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Different Considerations in Coastal Mapping*

Water penetration, refraction, and incomplete stereomodels are the three main problems in photogrammetric mapping of coastal waters.

Introduction

Photogrammetric mapping of coastal waters can be more economically carried out than mapping with traditional shipboard sounding methods. Moreover, additional information, such as the topography and planimetric detail of the coastal zone, can be directly delineated from the photographs. There are, however, three main problems involved in using photogrammetric mapping techniques in coastal waters, which are not usually present when mapping land areas. These are

- degree of water penetration—this is a function of a number of different factors including the type of film/filter combination used, the surface reflectance of the water, sun angle, turbidity, etc.;
- refraction of imaging rays—the degree of refraction experienced by an imaging ray at the water/air interface will depend on the inclination of the surface normal, the salinity and temperature of the water, and the incident angle the ray makes with the water surface normal; and
- incomplete stereomodels—it is not always possible to fly photography for coastal mapping so that the configuration of land areas in a stereomodel provides points adequate for relative and absolute orientation.

For a universal coastal mapping system, the three problems must be considered as being related, and treated as such. For example, additional depth measurements from laser bathymetry (LIDAR), in areas where the depth penetration of the photography is limited, may partly overcome the effects of the first problem and help greatly with the orientation problem due to incomplete stereomodels. This can be achieved by using the LIDAR measurements as control to adjust the orientation of the stereomodel or the depth measurements obtained from it and, consequently, reduce its deformation errors.

The first problem has been the subject of a number of publications [Lockwood, 1974; Anderson, 1972] and will not be discussed here, except with reference to the LIDAR. The subject matter of this paper is concerned with different configurations of a universal coastal mapping system and their implica-

Abstract: *Water penetration, refraction, and incomplete stereomodels are the three main problems in photogrammetric mapping of coastal waters. These problems, together with the different means for overcoming them, are discussed. In particular, the use of simultaneous photography from two aircraft, inertial navigation, black-and-white infrared photography, and* LIDAR *are analyzed. The objective of the analysis is to acquire an insight into the configuration and requirements of a universal coastal mapping system.*

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tions as far as they relate to the second and third problems.

The Refraction of Imaging Rays

The refraction of the imaging rays at the water/air interface is a distinct refraction as shown in Figure 1. The imaging ray, PA_1p_1, is refracted at the surface point, A_1. The position of the point, A_1, is such that

- the ray after its refraction passes through the camera perspective center, S_1, and
- the refraction at point A_1 follows Snell's Laws of Refraction.

If the orientations of the photographs and the shape of the water surface are known, the position of the point, *P*, on the bottom can be rigorously calculated.

The shape of the surface can be determined using simultaneous photography from two aircraft. The determination of the position of point *P*, however, may still be quite involved. This is mainly because, as is explained in detail in the next section, the surface usually deviates from one made up of ideal wave forms. Thus, a considerable number of observations may be required to describe the surface accurately, and an iterative procedure to determine the poisition of *P* may prove necessary.

Deviation of the Water Surface from a Simple Waveform

A number of factors may contribute to the deviation of the water surface from a simple form which could be represented by a simple mathematical function, facilitating the refraction computations. These factors are dealt with in detail in Sverdrup *et al.* (1942), Smith (1973), and Bigelow and Edmondson (1947). Actual profiles of the water surface were determined from stereo photographs taken from a surface vessel. The profiles obtained showed that the water surface can deviate greatly from simple wave forms. Figure 2 illustrates the deviation of an actual profile from the best fitting trochoidal waveform. It can be seen from the figure that the deviation between the directions of the normals to the actual and mathematical surfaces can be quite large, even where the vertical deviation is relatively small. Additional factors, particular to shallow coastal waters, may contribute to this deviation. Some of these are

- When a long wave crest enters shallow water at an angle to the shore, the end nearer the shore will enter shallow water first and will be slowed down. The rest of the wave, still in deep water, retains its original speed until it, in turn, reaches shallow water and slows down. This results in a bending or refraction of the wave front, as illustrated by Figure 3.
- The slowing down of waves entering shallow water results in a gradual reduction in the wavelength accompanied by an increase in the wave height, as shown in Figure 4.

