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A Closed Solution for Space Resection

A set of equations has been derived in which the three elements of camera rotation are eliminated, thus giving linear equations for the exposure station coordinates.

INTRODUCTION

THE BASIC EQUATIONS for photogrammetric resection are the projective equations (Thompson, 1966) connecting photo and ground coordinates. These equations contain six unknowns per photo, the three exposure station coordinates X_0, Y_0, and Z_0 and the three rotation elements κ, ϕ, and ω. These equations are non-linear and the solution for X_0, Y_0, Z_0, κ, ϕ, and ω is obtained in an iterative manner with some assumed initial values.

The object of this paper is to derive a set of new equations which are linear and therefore require no initial approximate values. These new equations, as is shown in the subsequent paragraphs, do not contain the rotational elements κ, ϕ, and ω but have only the exposure station coordinates, reducing the number of parameters from six to three. The equations developed by Church (Baker, 1960) also solve for X_0, Y_0, and Z_0 but these are non-linear and require initial approximate values. The rate of convergency of the solution depends upon nearness of initial values to true values. Church's method requires a minimum of three ground control and photo-points and considers the photo pyramid formed by the three ground and photo points at the exposure station. The apex angles are determined first and then the exposure station coordinates are obtained by iteration.

KEY WORDS: **Iteration; Nonlinear differential equations; Solutions; Space resection**
ABSTRACT: The existing solutions for space resection involve iterative procedures because of the non-linear projective equations. A new set of equations had been derived in which the three elements of camera rotation are eliminated, thus giving linear equations for the exposure station coordinates. A numerical example has been worked to demonstrate the solution.
REFERENCE: Rampal, K. K., "A Closed Solution for Space Resection," *Photogrammetric Engineering and Remote Sensing,* Journal of the American Society of Photogrammetry, ASP, Vol. 45, No. AP9, September, 1979

MATHEMATICAL FORMULATION

The basic equation to solve for the elements of exterior orientation are the projective equations of the type

$$x-x_0 = C\frac{a_{11}(X-X_0) + a_{12}(Y-Y_0) + a_{13}(Z-Z_0)}{a_{31}(X-X_0) + a_{32}(Y-Y_0) + a_{33}(Z-Z_0)}$$
$$y-y_0 = C\frac{a_{21}(X-X_0) + a_{22}(Y-Y_0) + a_{23}(Z-Z_0)}{a_{31}(X-X_0) + a_{32}(Y-Y_0) + a_{33}(Z-Z_0)} \qquad (1)$$

where C is the principal distance and the a's contain the rotational elements κ, ϕ, and ω. For vertical photography with $\kappa = \phi = \omega = 0$, Equations 1 reduce to

$$x-x_0 = C\,\frac{(X-X_0)}{(Z-Z_0)}$$
$$y-y_0 = C\,\frac{(Y-Y_0)}{(Z-Z_0)} \qquad (2)$$

PHOTOGRAMMETRIC ENGINEERING AND REMOTE SENSING,
Vol. 45, No. 9, September 1979, pp. 1255-1261.

Equations 2 are linear equations for the solutions of X_0, Y_0, and Z_0. This suggests that, if we are to get linear equations for the solution of space resection elements X_0, Y_0, and Z_0, we must attempt to eliminate the rotational elements from the projective equations.

Equations 1 are non-linear and have to be solved for X_0, Y_0, Z_0, κ, ϕ, and ω by a process of successive iterations. Normally three to four iterations yield quite successful and converging results. Such a solution, however, requires initial approximate values of the unknowns. In near vertical photography, this is no problem. Approximate values of X_0, Y_0, and Z_0 can be estimated graphically or otherwise. Some difficulty is encountered for determination of initial values and the signs of the rotational elements for a highly tilted photo or for terrestrial photography. A wrong sign for the estimated values of κ, ϕ, and ω may fail to yield a converging solution. This can cause a lot of frustration and considerable difficulty. This paper shows how such a difficulty can be totally eliminated by a new set of projective equations which are linear in X_0, Y_0, and Z_0 and which give an exact solution without any need for initial approximate values. Moreover, the rotational elements do not enter into such linear equations and, hence, offer no problem of sign or magnitudes. The derivation follows on the strict assumption that collinearity is valid.

