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Orientation and Construction of Models

Part III: Mathematical Basis of the Orientation Problem of One-Dimensional Central Perspective Photographs*

The orientation problem of one-dimensional central perspective photographs, e.g., strip photographs and other line array images, is discussed, first algebraically and then geometrically.

Introduction

A unit of strip imagery is a one-dimensional central perspective photograph. Multispectral Scanner (MSS) imagery also has the same geometrical characteristics under the reasonable assumption that exterior orientation parameters of the MSS are constant along a scan line. Thus, in considering the orientation problem of strip and MSS imagery, we must begin with the investigation of the orientation problem for one-dimensional central perspective photographs.

In the time since the rigorous analytical reduction of strip photography was published by Case[1], many

Abstract: *The orientation problem of one-dimensional central perspective photographs is discussed, first algebraically and then geometrically. The algebraic approach enables us to understand easily the general characteristics of the orientation problem of central perspective photographs. Thus, many new facts are revealed regarding the orientation problem of one-dimensional photographs. However, only using the algebraic approach, we can not fully grasp the photogrammetric meaning of that for central perspective photographs. The geometrical approach is very helpful for this purpose, if the concept of multidimensional space having more than three dimensions is introduced. Also, some new facts regarding the photogrammetric orientation problem of one-dimensional photographs based on geometrical considerations are clarified. From the mathematical basis given here, some practically useful orientation techniques may be developed for strip and MSS imagery.*

orientation techniques for strip and MSS imagery have been developed, e.g., by Konecny[2,3], Derenyi[4], and others. However, very little has been written that would provide a general approach to the orientation problem of such imagery, particularly to that of their stereoscopic pairs. Therefore, this paper treats the general orientation problem of one-dimensional central perspective photographs based on the mathematical basis[5] in the orientation problem of two-dimensional (conventional) central perspective pictures.

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Mathematical Discussions for the Orientation Problem of One-Dimensional Central Perspective Photographs

For Hilly Terrain

(1) *Preparations.*

The orientation problem of one-dimensional central perspective photographs can be treated as a special case[1] of that for two-dimensional photographs, i.e., in which the x_c-coordinate of a measured image point is considered to equal zero. It means that the central projection is generated in a plane in which the photographed ground surface is included (see Figure 18). Also, the plane can be uniquely determined in the object-space coordinate system (X,Y,Z) in the case where the photographed terrain has relief.

The general collinearity equations for a one-dimensional central perspective photograph are expressed as

$$x_c = 0 = \frac{A_1X + A_2Y + A_3Z + A_4}{A_9X + A_{10}Y + A_{11}Z + 1}$$
$$y_c = \frac{A_5X + A_6Y + A_7Z + A_8}{A_9X + A_{10}Y + A_{11}Z + 1} \,. \tag{81}$$

We will investigate how many independent parameters Equation 81 has. The first part of Equation 81 indicates a plane in the object-space coordinate system (X,Y,Z); i.e.,

$$A_1X + A_2Y + A_3Z + A_4 = 0 \tag{82}$$

and has three independent parameters. The second part of Equation 81 is the collinearity condition between an object point $P(Y',Z')$ and the measured image point $p_c(y_c)$ on the plane in the form

$$y_c = \frac{A'_1Y' + A'_2Z' + A'_3}{A'_4Y' + A'_5Z' + 1} \tag{83}$$

where the Y'-Z' plane of the new object-space coordinate system (X',Y',Z') lies on the plane. Also, designating the measured coordinate of the principal point H as y_H, Equation 83 has five photogrammetric orientation parameters $(\omega',Y'_0,Z'_0,c,y_H)$. However, there is an obvious geometrical relationship between these five elements. Thus, Equation 83 has four independent parameters. This will be more rigorously investigated as follows. Equation 83 seems to have five independent elements in the case where the y_c-axis of the comparator coordinate system (y_c,z_c) is not parallel to the y-axis of a one-dimensional photograph (see Figure 19). Now we will consider this problem for a stereoscopic pair of one-dimensional photographs. The relationship between an object point $P(Y',Z')$ and its measured image point $p_c(y_c)$ is described as

$$y_{c1} = \frac{{}_1A'_1Y' + {}_1A'_2Z' + {}_1A'_3}{{}_1A'_4Y' + {}_1A'_5Z' + 1} \tag{84}$$