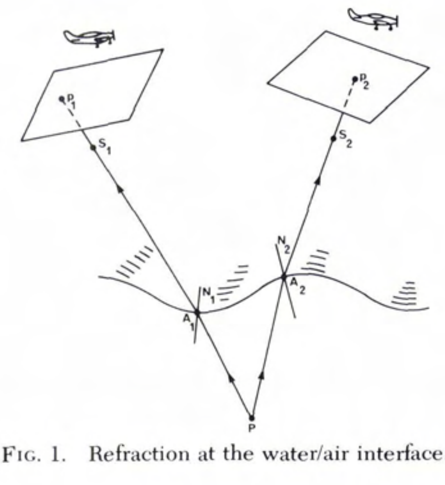

Fig. 1. Refraction at the water/air interface.

Consideration of Actual Surface in Refraction Calculations

The actual water surface can be established by using a stereo pair of photographs taken simultaneously from two aircraft. As mentioned above, the direction of the normal to the water surface may not be accurately described by fitting a simple mathematical surface to the water surface over the stereomodel area. An iterative approach to determine the normal, *N*, (Figure 5) to the water surface and the direction of the imaging ray, PA_1p_1, could be adopted. The procedure would be as follows:

- An approximate point, A', is determined, which is the intersection of the line, S_1P, with a horizontal plane with mean wave elevation. Point A' gives the general vicinity of the water surface where the actual point A_1 lies. (To start with, point *P* is not known but the procedure is iterative as explained in Masry (1975).)
- Points in the vicinity of A' are observed on the water surface (for example, points 1 to 5 in the figure).
- The points observed in the previous step provide sufficient information to allow a mathematical surface to be fitted to a portion of the wave surface encompassing point A_1.
- The direction of the normal to the surface *N*, can be determined, allowing the imaging ray, PA_1p, to be established.

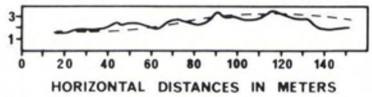

FIG. 2. Actual wave profile from stereo photos (solid lines) and theoretical trochoidal wave form given by

$$h = (L/2\pi - K)(1 + \cos 2n\pi)$$

where h is the height of a point on the wave profile above the lowest trough point, L is the wave length, K is a constant, and the value of the constant, n, ranges between zero and 1. From Sverdrup *et al.* (1942), reprinted by permission of Prentice Hall Inc., Englewood Cliffs, New Jersey.

A similar procedure would be carried out for the ray from the other photograph. Such calculations may be carried out on-line using the computer of an analytical plotter. Practically, however, the procedure may prove tedious. In addition, in areas where water penetration takes place, the water surface itself may not be clearly imaged and, consequently, observed.

The refraction calculations become much simpler if the water surface is taken to be a plane, as we shall see below.

APPROXIMATION OF THE WATER SURFACE BY A PLANE

The water surface may be approximated by a plane surface having the mean height of the wave crests and troughs. The computations of the refraction corrections are, consequently, much simplified. The approximation errors are dealt with first, followed by procedures for refraction calculations.

APPROXIMATION ERRORS

Estimates of the approximation errors due to assuming a plane water surface have been given (Tewinkel, 1963). It is the opinion of the authors that a general estimate is meaningless because:

- As discussed above, the surface may deviate greatly from a simple wave form used in such estimation;
- The deviations are a function of time and location and can be considered practically random; and
- Within one model, the approximation error varies as a function of the position of the point observed and the orientation of the wave at the point.

To give only some idea of the magnitude of the approximation errors, the following specific case was assumed and the errors were evaluated for a number of points within a stereomodel. A wave with a wave-height-to-wave-length ratio of 1:25 was assumed. A model with 0.6 base/height ratio was taken. Flying height was 1520 m, the base was 910 m, and the dimensions of the model were 910 m in X and 1820 m in Y. The refractive index of seawater was taken as 1.33 and the mean depth of water at the points of evaluation was 18.3 m.