Let us consider two given ground control points labeled 1 and 2 as in Figure 1 with their coordinates X_1, Y_1, Z_1 and X_2, Y_2, Z_2 referred to an arbitrary origin O′. It is not necessary to assume knowledge of flight direction. All that is required is a knowledge of the photo coordinates referred to the photo principal point x_0, y_0. Let x_1, y_1 and x_2, y_2 be the observed coordinates of the corresponding photo images.

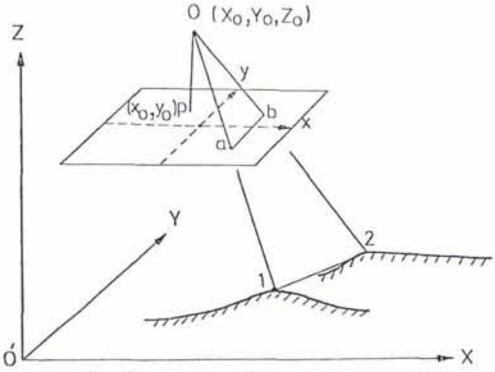

FIG. 1. Geometry of the space resection.

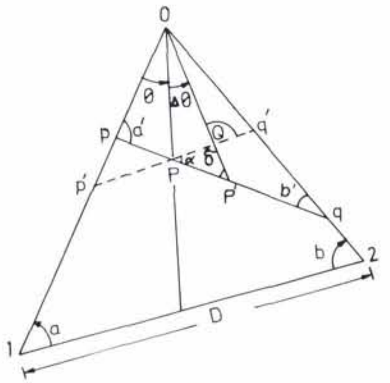

FIG. 2. Ground and photo distances.

From the definition of scale of a photograph at a point and referring to Figure 1 and Figure 2, we can write the basic linear differential equation of the first order for this case as

$$dS = \frac{\left[(X-X_0)^2 + (Y-Y_0)^2+(Z-Z_0)^2\right]^{1/2}}{\left[(x-x_0)^2 + (y-y_0)^2+C^2\right]^{1/2}}\, ds \tag{3}$$

where

dS = an element of ground length,
ds = an element of photo length parallel to ground length 'D' (see Figure 2), and
C = principal distance.

We can write Equation 3 as

$$\frac{dS}{\left[(X-X_0)^2+(Y-Y_0)^2+(Z-Z_0)^2\right]^{1/2}} = \frac{ds}{\left[(x-x_0)^2+(y-y_0)^2+C^2\right]^{1/2}}$$

Hence, in going from point 1 to point 2,

$$\int_1^2 \frac{dS}{\left[(X-X_0)^2+(Y-Y_0)^2+(Z-Z_0)^2\right]^{1/2}} = \int_1^2 \frac{ds}{\left[(x-x_0)^2+(y-y_0)^2+C^2\right]^{1/2}} \tag{4}$$

It has been shown by Rampal (1975) that the left hand side of Equation 4 reduces to the form (see Appendix)

$$\frac{1 + D_3+D_5}{D_6} = \frac{\Delta X\Delta X_2+\Delta Y\Delta Y_2+\Delta Z\Delta Z_2+DS_2}{\Delta X\Delta X_1+\Delta Y\Delta Y_1+\Delta Z\Delta Z_1+DS_1} = \cot\left(\frac{a'-\alpha}{2}\right)\cot\left(\frac{b'+\alpha}{2}\right)=k \tag{6}$$

