δv, the two constant terms, two of the four possible linear terms, and two of the six possible second-degree terms usually will be too heavily correlated with the six orientation elements of each photograph and will have to be omitted. This leaves six terms of up to and including the second degree, e.g.,

$$\begin{aligned} \Delta u &= b_1 uv + b_2 v^2 \\ \Delta v &= a_1 v + a_2 u + b_3 u^2 + b_4 uv \end{aligned} \tag{1}$$

The linear terms correct for affine deformation of the photographs. It is immaterial whether these terms are used to correct v, as here, or u, as in Brown (1976) or both, as in Ebner (1976), or one coordinate each, as in Gotthardt (1975). The second-degree terms are used in the above form by Brown, with a minor modification by Ebner, and they are proposed also by Gotthardt.

The two terms with the product uv could be replaced by the two omitted second-degree terms without noticeably affecting the result of the adjustment except perhaps in the case of extreme terrain height variations (Schut, 1974). Even in such a case they cannot be expected to make the result worse. However, no writer has used those missing terms.

Of the eight possible third-degree terms, the four terms with u^3 and v^3 cannot be used if only nine image points are measured in a grid pattern in each photograph because their parameters will be highly correlated with the parameters a_1 and a_2. Brown uses the remaining four terms and two of the fourth degree, in addition to those of Equation 1, i.e.,

$$\begin{aligned} \Delta u &= c_1 u^2 v + c_2 uv^2 + d_1 u^2 v^2 \\ \Delta v &= c_3 u^2 v + c_4 uv^2 + d_2 u^2 v^2 \end{aligned} \tag{2}$$

Ebner has used essentially the 12 parameters of Equations 1 and 2, but has made a linear transformation of the parameters to obtain an orthogonal (correlation-free) set. As Brown has already remarked, this is not necessary because the correlation between the above 12 is not so high that it would make the normal equations ill-conditioned.

In the case of a block with 60 percent overlap between photographs in two directions, but measuring only nine points per photograph, 12 parameters is the maximum that can be used and the parameters of Equations 1 and 2 or a selection from them form a suitable set.

If the density of the image points is considerably greater, more parameters can be used and the use of the third-degree terms omitted in Equation 2 can be reconsidered. Brown (1976) adds to the above 12 parameters six which apply radial corrections to the photograph coordinates. Each of the six contributes terms with powers and/or products of u and v to both Δu and Δv. The first of the six occurs in combination with a term with u^3 in Δu and a term with v^3 in Δv. Schut (1974) makes use of the same two terms but in a way, described below, which does not increase the number of parameters. The other two third-degree terms have been tried out by Grün (1976, 1978), who has experimented with sets of parameters suggested by Brown and by Ebner and with various other sets. In one experiment only, he found the term with v^3 in Δu to be significant. Of the ten fourth-degree terms omitted in Equation 2, Brown has earlier considered eight, but has omitted them in his latest error model (Brown 1973, 1975, 1976).

Therefore, assuming that the polynomials can reasonably be restricted to terms of no higher than the fourth degree, and having a high density of image points, perhaps only two terms of the third degree should be added to those in Equation 2:

$$\begin{aligned} \Delta u &= c_1u^2v + c_2uv^2 + d_1u^2v^2 + c_5u^3 \\ \Delta v &= c_3u^2v + c_4uv^2 + d_2u^2v^2 + c_6v^3 \end{aligned} \quad (3)$$

The parameters in the above equations have been used in some of the experiments reported on below.

Alternatively, it is possible to devise corrections for systematic deformation in which errors or corrections, Δu and Δv, cannot be written as polynomials in u and v. Two formulations have been published which base their corrections on a polar coordinate system (r,ϕ) with origin in the principal point. In this system one formulates radial corrections, Δr, and tangential corrections (perpendicular to the radius vector of an image point), $\Delta t = r\Delta\phi$.

Bauer, who was the first to use added parameters in a bundle adjustment (Bauer and Müller, 1972; Bauer, 1975), makes use of two parameters for affine deformation, of two parameters for corrections, Δr, proportional to powers of r, and of corrections Δr and Δt which are products of trigonometric functions of the azimuth, ϕ, by the radius vector, r. The latter corrections increase only linearly along any radial line.