In the error analysis, the approximation error was considered to have two components; one due to the deviation of the normal to the wave surface from the vertical direc-

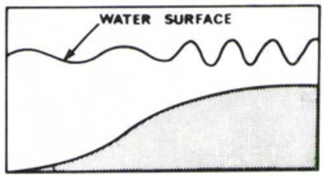

FIG. 4. Decrease in wavelength and increase in wave height as waves enter shallow water.

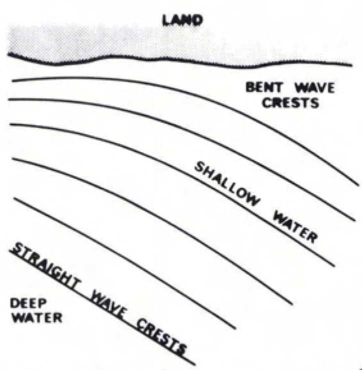

FIG. 3. Refraction of waves entering shallow water.

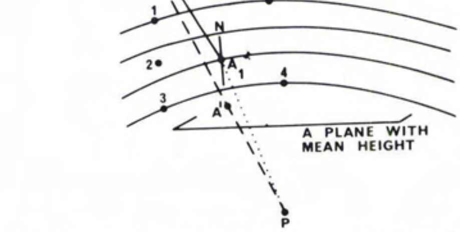

FIG. 5. Iterative procedure to determine surface point A_1.

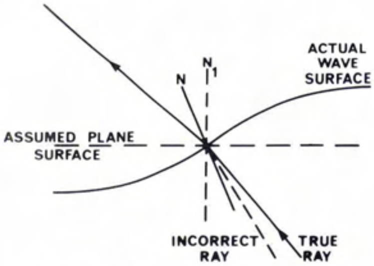

FIG. 6. Effect of assuming a plane air/water interface on the angle of refraction; component due to inclination of the normal to the surface.

tion (Figure 6), and the other due to the possible increase or decrease in the depth of water through which the imaging ray is considered to have traveled (Figure 7).

For the first component, the inclination of the wave surface normal, N, (Figure 6) was taken to be ±4.5, which is the average slope for the wave under consideration, always in the vertical plane containing the imaging ray and the vertical through the perspective center. A summary of the errors is as follows:

- Maximum error in depth occurred in the center of the model with a magnitude of approximately ±1.8 m; and
- Errors around the edge of the model were approximately 66 percent of the magnitude of those at the center of the model, being approximately ±1.2 m.

To evaluate the error component due to the variation of the actual depth of water, a wave height of 1.8 m and, consequently, deviations of wave crests and troughs from the mean of 0.9 m were assumed. Rays were assumed to intersect the water surface at wave crests and wave troughs where no deviation of the surface normal occurs. A summary of the errors is as follows:

- Errors in the center of the model were smallest with a magnitude of approximately ±0.3 m, and
- Errors around the edges of the model were larger than those at the center of the model and ranged approximately between ±0.5 and ±0.4 m.

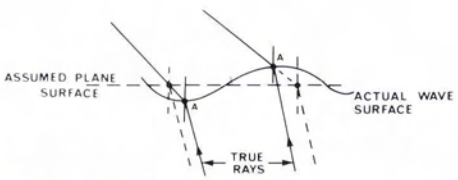

FIG. 7. Effect of assuming a plane air/water interface on the position of the surface point A; component due to variation in water depth.

REFRACTION CALCULATIONS

If one assumes the water surface to be a plane, refraction calculations can be carried out in real-time using an analytical plotter. The calculations involve the solution of a 4th degree equation of the form

$$R^2\,[(n^2 - 1)(L - R)^2 - Z^2] + n^2 h_1^2\,(L - R)^2 = 0.$$

where n is refractive index and all other quantities are as illustrated in Figure 8. More detail about the method can be found in Masry *et al.* [1970].

This procedure has been programmed and tested on the analytical plotter at University of New Brunswick. The results of tests carried out to compare the accuracy of spot depth readings with actual soundings, assuming a plane water surface, are reported in detail in Masry [1975].