where k is defined by the equality sign and other notations as in Rampal (1975). Equation 6 expresses the relationship between photo and ground coordinates for a set of two points labeled as '1' and '2'. The right hand side of Equation 6 involves observed and computed data as obtained from photo coordinates and the elements of interior orientation x_0, y_0, and C. The left hand side contains the ground data and the unknowns X_0, Y_0, and Z_0. This is then the projective equation for two points as opposed to that of a single point as expressed by Equation 1. Equation 6 obviously is independent of rotational elements κ, ϕ, and ω and contains only the exposure station coordinates. Thus, in going from Equation 1 to Equation 6 we have eliminated κ, ϕ, and ω and, hence, initial values and their signs are not required. Equation 6 is still, however, non-linear so far as X_0, Y_0, and Z_0 are concerned because S_2 and S_1 contain X_0, Y_0, and Z_0 in quadratic form. Now it will be shown how this nonlinearity can also be removed by determination of a and b, the base angles as shown in Figure 2, and hence S_1 and S_2 from ground and photo data.

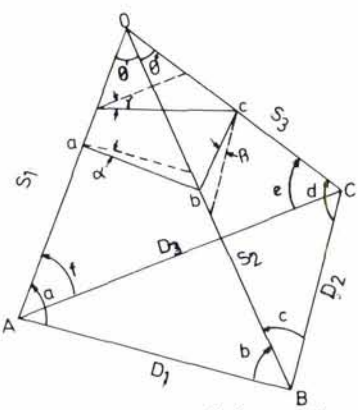

FIG. 3. Geometry of photo angles.

From Figure 3, making use of the common side of the triangles in the figure, it is easy to write the condition equations:

$$S_1 = \frac{D_1}{\sin\theta'}\sin(b'+\alpha) = \frac{D_3}{\sin\theta'''}\sin(e'-\gamma) \tag{7}$$

$$S_2 = \frac{D_1}{\sin\theta'}\sin(a'-\alpha) = \frac{D_2}{\sin\theta''}\sin(d'+\beta) \tag{8}$$

$$S_3 = \frac{D_2}{\sin\theta''}\sin(c'-\beta) = \frac{D_3}{\sin\theta'''}\sin(f'+\gamma) \tag{9}$$

Since α, β, and γ are small (unless we are dealing with extremely tilted photos or very high altitude differences), we can write Equations 7, 8, and 9 in a linear form as

$$a_1\alpha + b_1\beta = C_1 \tag{10}$$
$$a_2\alpha + b_2\gamma = C_2 \tag{11}$$
$$a_3\beta + b_3\gamma = C_3 \tag{12}$$

which gives

$$\alpha = \frac{a_3b_2C_1 + b_3C_2b_1 - b_1b_2C_3}{a_1a_3b_2 + a_2b_1b_3} \tag{13}$$

where

$$a_1 = D_1\cos a'\sin\theta''$$
$$b_1 = D_2\cos d'\sin\theta'$$

$$
\begin{aligned}
C_1 &= D_1 \sin a' \sin \theta'' - D_2 \sin d' \sin \theta' \\
a_2 &= D_1 \cos b' \sin \theta'' \\
b_2 &= D_3 \cos e' \sin \theta' \\
C_2 &= D_3 \sin c' \ \theta' - D_1 \sin b' \sin \theta''' \\
a_3 &= D_2 \cos c' \sin \theta''' \\
b_3 &= D_3 \cos f' \sin \theta'' \\
C_3 &= D_2 \sin c' \ \theta''' - D_3 \sin f' \sin \theta''
\end{aligned}
$$