Experimentation with spherical harmonics has been predicted, perhaps facetiously, by Brown (1976) and has since been carried out by El Hakim and Faigh (1977). This has resulted in a purely radial correction written as a function of the polar coordinates. Although not shown by the authors, this function can be written also as a function of the u- and v-coordinates. Making a linear transformation of the parameters, it turns out to be nothing else than a complete polynomial in u and v. With the ten parameters used by the authors, the function becomes a complete polynomial of the third degree.

Selection of a Set of Seven Parameters

In the case of a block with 20 percent side overlap between strips, the use of a set of 12 parameters can lead to the danger of introducing deformation into the block instead of correcting for it. To derive a parameter set that can be safely used here, the concept of correction for strip deformation developed by Schut (1974) is useful. Here again, the errors Δu and Δv are written as polynomials in u and v. As before, the first parameters to be used will be the two that correct for affine deformation, i.e.,

$$\begin{aligned} \Delta u &= 0 \\ \Delta v &= a_1v + a_2u \end{aligned} \quad (4)$$

The first parameter corrects for a difference in scale of the photographs in the u- and v-directions and the second parameter corrects for twist or nonorthogonality of the axes.

The parameters b_1 to b_4 of Equation 1 will be replaced by two parameters which serve to correct a triangulated strip for a linear variation of scale and of azimuth. In Equation 1 the parameter b_4 corrects a constant change of scale throughout a triangulated strip and the parameter b_3 corrects a constant change of azimuth. Writing b_1 as a suitable function of b_3 and b_2 as a suitable function of b_4, the corrections Δu and Δv will apply a conformal transformation to the photograph coordinates. Choosing two other suitable functions, the corrections will apply a parabolic transformation. In either way, the number of parameters is reduced by two. Further, it is of interest that, when adding a multiple of b_4uv to Δv and the same multiple of b_4u^2 to Δu, the effect of the added parameters upon a triangulated strip of flat terrain remains virtually unchanged. Only the orientation elements of the photographs are affected. The same applies if a multiple of b_1uv is added to Δu and the same multiple of b_1v^2 is added to Δv. In either case, if this is done starting from a two parameter conformal or parabolic transformation, the transformation changes its character and might now be called pseudo-conformal or pseudo-parabolic.

This offers quite a few possible variations in the choice of the next two parameters. The conformal transformation and two pseudo-conformal transformations are of special interest, i.e.,

$$\begin{aligned} \Delta u &= b_1(-u^2 + v^2) - 2b_2uv \\ \Delta v &= \quad -2b_1uv + b_2(u^2 - v^2) \end{aligned} \quad (5a)$$

$$\begin{aligned} \Delta u &= b_1(u^2 + v^2) \\ \Delta v &= b_2(u^2 + v^2) \end{aligned} \quad (5b)$$

$$\begin{aligned} \Delta u &= b_1(3u^2 + v^2) + 2b_2uv \\ \Delta v &= 2b_1uv + b_2(u^2 + 3v^2) \end{aligned} \quad (5c)$$

The conformal transformation is the first one. The third one is known as the correction for decentering lens distortion and has been used by Brown (1973, 1975). The second one stands midway between the other two and is of interest because it involves somewhat less computation. It may be noted that in the conformal transformation the signs of the terms with b_1 are the opposite from the signs that are obtained when this transformation is derived from the conventional formulation in complex variables. This serves to facilitate the derivation of Equations 5b and 5c from the conformal transformation and to obtain in each case the same sign for the computed value of b_1.

A parabolic transformation is obtained by omitting the terms with v^2 in the conformal transformation. Using now the conventional signs, this gives

$$\begin{aligned} \Delta u &= b_1u^2 - 2b_2uv \\ \Delta v &= 2b_1uv + b_2u^2 \end{aligned} \tag{5d}$$

and a simple pseudo-parabolic one is—

$$\begin{aligned} \Delta u &= 0 \\ \Delta v &= b_1uv + b_2(u^2 + 2v^2) \end{aligned} \tag{5e}$$

In each of these five transformations, the terms with b_1 correct a constant change of scale in a triangulated strip and the terms with b_2 correct a constant change of azimuth. The simplest transformation that has this effect is, of course, obtained by using only two contributions to Δv, i.e.,

$$\begin{aligned} \Delta u &= 0 \\ \Delta v &= b_1uv + b_2u^2 \end{aligned} \tag{5f}$$