Another approach to carry out the refraction calculations is presented here for the first time in an Appendix. This approach seems to involve less calculations than the one given above and may, therefore, be more suitable for real-time calculations.

INCOMPLETE STEREOMODELS

Depending on the shape of the coast, aerial photographs flown for coastal mapping may show mainly water and result in incomplete stereomodels. In such cases, the distribution of orientation points (relative and absolute) can be poor. This results in inaccurate measurements from stereomodels and weak connections between adjacent models. Simultaneous photography from two aircraft can help partly to overcome this problem by defining the water surface, as discussed above in relation to the refraction aspect. With such photography, points may appear clearly defined on the surface and, at least over some areas of a stereomodel, can be

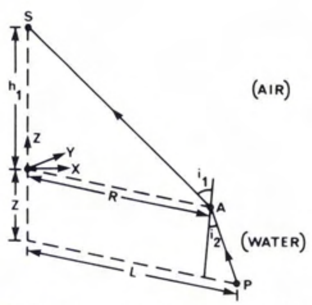

FIG. 8. Refraction calculations involve the calculation of surface point A through the solution of a 4th degree equation.

used in relative orientation. A model can also be leveled using the water surface, taking into consideration the shape and height of the waves. Connection of stereomodels may not, however, be improved through the use of this type of photography.

Another consideration when taking simultaneous photography is the tolerance of synchronizing the exposure time of the cameras in the two aircraft. The tolerance should obviously be such that the water surface motion at photo scale can be neglected.

There are two speeds to be considered here (see Figure 9): (a) the orbital speed of water particles at the surface, and (b) the wave speed.

For stable waves (non-breaking waves), the speed of the wave is higher than the orbital speed and is, therefore, the governing speed. In shallow waters, i.e., in water with depth equal to or less than half the wave length, this speed is governed by the water depth only, and is given theoretically by Sverdrup (1942); i.e.,

$$c = \sqrt{gh}$$

where c is the wave speed (in m/s),
h is the water depth (in m), and
g is the force of gravity (in m/s^2).

As an example, consider the depth of 18 m, which is the usual maximum water penetration depth from aerial photography in most areas. The wave speed at this depth is about 13 m/sec. The relative image motion between a stereopair of photographs with a scale of 1:3000, due to a synchronization tolerance of 1 millisecond, is of the order of 4 μm. For smaller scales and shallower water, the tolerance can be less stringent.

Alternative System

Absolute orientation and the connection between models can be improved by using control points in the water area [Wenyon, 1978]. An alternative to that and simultaneous photography is to use auxiliary aids to determine the six exterior orientation parameters of the aerial camera at each exposure station. Knowing the six orientation parameters of each photograph, mapping of the coastal zone can be accomplished, theoretically, regardless of the presence of any land coverage. Recent advances in the field of Inertial Navigation Systems (INS) indicate that suitable accuracy of the exterior orientation parameters can be obtained using these systems.

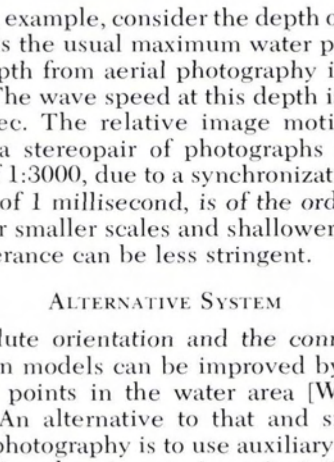

Fig. 9. Wave velocity/orbital velocity.

Research was carried out at University of New Brunswick, in cooperation with the Canada Centre for Remote Sensing and the Canadian Hydrographic Service, to determine the accuracy requirements of such a system. A detailed analysis of the accuracy requirements was presented in Masry [1977]. The objective of the analysis was to find answers to two questions:

- What are the necessary accuracies of the six exterior orientation parameters so that a certain water depth measurement accuracy, say 0.6 m can be obtained? and
- If the accuracy requirements are too high to be practically obtained, can the photogrammetric measurements be improved by using additional information such as water line points from black-and-white infrared photography and LIDAR measurements?