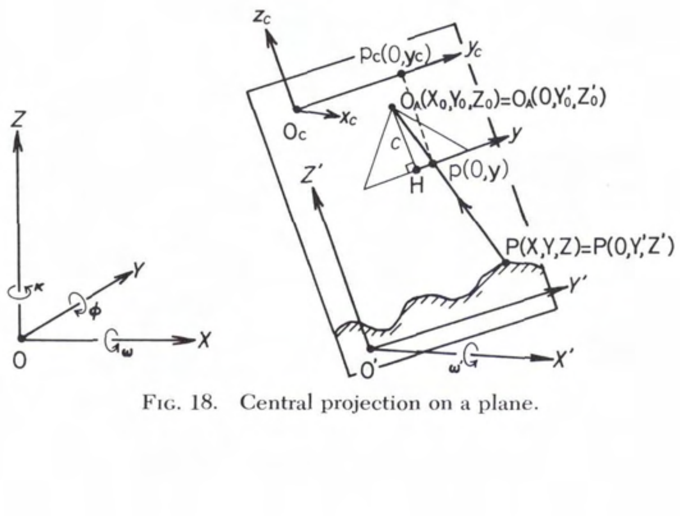

FIG. 18. Central projection on a plane.

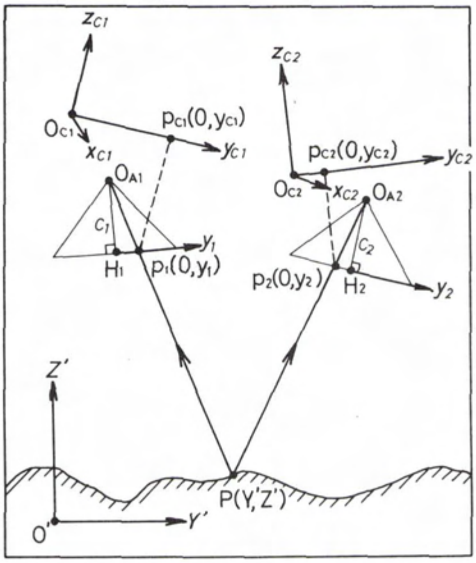

FIG. 19. A stereoscopic pair of one-dimensional photographs.

for the left photograph and

$$y_{c2} = \frac{{}_2A_1'Y' + {}_2A_2'Z' + {}_2A_3'}{{}_2A_4'Y' + {}_2A_5'Z' + 1} \tag{85}$$

for the right one, respectively. Also, it must be noted that central projection for the stereoscopic pair of one-dimensional photographs is generated on the same plane, if the photographed terrain has relief. Furthermore, an object point $P(Y',Z')$ must be uniquely obtained from Equations 84 and 85, which means that the object point can be inverse-transformed from its left and right measured image coordinates (y_{c1},y_{c2}). It follows that both Equations 84 and 85 must, with respect to (Y',Z'), have a solution in the form

$$\begin{aligned} Y' &= \frac{B_1y_{c1} + B_2y_{c2} + B_3}{B_7y_{c1} + B_8y_{c2} + 1} \\ Z' &= \frac{B_4y_{c1} + B_5y_{c2} + B_6}{B_7y_{c1} + B_8y_{c2} + 1}\ . \end{aligned} \tag{86}$$

From this fact we can see that Equations 84 and 85 have only eight independent parameters. Accordingly, the relationship (Equation 83) can be uniquely determined with four independent elements. Consequently, the general collinearity equations (Equation 81) of a one-dimensional photograph have only seven independent parameters.