Of the eight possible parameters in third-degree terms, six can be equated in pairs to produce corrections of triangulated strips for longitudinal height curvature, torsion, and transversal height curvature, respectively, i.e.,

$$\begin{aligned} \Delta u &= c_1u^3 + c_2u^2v + c_3uv^2 \\ \Delta v &= c_1u^2v + c_2uv^2 + c_3v^3 \\ \text{or } \Delta r/r &= c_1u^2 + c_2uv + c_3v^2 \end{aligned} \tag{6a}$$

As shown by the third of these equations, these parameters apply purely radial corrections. Equating further the first and the third parameter, a spherical height curvature correction and a torsion correction are produced, i.e.,

$$\begin{aligned} \Delta u &= c_1ur^2 + c_2u^2v \\ \Delta v &= c_1vr^2 + c_2uv^2 \\ \text{or } \Delta r/r &= c_1r^2 + c_2uv \end{aligned} \tag{6b}$$

The two parameters of Equation 4, the two in one of the pairs of Equation 5 and the two in Equation 6b together form a set of six parameters that can be safely used for blocks with 60 percent longitudinal overlap in the strips and 20 percent overlap between strips. In the NRC program for bundle adjustment, such a set has been incorporated by means of a subroutine as follows:

$$\begin{aligned} \Delta u &= \qquad\qquad\quad b_1(u^2 + v^2) + c_1ur^2 \\ &\quad + c_2u^2v \\ \Delta v &= a_1v + a_2u + b_2(u^2 + v^2) + c_1vr^2 \\ &\quad + c_2uv^2 \end{aligned} \tag{7}$$

If one specifies that p added parameters shall be used in an adjustment, only the first p parameters in Equation 7 are used. One of the (pseudo-) conformal transformations in Equation 5 has been preferred to one of the others because only the former apply constant scale and azimuth corrections in the strip direction regardless of whether one chooses the u-axis or the v-axis in that direction. Similarly, Equation 6b applies a longitudinal height curvature correction to a triangulated strip independent of the choice of u- and v- axes.

Unless only the first two parameters are used, care must be taken that in all photographs the u- and v-axes have the same positive directions with respect to the direction of flight. The two parameters of Equation 6b may be replaced by the three of Equation 6a only if sufficient vertical control is available to correct the strips separately for longitudinal and transversal height curvature.

Experiments

In order to evaluate the effect of the sets of parameters in the preceding section, they have been used in the adjustment of two blocks. One of these is a block of two strips with 14 photographs at a scale of 1:15 000 taken for the St. Faith experiment (Thompson, 1963). The photographs were taken with a Hilger and Watts camera with a 6 inch Wild Aviogon wide-angle lens on Ilford H.R.A. film. Side overlap is about 20 to 30 percent. The lens has a reseau but in this investigation the reseau has not been used. This block has relatively few check points for height and, moreover, with a few exceptions those points are situated in three bands across the strips. The other block contains three strips with 29 photographs at a scale of 1:21 000 taken over the Crysler test area of the NRC. These photographs were taken with a Wild RC8 camera with a 6 inch Aviogon wide-angle lens without reseau on Kodak Aerographic film with estar base. Side overlap is about 60 percent. All measurements have been corrected for symmetric radial lens distortion. In all adjustments, a three-dimensional coordinate system consisting of easting, northing, and terrain height has been used and the object rays have been corrected for the effects of Earth curvature and refraction.

Tables 1, 2, and 2a list the root-mean-square values of the residuals obtained in various adjustments of these two blocks. The RMS values of the residuals at the image points have been computed in the conventional way as $m = (m_u^2 + m_v^2)^{\frac{1}{2}}$, and those in

TABLE 1. HILGER AND WATTS 105 CAMERA WITH 6 INCH WILD AVIOGON WIDE-ANGLE LENS AND RESEAU
BLOCK OF TWO STRIPS WITH 14 PHOTOGRAPHS AT A SCALE OF 1:15,500 FLOWN OVER
ST FAITH TEST AREA (10 cm = 6.5 μm)
TEN PLANIMETRIC AND 14 HEIGHT CONTROL POINTS