Summary of Analysis

The variances in the X, Y, Z position of a point were determined as functions of the errors in the exterior orientation parameters of the two photographs of a stereo pair, viz:

$$\begin{aligned} S_X^2 = {} & (1 - (X/B))^2 S_{bX1}^2 \\ & + (1 - (X/B))^2 (X/Z)^2 S_{bZ1}^2 \\ & + (1 - (X/B))^2 Y^2 S_{\kappa 1}^2 \\ & + (1 - (X/B))^2 (Z + (X^2/Z)) S_{\phi 1}^2 \\ & + (1 - (X/B))^2 (XY/Z)^2 S_{\omega 1}^2 \\ & + (X/B)^2 S_{bX2}^2 + (X/B)^2 ((X-B)/Z)^2 S_{bZ2}^2 \\ & + (X/B)^2 (Y)^2 S_{\kappa 2}^2 \\ & + (X/B)^2 [Z + (X-B)^2/Z]^2 S_{\phi 2}^2 \\ & + (X/B)^2 [((X-B)/Z)\, Y]^2 S_{\omega 2}^2 \end{aligned} \qquad (1)$$

$$\begin{aligned} S_Y^2 = {} & S_{bY1}^2 + (Y/B)^2 S_{bX1}^2 \\ & + (Y/Z)^2 (1 - (X/B))^2 S_{bZ}^2 \\ & + (X + (Y/B) \cdot Y)^2 S_{\kappa 1}^2 + [-(Y^2/Z) + Z) \\ & + (Y/B) \cdot (XY/Z)]^2 S_{\omega 1}^2 \\ & + [(XY/Z) - (Y/B)\,(Z + (X^2/Z)]^2 S_{\phi 1}^2 \\ & + (Y/B)^2 S_{bX2}^2 + (Y/B^2((X-B)/Z)^2 S_{bZ2}^2 \\ & + (Y/B)^2\, Y^2\, S_{\kappa 2}^2 \\ & + (Y/B)^2 [Z + (X-B)^2/Z)]^2 S_{\phi 2}^2 \\ & + (Y/B)^2 [(X-B)/Z)\, Y]^2 S_{\omega 2}^2 \end{aligned} \qquad (2)$$

The variance in height/depth will be given by:

$$\begin{aligned} S_Z^2 = {} & S_{bZ1}^2 + (Z/B)^2 S_{bX1}^2 + (Z/B)^2 (X/Z)^2 S_{bZ1}^2 \\ & + (Z/B)^2 Y^2 S_{\kappa 1}^2 + (Z/B)^2 (Z + (X^2/Z))^2 S_{\phi 1}^2 \\ & + (Z/B)^2 (XY/Z)^2 S_{\omega 1}^2 + (Z/B)^2 S_{bX2}^2 \\ & + (Z/B)^2 ((X-B)/Z)^2 S_{bZ2}^2 + (Z/B)^2\, Y^2\, S_{\kappa 2}^2 \\ & + (Z/B)^2 [Z + (X-B)^2/Z)]^2 S_{\phi 2}^2 \\ & + (Z/B)^2 [(X-B)Y/Z)]^2 S_{\omega 2}^2 \end{aligned} \qquad (3)$$

where 1 and 2 in the suffixes indicate photo 1 and photo 2, respectively; S^2_{bX}, S^2_{bY}, and S^2_{bZ} are the variances of a photo perspective center position (bX_1 relating to photo 1 in X and bY_1 to photo 1 in Y, etc.); S^2_{κ}, S^2_{Φ}, and S^2_{ω} are the variances of the rotations; and B is the base vector between the two perspective centres.

Since the Z position (depth/height) is quite important in coastal mapping, Equation 3 was further analyzed to determine any possible improvements in accuracy which might be achieved using additional information such as bathymetric depth measurements from a LIDAR, or points on the water line which can be obtained from black-and-white infrared photography.