(2) *General orientation problem of a stereoscopic pair of one-dimensional photographs.*

The general collinearity equations (Equation 81) of a one-dimensional central perspective photograph, needless to say, cannot be central projectively inverse-transformed. Thus, an object point $P(X,Y,Z)$ can not be uniquely determined from the measured image point $p_c(0,y_c)$. Therefore, a stereoscopic pair of one-dimensional photographs must also be introduced in order to determine uniquely the object point from the measured image coordinates. The general collinearity equations are

$$\begin{aligned} 0 &= \frac{{}_1A_1X + {}_1A_2Y + {}_1A_3Z + {}_1A_4}{{}_1A_9X + {}_1A_{10}Y + {}_1A_{11}Z + 1} \\ y_{c1} &= \frac{{}_1A_5X + {}_1A_6Y + {}_1A_7Z + {}_1A_8}{{}_1A_9X + {}_1A_{10}Y + {}_1A_{11}Z + 1} \end{aligned} \tag{87}$$

for the left photograph, and

$$
\begin{aligned}
0 &= \frac{{}_2A_1X + {}_2A_2Y + {}_2A_3Z + {}_2A_4}{{}_2A_9X + {}_2A_{10}Y + {}_2A_{11}Z + 1} \\
y_{c2} &= \frac{{}_2A_5X + {}_2A_6Y + {}_2A_7Z + {}_2A_8}{{}_2A_9X + {}_2A_{10}Y + {}_2A_{11}Z + 1}
\end{aligned}
\tag{88}
$$

for the right one, respectively. Equations 87 and 88 can also be expressed in the linear form with respect to the object-space coordinates (X,Y,Z) as

$$
\begin{aligned}
&{}_1A_1X + {}_1A_2Y + {}_1A_3Z + {}_1A_4 = 0 \\
&(y_{c1}\cdot{}_1A_9 - {}_1A_5)X + (y_{c1}\cdot{}_1A_{10} - {}_1A_6)Y + (y_{c1}\cdot{}_1A_{11} - {}_1A_7)Z + (y_{c1} - {}_1A_8) = 0 \\
&{}_2A_1X + {}_2A_2Y + {}_2A_3Z + {}_2A_4 = 0 \\
&(y_{c2}\cdot{}_2A_9 - {}_2A_5)X + (y_{c2}\cdot{}_2A_{10} - {}_2A_6)Y + (y_{c2}\cdot{}_2A_{11} - {}_2A_7)Z + (y_{c2} - {}_2A_8) = 0\,.
\end{aligned}
\tag{89}
$$

The condition that Equation 89 is satisified for an arbitrary object point $P(X,Y,Z)$ is derived in the following determinant form:

$$
\begin{vmatrix}
{}_1A_1 & {}_1A_2 & {}_1A_3 & {}_1A_4 \\
y_{c1}\cdot{}_1A_9 - {}_1A_5 & y_{c1}\cdot{}_1A_{10} - {}_1A_6 & y_{c1}\cdot{}_1A_{11} - {}_1A_7 & y_{c1} - {}_1A_8 \\
{}_2A_1 & {}_2A_2 & {}_2A_3 & {}_2A_4 \\
y_{c2}\cdot{}_2A_9 - {}_2A_5 & y_{c2}\cdot{}_2A_{10} - {}_2A_6 & y_{c2}\cdot{}_2A_{11} - {}_2A_7 & y_{c2} - {}_2A_8
\end{vmatrix} = 0.
\tag{90}
$$

On the other hand, the space coordinates (X,Y,Z) of an object point must be uniquely determined by means of the second, third and fourth equations of Equation 89 under the condition of Equation 90. It means that the measured image coordinates $(y_{c1},0,y_{c2})$ and the object-space coordinates (X,Y,Z) must be central projectively transformed from each other, if the condition (Equation 90) is satisfied. Thus, the following three equations with respect to (X,Y,Z), i.e.,