added parameters			RMS values of residuals			
			at check points		at image points	
number	Equations	type	plan (cm) (151)	height (cm) (29)	control (μm) (46)	non-control (μm) (619)
0			15.8	33.1	11.2	6.5
0*			16.5	33.9	11.7	6.6
0***			14.1	33.0	10.8	6.5
2	4**	affine	14.8	32.3	9.8	6.5
4	5a	conformal terms	14.5	32.3	9.5	6.5
4	5b	pseudo-conformal	14.5	32.3	9.5	6.5
4	5c	decentering corr.	14.4	32.2	9.5	6.5
4	5d	parabolic terms	14.3	32.1	9.4	6.5
4	5e	pseudo-parabolic	14.3	32.2	9.4	6.5
4	5f	mixed 2nd-degree	14.8	32.1	9.5	6.5
3	c_1 of 6a	long. height corr.	14.9	31.8	9.5	6.1
4	c_1, c_2 of 6a	two height corr.	13.6	33.3	9.0	6.1
5	6a	three height corr.	13.1	34.5	9.0	6.0
3	c_1 of 6b	radial corr.	14.0	34.6	9.5	6.1
4	6b	two height corr.	12.9	35.6	9.0	6.0
5	5e and c_1 of 6b		13.4	34.2	9.1	6.1
6	5e and 6b		12.5	35.2	8.5	6.0

* after correction for an average differential scale change of 20 parts in 100,000, as computed from measurements of four reseau marks
** used also in all following adjustments
*** after correction for an average differential scale change of 6 parts in 100,000, as computed in the adjustment with two added parameters

planimetry at the check points have been computed similarly.

The first lines in the tables show the results obtained first without using additional parameters and then with the use of the two parameters for correction of affine deformation. If no additional parameters are used, it should be useful to give the photograph coordinates corrections for affine distortion, as derived from measurements of the fiducial marks. In particular, an average value of the differential scale change can easily be determined from those measurements and be corrected for in advance of the adjustment.

Table 1 shows that in the case of the Hilger and Watts block this correction for differential scale change does not improve the accuracy. The correction has here been computed from measurements of the four reseau crosses in the middle of the four sides. It varies for the 14 photographs from 9 to 26 parts in 100,000. Correction of all measured photograph coordinates by the average of 20 parts in 100,000 actually increases the root-mean-square values of the residuals in planimetry and height. The actual value of the differential scale error, as computed by the adjustment with two added parameters, is only 6 parts in 100,000, smaller than the measured value in any of the photographs. Adjustment with that value does somewhat improve the results.

Tables 2 and 2a list results obtained with and without the use of these corrections in the case of the Wild RC8 block. Here, the differential scale error has been determined for each photograph from measurements of the four corner fiducial marks. The measured values vary from 7 to 30 parts in 100,000 with an average of a little less than 20 parts in 100,000. This is very close to the value of 15 parts in 100,000 computed by the adjustment with two added parameters. Here, the use of these two parameters gives an improvement in accuracy of 30 to 40 percent.

Next, Tables 1 and 2 list results obtained with four added parameters. Besides the affine transformation of Equation 4, here the second-degree transformations of Equations 5a to 5f are used. As predicted already on

TABLE 2. WILD RC8 CAMERA WITH 6 INCH AVIOGON WIDE-ANGLE LENS
BLOCK OF 3 STRIPS WITH 29 PHOTOGRAPHS AT A SCALE OF 1:21,000 FLOWN
OVER CRYSLER TEST AREA (10 cm = 5 μm)
EIGHT PLANIMETRIC AND 11 HEIGHT CONTROL POINTS

added parameters			RMS values of residuals			
			at check points		at image points	
			plan (cm)	height (cm)	control (μm)	non-control (μm)
number	Equations	type	(41)	(41)	(40)	(526)
0			43.2	35.6	11.3	7.2
0*			25.5	26.3	8.9	7.0
2	4**	affine	24.5	23.2	7.8	7.0
4	5a	conformal terms	23.4	23.2	7.5	6.6
4	5b	pseudo-conformal	23.3	23.1	7.5	6.6
4	5c	decentering corr.	23.3	23.0	7.5	6.6
4	5d	parabolic terms	23.4	23.3	7.7	6.6
4	5e	pseudo-parabolic	23.4	23.2	7.8	6.6
4	5f	mixed 2nd-degree	22.5	22.6	7.4	6.5
3	c_1 of 6a	long. height corr.	24.7	22.9	7.8	7.0
4	c_1,c_2 of 6a	two height corr.	24.9	23.2	7.8	7.0
5	6a	three height corr.	24.9	23.2	7.9	7.0
3	c_1 of 6b	radial corr.	24.3	22.9	7.8	7.0
4	6b	two height corr.	24.5	23.2	7.8	7.0
5	5f and c_1 of 6b		22.4	22.3	7.4	6.5
6	5f and 6b		22.4	22.3	7.3	6.5
10	1 and 3 (first ten parameters)		22.8	22.3	7.2	6.5
14	1 and 3		22.4	22.2	7.1	6.5