Here D_1, D_2, and D_3 are the ground distances which are computed from the ground coordinates X_1, Y_1, Z_1; X_2, Y_2, Z_2; and X_3, Y_3, Z_3 of the three given ground points. The angles a', b', c', d', e', and f' are photo angles and are computed, along with θ', θ'', and θ''', from photo data. As an example we have

$$\cos \theta' = \frac{s_1^2 + s_2^2 - d^2}{2\, s_1 s_2}$$

where

$$
\begin{aligned}
s_1^2 &= (x_1-x_0)^2 + (y_1-y_0)^2 + C^2 \\
s_2^2 &= (x_2-x_0)^2 + (y_2-y_0)^2 + C^2 \\
d^2 &= (x_2-x_1)^2 + (y_2-y_1)^2
\end{aligned}
\tag{15}
$$

Similarly,

$$\cos a' = \frac{s_1^2 + d^2 - s_2^2}{2 s_1 d} \tag{16}$$

$$\cos b' = \frac{s_2^2 + d^2 - s_1^2}{2 s_2 d}$$

Equations 7 to 14 make the determination of α, β, and γ possible by using photo and ground data. It is then possible to solve for the sides S_1, S_2, and S_3 by the simple sine formula,

$$\frac{S_1}{\sin(a'-\alpha)} = \frac{D_1}{\sin \theta'} \tag{17}$$

The computation of S_2 and S_3 follows in the same manner. Once S_1 and S_2 are known, Equation 6 reduces to a linear form very easily. This can be shown as follows:

Using Equation 6 we have

$$\Delta X(X_2-X_0) + \Delta Y\,(Y_2-Y_1) + \Delta Z\,(Z_2-Z_0) = D\,(kS_1-S_2)$$

Since $X_2 = X_1 + \Delta X$, $Y_2 = Y_1 + \Delta Y$, and $Z_2 = Z_1 + \Delta Z$.

Further simplification yields:

$$\Delta X \cdot X_0 + \Delta Y \cdot Y_0 + \Delta Z \cdot Z_0 = \Delta X X_1 + \Delta Y Y_1 + \Delta Z Z_1 - \frac{D_1(D_1 + D_2 - kS_1)}{k-1}$$

Writing $\bar{D}_1 = D_1\,(D_1+D_2 - kS_1)\,/\,(k-1)$

we get

$$\Delta X \cdot X_0 + \Delta Y \cdot Y_0 + \Delta Z \cdot Z_0 = \Delta X X_1 + \Delta Y Y_1 + \Delta Z\, Z_1 - \bar{D}_1. \tag{18}$$

Equation 18 is the linear equation for direct (closed) solution of the exposure station coordinates. One can note that

- It requires no knowledge of the initial values of the parameters,
- No knowledge of the initial values or the signs of the rotational elements is required,
- It is applicable to both aerial and terrestrial types of photography, and
- Only photo data along with the necessary ground information is used.

From the nature of Equation 18 it can be inferred that we need at least four ground and photo points (and not a minimum of three) for independent equations, unless we separate planimetry from height. The other case is that when we have a very nearly flat terrain. A numerical example has been worked out, as below, to illustrate this case. This is obvious from the fact that $\Delta X_{12} + \Delta X_{23} + \Delta X_{31} = 0$. Similarly, it is true with ΔY and ΔZ.

Numerical Example

The data as given in Table 1 are from the Casa Grande Test Range (Arizona) and are published by the Defense Mapping Agency Hydrographic/Topographic Center, Washington

TABLE 1. GROUND AND PHOTO DATA—CASA GRANDE TEST RANGE
CAMERA FOCAL LENGTH = 152.01 mm
PHOTO NO. 80

Point No.	Point Identification	Ground Data in Metres			Photo Data in mm	
		X	Y	Z	x	y
1.	AE-46	430,823.492	3,634,795.016	432.036	−53.5492	50.0729
2.	AF-46	432,435.126	3,634,763.853	435.731	−1.8000	49.9025
3.	AF-45	432,447.333	3,636,323.557	432.940	−1.8029	100.7271
4.	AE-47	430,771.704	3,633,046.953	433.768	−54.5791	−6.0726

D.C. The photo coordinates were observed by the author during his stay at The Ohio State University (Department of Geodetic Science) using a Zeiss-PSK Stereocomparator. The program TRANC-4 was used to transform comparator coordinate to photo coordinates. Glass 9 by 9 in. diapositives were available for observations.