Several cases of errors in the exterior orientation parameters and corresponding errors in Z were evaluated, without, and with, additional information. Table 1 shows a sample case. The distribution of the points is as shown in Figure 10. Columns (5), (6), (7), and (8) show the standard error in Z improved using LIDAR, datum shift corrections only, leveling in the X-direction, and leveling in the X- and Y-directions. The latter three cases depend on the orientation and extent of the water line in the stereomodel as shown in Figure 11.

From an analysis of the results of all the cases tested, the following conclusions were drawn:

- The accuracy of the rotations should be known to about ±25 seconds of arc. Better accuracy may improve the depth measurements but may prove difficult to attain with the type of Inertial Navigation System (Litton LTN-51) which was used in the tests. In addition, it is doubtful that the rotations of the camera, which are used to calibrate the INS, can be determined with much better accuracy.
  Accuracies less than ±25 seconds of arc and up to ±35 seconds of arc, although they do not cause significant increases in the theoretical errors, may cause additional errors in practice. Such additional errors are mainly due to errors in the interpretability of the height, or depth, in the presence of large Y-parallaxes. This was practically tested and the results presented in Masry [1977].
- An accuracy of ±1.5 m in the position of the perspective center of a photograph seems, from preliminary results, to be obtainable in practice. However, this accuracy results in large errors in Z, which are not acceptable. The analysis showed that improvement of the position determination of the perspective centre to ±1.0 m would not improve the accuracy of depth measurements to a tolerable accuracy.
  Elimination of datum shift errors, which can be achieved if a limited length of water line appears on the photography (Figure 11d), improves the accuracy to an average of ±1.0 m and a maximum of ±1.7 m, which means that leveling of the model using LIDAR data and/or the water line is still necessary if an accuracy of ±0.6 m, the accuracy aimed for in the project, is to be achieved.
- Water line points are, as discussed in the first two conclusions above, of prime importance in achieving the target accuracy. For the mapping system to be universal,

TABLE 1. SAMPLE STANDARD ERRORS IN X, Y, Z COORDINATES OF A DETERMINED POINT FOR THE CASE OF FLYING HEIGHT 1524 m ABOVE SEA LEVEL, ASSUMED ERRORS OF ±1.5 m IN X, Y AND Z POSITION, ±25 SECONDS OF ARC ERROR IN ATTITUDE FOR EACH PHOTOGRAPH. COLUMNS (5) TO (8) SHOW THE EFFECT ON THE STANDARD ERROR IN Z OF USING ADDITIONAL INFOMATION.

Point No. (1)	INS Data Only S_X (2)	S_Y (3)	S_Z (4)	(5)	(6)	(7)	(8)
1	1.5	1.5	4.1	0.1	0.1	0.2	0.2
2	1.5	3.4	4.1	0.2	0.4	0.4	0.3
3	1.1	3.0	4.0	0.1	1.3	0.4	0.2
4	1.9	3.0	4.2	0.2	2.6	0.5	0.3
5	1.9	1.5	4.2	0.2	2.5	0.2	0.2
6	1.9	3.0	4.2	0.2	2.6	0.5	0.3
7	1.1	3.0	4.0	0.1	1.3	0.4	0.2
8	1.5	3.4	4.1	0.2	0.4	0.4	0.3

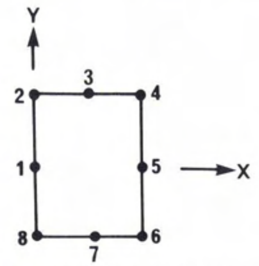

FIG. 10. Point distribution in a stereomodel at which standard errors were computed. Values are as shown in Table 1.

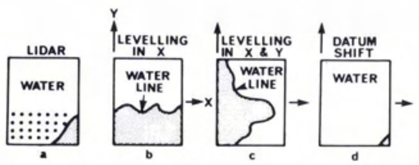

FIG. 11. Types of additional information for improvement of the Z accuracy.

we must consider the possibility that the orientation and length of the water line in the stereomodel may be such that it can not be used accurately in the leveling of the model. Another possibility is that the amount of penetration for some areas of a stereo-model may be quite limited. These two factors indicate that universality of the system can be achieved through the use of a LIDAR. The components of the system are then as illustrated in Figure 12. The position and attitude obtained from the INS is used also to determine the position of LIDAR bathymetric measurements.