* after correction for an average differential scale change of 20 parts in 100,000
** used also in all following adjustments

TABLE 2a. SAME PHOTOGRAPHY AS IN TABLE 2
EIGHTEEN PLANIMETRIC AND 22 HEIGHT CONTROL POINTS

added parameters		RMS values of residuals			
		at check points		at image points	
		plan (cm)	height (cm)	control (μm)	non-control (μm)
number	Equations	(30)	(30)	(70)	(496)
0		31.3	33.5	12.9	7.7
0*		21.2	22.8	9.3	7.3
2	4**	19.6	23.1	8.2	7.2
4	5f	18.9	23.4	7.9	6.7
5	5f and c_1 of 6b	18.7	23.2	7.9	6.7
6	5f and 6b	18.7	23.8	7.8	6.7
8	5f, 6b, and 8	18.6	23.6	7.9	6.7

* after correction for average differential scale change of 20 parts in 100,000
** used also in all following adjustments

theoretical grounds, the results obtained with the three different (pseudo-) conformal transformations are virtually the same, and those obtained with the two different (pseudo-) parabolic transformations are virtually the same. However, different types of second-degree transformation give some distinctly different results. Interestingly, the transformation of Equation 5f gives the best results both in planimetry and height in

Table 2 but in Table 2a it improves only the planimetry and in Table 1 it gives the worst results in planimetry.

The next five lines in Tables 1 and 2 show results obtained by replacing the second-degree transformations by various transformations of the third degree designed to correct errors that produce height deformations in triangulated strips. Rather surprisingly, they do not show an expected improvement of the heights but, in the case of Table 1, they show instead an appreciable improvement in the planimetric accuracy.

The following lines of Tables 1, 2, and 2a show results obtained with a combination of second and third-degree terms and, in the case of Table 2a, the two fourth-degree terms retained in Brown (1976), i.e.,

$$\begin{aligned} \Delta u &= d_1 u^2 v^2 \\ \Delta v &= d_2 u^2 v^2 \end{aligned} \tag{8}$$

The largest improvement in the accuracy over the adjustment with two added parameters amounts to 15 percent in planimetry, but this is associated with a 10 percent decrease in accuracy of the heights. Adjustments of the Wild RC8 block with the parameters of Equations 1 and 3 have not improved upon these results.

It may be noticed that in two cases in Tables 2 and 2a the RMS values of residuals at image points increase from 7.8 to 7.9 μm when the number of added parameters is increased. This apparently illogical result is caused by minor errors introduced in the computation because the program rounds off the coordinates of the projection centers in centimeters before intersected image coordinates and image residuals are computed. This adds minor errors to the residuals and affects different adjustments slightly differently.

Further experiments have been performed with two sets of 3 × 3 photographs with 60 percent overlap in both directions taken in 1974 at a scale of 1:7750 over the Sudbury test area, simultaneously with a Wild RC8 camera with a 6 inch Aviogon wide-angle lens and a Zeiss RMK camera with 6 inch Pleogon wide-angle lens, both on Kodak Aerographic film with estar base. The adjustments have been performed using 12 complete control points along the perimeter of the block and an additional height control point in the center. The four control points in the corners of the block each occur in only one photograph.

Both these cameras have a reseau, but for the present purpose the reseau has been used only to determine the average differential scale change of the film. This scale change amounts for both sets of photographs to no more than about 5 parts in 100,000. However, in the case of the Wild photographs, the two sides of the photographs parallel to the long sides of the film roll show a differential scale change of between 35 and 50 parts in 100,000, while the other two sides show no differential scale change. In the case of all but one of the Zeiss photographs, the first two sides show a differential scale change of about 20 parts in 100,000 and the latter two show one that varies between 25 and 45 parts in 100,000.