The photo and ground data yield the following:

$$\theta' = 17°.552 \qquad D_1 = 1611.9354 \text{ metres}$$
$$\theta'' = 15°.3550 \qquad D_2 = 1559.754 \text{ metres}$$
$$\theta''' = 23°.3910 \qquad D_3 = 2230.0891 \text{ metres}$$
$$a' = 71°.4410 \qquad b' = 90°.7038 \quad c' = 108°.1732 \quad d' = 56°.4716$$
$$e' = 67°.3674 \qquad f' = 89°.2416$$

Using Equation 13 we get

$$\alpha = 1°.40696$$

Substituting this value of α in Equations 10 and 11 we get

$$\beta = 0°.55618$$
$$\gamma = -1°.89419$$

Equations 7, 8, and 9 are now expanded by Taylor's series around the values of α, β, and γ as obtained to improve their values. This results in the following:

$$\alpha = 1°.48141$$
$$\beta = 0°.49456$$

Finally we get:

$$a = 69°.9340 \qquad b = 92°.1104 \qquad c = 107°.61702$$
$$d = 57°.02778 \qquad e = 69°.26159 \qquad f = 87°.34741$$

The base angles, $a, b, c, \ldots$, having been determined, it is now easy to compute S_1, S_2, S_3 from D_1, D_2, D_3 and θ', θ'', θ''', etc.

To separate height from planimetry, we rewrite Equation 18 for points 1, 2, and 3 as

$$\begin{aligned} \Delta X_{21} X_0 + \Delta Y_{21} Y_0 &= \Delta Y_{21} Y_1 + \Delta Y_{21} Y_1 + \Delta Z_{21} H - \bar{D}_1 \\ \Delta X_{32} X_0 + \Delta Y_{32} Y_0 &= \Delta Y_{32} Y_2 + \Delta Y_{32} Y_2 + \Delta Z_{32} H - \bar{D}_2 \end{aligned} \tag{19}$$

where $H = (Z_{\text{average}} - Z_0)$

$$Z_{\text{average}} = \frac{Z_1 + Z_2 + Z_3}{3}$$

Since the terrain under consideration is flat, we can obtain the value of H from the focal length and the mean photo scale λ from:

$$\lambda = \frac{C}{H} = \frac{C}{Z - Z_0}$$

The mean photo scale is computed from the known photo and ground distances of points 1, 2, and 3.

With Equation 17 and 19 and using the data of Table 1, we get $S_1 = 5253.3998$
$S_2 = 4938.7064$
$S_3 = 5612.1550$
$S_4 = 5046.9696$

$$11.634\, X_0 - 31.163\, Y_0 = 4{,}063{,}300.6$$
$$12\, X_0 + 1559.704\, Y_0 = 5{,}134{,}045.8$$

The solution gives

$$X_0 = 432{,}584.4881$$
$$Y_0 = 3{,}633{,}271.4521$$

Z_0 now can be computed from any of the slant distances S_1, S_2, etc., from the relation

$$S_1 = \left[(X_1-X_0)^2 + (Y_1-Y_0)^2 + (Z_1-Z_0)^2\right]^{1/2} \tag{20}$$

The values of Z_0 as obtained by using Equation 20 with the computed values of X_0, Y_0 and S_1, S_2, S_3, S_4 give Z_0 as 5141.1606, 5141.1800, 5140.6069, 5138.5864, respectively, with a mean of 5140.3832. Thus, the final results of the computations, done with a small pocket calculator and using the linear solution, are

$$X_0 = 432{,}584.4881$$
$$Y_0 = 3{,}633{,}271.4521$$
$$Z_0 = 5140.3832$$

The results, using the RESEC-1 program on the IBM/370 of the Department of Geodetic Science, The Ohio State University, Columbus, Ohio, and using the well known projective equations with a least squares solution, gave the results as Rampal (1973):

$$X_0 = 432585.1824 \pm 2.3949$$
$$Y_0 = 3633272.8913 \pm 0.9075$$
$$Z_0 = 5140.1709 \pm 0.6335$$