OPERATIONAL PROCEDURE

An envisaged operational procedure with such a system is as follows: Ground control is established in the area to be mapped. Distribution and spacing between control points is established as a result of experience with the behaviour characteristics of the INS. The INS data is transferred to the control computer of an analytical plotter for processing.

Water line points are transferred from the black-and-white infrared photography to the color photography (presently, color photography gives best penetration). The following points are then observed on the color photography: water-line points, orientation points on land, orientation points underwater, points at which depth measurements from the LIDAR are available (if also penetrated by the color photography), and control points. Adjustment of the INS data is then carried out on the control computer of the plotter to achieve the finally adjusted exterior orientation parameters. Delineation of information from the photographs, such as bathymetric measurements and high water line, etc., is then carried out from stereopairs of photographs.

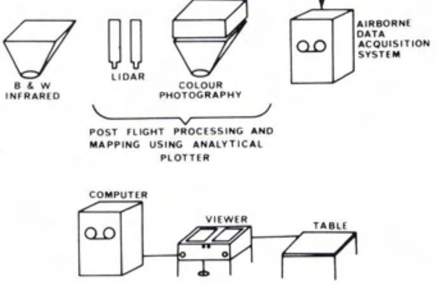

FIG. 12. Configuration of a universal coastal mapping system.

For integration with bathymetric measurements in deeper waters, it is advantageous to store the delineated information in digital form. We note here that the bathymetric depth measurements will consist of those determined from stereomodels as well as those provided by the LIDAR.

STATUS OF RESEARCH

Preliminary accuracy results of the INS were reported in Masry [1978]. Correction of refraction using the analytical plotter has been developed and thoroughly tested, as reported in Masry [1975]. Work is in progress to develop algorithms and procedures to transfer LIDAR measurements and infrared coastline points to the color photographs as well as work on the adjustment of the INS data.

CONCLUSIONS

- Refraction calculations can be carried out rigorously using simultaneous photography: however, the work involved can be tedious;
- General estimates of approximation errors in water depth measurements, made assuming a planar water surface, can be meaningless;
- Simultaneous photography may not provide, without additional information in the form of position, a universal coastal mapping system, due to possible problems of model connections; and
- The accuracy requirements of exterior orientation parameters provided by auxiliary aids such as INS have been analyzed. Additional information, such as water line points and LIDAR measurements, can help reduce the accuracy requirements.

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Appendix

This Appendix deals with a new approach for refraction calculations assuming that the water surface is a plane. The approach is iterative and is illustrated in Figure A.1. It consists of the following steps:

(a) Determine the intersection point, A', on the plane with an undeviated ray, S_1P, where points P and S_1 are the underwater point under consideration and the camera perspective center, respectively.

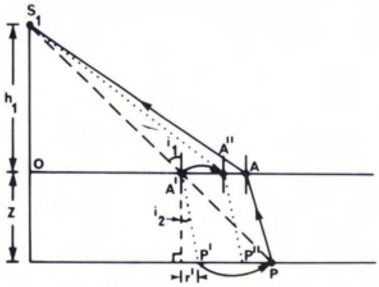

Fig. A.1. Iterative method of finding surface point *A*.

(b) The position of a point, P', is calculated so that a ray along $P'A'$ will be refracted to coincide with $A'S_1$. The calculation involves the computation of the distance, r', in the figure given by

$$r' = \frac{ZR'}{[n^2h_1^2 + R'^2\,(n^2 - 1)]^{1/2}}$$

where n is refractive index, R' is the distance OA', and other distances are as shown in the figure.

(c) The position of the surface point, A', is adjusted by the distance PP' to obtain a point, A'', and the corresponding point, P'', is calculated following step (b) above. (The adjusted surface point, A'', can be on the other side of A to that of A'.)

(d) Steps (b) and (c) are repeated until the positions of points P'' and P are within an acceptable tolerance.

It was found that the procedure above gave the position of the surface point A to a practically acceptable tolerance after two iterations.

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