The sizes of these various differential scale changes are reflected in the results of the adjustments with added parameters, shown in Tables 3 and 4. The affine transformation of Equation 4 necessarily improves the residuals at the image points, but

TABLE 3. WILD RC8 CAMERA WITH 6 INCH AVIOGON WIDE-ANGLE LENS AND RESEAU
BLOCK OF 3 × 3 PHOTOGRAPHS FLOWN AT SCALE 1:7750 OVER THE SUDBURY TEST AREA (10 cm = 13 μm)
TWELVE PLANIMETRIC AND 13 HEIGHT CONTROL POINTS

added parameters		RMS values of residuals			
		at check points		at image points	
number	in Equations	plan (cm) (173)	height (cm) (173)	control (μm) (26)	non-control (μm) (591)
0		15.8	18.8	13.2	5.1
2	4	16.5	19.7	11.6	5.0
4	7 (first 4)	12.2	14.1	10.2	4.2
6	7	6.2	9.1	5.2	3.8
6	1	12.1	13.5	10.4	4.1
10	1 and 3 (first 10)	8.4	9.5	7.0	4.0
14	1 and 3	5.5	8.2	4.9	3.7

TABLE 4. ZEISS RMK CAMERA WITH 6 INCH PLEOGON WIDE-ANGLE LENS AND RESEAU
BLOCK OF 3 × 3 PHOTOGRAPHS FLOWN AT SCALE 1:7750 OVER THE SUDBURY TEST AREA (10 cm = 13 μm)
TWELVE PLANIMETRIC AND 13 HEIGHT CONTROL POINTS

added parameters		RMS values of residuals			
		at check points		at image points	
		plan (cm)	height (cm)	control (μm)	non-control (μm)
number	in Equations	(165)	(165)	(26)	(573)
0		15.0	17.5	13.7	5.8
0*		15.3	17.8	13.4	5.8
2	4	16.3	19.0	9.5	5.6
4	7 (first 4)	7.2	9.1	5.5	4.4
6	7	7.6	9.4	5.4	4.4
6	4,5a,6b	7.6	9.5	5.3	4.4
6	1	6.2	8.5	5.3	4.3
10	1 and 3 (first 10)	6.6	8.9	5.0	4.2
14	1 and 3	6.5	8.8	4.9	4.2

* after correction for an average differential scale change of 4 parts in 100,000, as computed from measurements of eight reseau marks.

the residuals at the check points become even larger than without the use of the added parameters. On the other hand, in the case of the Zeiss photography, the use of the first four parameters of Equation 7 already reduces the residuals at the check points by about 50 percent. In the case of the Wild photography with its different differential scale changes, this reduction is achieved, and even exceeded in planimetry, when all six parameters of Equation 7 are used.

Having here photographs with 60 percent overlap in both directions and up to 100 measured targets in each photograph, it is possible to use the full set of 14 added parameters of Equations 1 and 3. This set gives still somewhat better results, with a maximum of 65 percent improvement in the residuals at check points.

The Wild RC8 camera used over the Crysler test area also was used to fly ten separate strips over the Sudbury test area, at three different scales. The photographs were taken on one roll of Kodak Aerographic film, in the same sequence as listed in Table 5. In each adjustment of each strip, all ground control points were used as such. Here also, the density of the measured points allows the set of 14 added parameters to be used.

The RMS values of the residuals obtained in the adjustments are listed in Table 5. The residuals are derived from the adjusted positions of the ground control points. Because the given coordinates were used with a weight which, although large, does not absolutely enforce the position, these positions differ from the given ones. However, in all cases they differ by less than one centimetre.

As Table 5 shows, the use of an average correction for differential scale change (7 parts in 100,000) computed from measurements of the fiducial marks and the use of the parameters of Equation 7 improve the residuals rather little. A further small but appreciable improvement is obtained by using the parameters of Equations 1 and 3. These results differ much from those obtained for the Crysler block. It is also noteworthy that in the adjustments with two parameters the value of the parameter a_1 for correction of differential scale change, even though it has the same sign for all ten strips, varies in absolute value along the one film-roll from 1 to 10 parts in 100,000.

DISCUSSION OF RESULTS

The results listed in Tables 1 to 5 are summarized in Table 6 in the form of percentages of reduction of RMS values obtained by the use of corrections for measured differential scale changes and by the use of additional parameters.

The table shows clearly that different blocks can have very different image deformations and require different additional parameters for their elimination. The improvement by the use of additional parameters varies from nil to about 60 percent. The latter value is close to one already obtained by Bauer and Müller (1972) at the check points (there called pass points) in the Oberschwaben block.