Conclusions

A new mathematical model has been derived which is linear in terms of the parameters of the exposure station, i.e., coordinates X_0, Y_0, and Z_0. The rotational elements κ, ϕ, and ω have been eliminated and, hence, the solution is independent of the orientation of the photo and ground coordinate system. The solution also requires no initial or approximate values of the parameters. This reduces the computation time considerably as there are no iterations involved. In fact, the solution is so simple that, with a minimum of three ground and photo points for flat terrain, a solution is possible with a pocket electronic calculator.

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Appendix

Consider two ground points 1 and 2 imaged at p and q (see Figure 2). Let P be a point on the line with photo coordinates x and y. Let P' be another point so that PP' is of infinitesimal length ds'. Through P draw a line $p'q'$ parallel to the ground distance 'D', intersecting OP' at Q. Let $PQ = ds$.

Let $PQP' = \delta$ and $QPP' = \alpha$ be the angles as shown in the figure. We have also the following relations:

$$a = a' - \alpha$$
$$b = b' + \alpha$$
$$\delta = a + \theta + \Delta\theta$$

From triangle OPQ

$$\frac{PQ}{\sin\Delta\theta} = \frac{OP}{\sin(180 - \delta)}$$

$$ds = \frac{OP \sin\Delta\theta}{\sin \delta} \tag{2}$$

Since $\Delta\theta$ can be chosen as small as we like, we can write Equation 2 as

$$\frac{ds}{OP} = \frac{d\theta}{\sin(a+\theta)}$$

but $OP = \left[(x-x_0)^2 + (y-y_0)^2 + C^2\right]^{1/2}$

Therefore, we have

$$\int \frac{ds}{\left[(x-x_0)^2+(y-y_0)^2+C^2\right]^{1/2}} = \int \frac{d\theta}{\sin\,(a+\theta)}$$

$$\int_0^{\theta'} \frac{d\theta}{\sin(a+\theta)} = \ln\tan\left|\frac{a+\theta}{2}\right|_0^{\theta'}$$

$$= \ln\tan\frac{\theta'+a}{2}\cot a/2$$

but $a + \theta' = 180-b$

$$\int_0^{\theta'} \frac{d\theta}{\sin\,(a+\theta)} = \ln\left[\cot b/2 \cot a/2\right].$$

Therefore, we have

$$\int \frac{ds}{\left[(x-x_0)^2+(y-y_0)^2+C^2\right]^{1/2}} = \ln\left[\cot(a/2)\cot(b/2)\right].$$

$$\text{Since} \int \frac{dS}{\left[(X-X_0)^2+(Y-Y_0)^2+(Z-Z_0)^2\right]^{1/2}}$$

$$= \ln\frac{\Delta X\Delta X_2+\Delta Y\Delta Y_2+\Delta Z\Delta Z_2 + DS_2}{\Delta X\Delta X_1+\Delta Y\Delta Y_1+\Delta Z\Delta Z_1 + DS_1}$$

Therefore,

$$\frac{\Delta X\Delta X_2+\Delta Y\Delta Y_2+\Delta Z\Delta Z_2 + DS_2}{\Delta X\Delta X_1+\Delta Y\Delta Y_1+\Delta Z\Delta Z_1 + DS_1} = \cot\left(\frac{a}{2}\right)\cot\left(\frac{b}{2}\right)$$

$$= \cot\left(\frac{a'-\alpha}{2}\right)\cot\left(\frac{b'+\alpha}{2}\right)$$

(Received December 6, 1977; revised and accepted March 31, 1979)

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