These high values are obtained by comparing the results with those of an adjustment in which no special corrections for systematic image deformation are applied in

TABLE 5. WILD RC8 CAMERA WITH 6 INCH AVIOGON WIDE ANGLE LENS
TEN STRIPS AT THREE SCALES FLOW OVER THE SUDBURY TEST AREA
RMS VALUES OF THE RESIDUALS AT THE CONTROL POINT IMAGES IN MICROMETRES.

strip	photographs	control points		added parameters			
		in terrain	images	0	0*	2	$10^5 a_1$
1:6,000							
line 1	11	97	381	6.2	6.0	5.9	5
2	12	90	375	8.1	7.7	7.6	6
3	10	83	313	8.1	7.3	7.1	10
1:10,000							
line 1	9	138	569	7.8	7.3	7.3	8
2	8	141	575	8.0	7.4	7.3	10
3	7	146	529	8.1	8.0	7.8	4
4	7	141	503	7.0	6.6	6.5	7
5	8	145	575	7.1	7.2	7.0	3
1:14,000							
line 1	5	196	620	7.7	7.9	7.5	1
2	3	184	414	6.8	6.3	6.2	8
mean				7.4^{9}	7.1^{7}	7.0^{2}	

strip	with parameters of Equation 7		with parameters of Equations 1 and 3		
	first 4	all 6	first 6	first 10	all 14
1:6,000					
line 1	5.9	5.8	5.8	5.5	5.5
2	7.6	7.6	7.2	6.6	6.5
3	6.8	6.7	6.6	6.1	6.1
1:10,000					
line 1	7.3	7.3	6.8	6.5	6.5
2	7.2	7.2	7.1	6.6	6.5
3	7.8	7.7	7.7	7.1	7.0
4	6.3	6.2	6.3	5.7	5.7
5	7.0	7.0	6.6	6.1	6.0
1:14,000					
line 1	7.5	7.4	7.5	6.9	6.8
2	6.2	6.2	6.0	5.7	5.6
mean	6.9^{6}	6.9^{1}	6.7^{6}	6.2^{8}	6.2^{2}

* after correction for an average differential scale change of 7 parts in 100,000

TABLE 6. PERCENTAGES OF REDUCTION OF RMS VALUES OF RESIDUALS IN PLANIMETRY AND HEIGHT (TABLES 1 TO 4) AND IN IMAGE COORDINATES (TABLE 5) OBTAINED BY THE USE OF ADDED PARAMETERS

tables	number of added parameters, and Equations used				
		2	4	6	14
	0*	(4)	(4) and (5b)	(7)	(1) and (3)
1	−4 and −2%	6 and 2%	8 and 2%	20 and −7%	
2	41 and 26	43 and 35	48 and 37**	48 and 37**	48 and 38%
2a	32 and 32	37 and 31	40 and 30**	40 and 29**	
3		−4 and −5	23 and 25	61 and 52	65 and 56
4	−2 and −2	−9 and −9	52 and 48	49 and 46	57 and 50
5	4	6	7	8	17

* after correction for average differential scale change
** Equations 5f used instead of 5b

advance. However, such corrections can be easily derived from measurements of fiducial marks and they have the advantage that their use hardly increases the computation time.

If in the case of the Crysler block of Tables 2 and 2a the comparison is made with the adjustment in which the correction for average differential shrinkage is applied, the further improvement obtained by the use of additional parameters varies from nil to less than 20 percent. In the case of the block adjustments of Bauer and Müller, also, the bulk of the improvement is obtained by the use of the two parameters which correct for affine distortion; presumably, this improvement could be achieved alternatively by an advance correction based upon measurements of fiducial marks. In the case of the Sudbury blocks of Tables 3 and 4, a correction for the more complicated deformation, mentioned earlier, could be derived from the measurements of four or eight reseau crosses and should give similar improvements.

It appears from this that the use of additional parameters can hardly improve the result of an adjustment by more than some 20 percent over that of an adjustment in which the determinable systematic errors have been corrected for in advance. Such corrections could be based upon measurements of eight fiducial marks.

More experimentation under better controlled conditions is needed to determine

- what are the causes of the different systematic errors observed in the different blocks;
- whether the photographs of the same filmroll should be corrected for their individual measured systematic deformation or for the average; and, perhaps,
- whether other parameters exist that can contribute significantly to the reduction of the residuals at check points.

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