$$
\begin{aligned}
y_{c1} &= \frac{{}_1A_5X + {}_1A_6Y + {}_1A_7Z + {}_1A_8}{{}_1A_9X + {}_1A_{10}Y + {}_1A_{11}Z + 1} \\
0 &= \frac{{}_2A_1X + {}_2A_2Y + {}_2A_3Z + {}_2A_4}{{}_2A_9X + {}_2A_{10}Y + {}_2A_{11}Z + 1} \\
y_{c2} &= \frac{{}_2A_5X + {}_2A_6Y + {}_2A_7Z + {}_2A_8}{{}_2A_9X + {}_2A_{10}Y + {}_2A_{11}Z + 1}
\end{aligned}
\tag{91}
$$

must have a solution in the form

$$
\begin{aligned}
X &= \frac{B_1y_{c1} + B_2y_{c2} + B_3}{B_{10}y_{c1} + B_{11}y_{c2} + 1} \\
Y &= \frac{B_4y_{c1} + B_5y_{c2} + B_6}{B_{10}y_{c1} + B_{11}y_{c2} + 1} \\
Z &= \frac{B_7y_{c1} + B_8y_{c2} + B_9}{B_{10}y_{c1} + B_{11}y_{c2} + 1}
\end{aligned}
\tag{92}
$$

where the coefficients $B_i(i = 1, \ldots, 11)$ are all independent. Also, these 11 independent parameters can be mathematically determined from the Y and Z coordinates of four object points and the X coordinates of three object points. These points can be independently selected in the object space, because it can be regarded as a plane, points on which are given three-dimensionally.

Consequently, three independent elements must be mathematically obtained from the condition (Equation 90), since 14 independent parameters in Equations 87 and 88 have to be known for the unique determination of the space coordinates of all photographed object points.

As for the general photogrammetric orientation problem of a stereoscopic pair of one-dimensional central perspective photographs, the condition (Equation 90) corresponds to the model construction, while Equation 92 is equivalent to the general central projective one-to-one correspondence between the model and object spaces. Also, the stereo model is constructed in a two-dimensional space. Thus, the general central projective one-to-one correspondence is described between the model space $(0,Y_M,Z_M)$ and the object space (X,Y,Z) in the form

$$
\begin{aligned}
0 &= \frac{E_1X + E_2Y + E_3Z + E_4}{E_{13}X + E_{14}Y + E_{15}Z + 1} = E'_1X + E'_2Y + E'_3Z + 1 \\
Y_M &= \frac{E_5X + E_6Y + E_7Z + E_8}{E_{13}X + E_{14}Y + E_{15}Z + 1} = \frac{E'_4Y' + E'_5Z' + E'_6}{E'_{10}Y' + E'_{11}Z' + 1} \\
Z_M &= \frac{E_9X + E_{10}Y + E_{11}Z + E_{12}}{E_{13}X + E_{14}Y + E_{15}Z + 1} = \frac{E'_7Y' + E'_8Z' + E'_9}{E'_{10}Y' + E'_{11}Z' + 1}
\end{aligned} \tag{93}
$$

or inversely as

$$
\begin{aligned}
X &= \frac{G_1Y_M + G_2Z_M + G_3}{G_{10}Y_M + G_{11}Z_M + 1} \\
Y &= \frac{G_4Y_M + G_5Z_M + G_6}{G_{10}Y_M + G_{11}Z_M + 1} . \\
Z &= \frac{G_7Y_M + G_8Z_M + G_9}{G_{10}Y_M + G_{11}Z_M + 1} .
\end{aligned} \tag{94}
$$

The general central projective one-to-one correspondence (Equation 93) between the model space $(0,Y_M,Z_M)$ and the object space (X,Y,Z) can also be explained as follows. The first part of Equation 93 indicates the object plane which can be fixed with three points given in the object-space coordinate system (X,Y,Z). Thus, the three independent coefficients, E'_1, E'_2, and E'_3, are determined. Also, if the equation of the object plane is known, object points can be expressed in the new object-space coordinate system (X',Y',Z'). Accordingly, we can discuss the general central projective one-to-one correspondence between the model plane (Y_M,Z_M) and the object plane (Y',Z'). It is described by means of the second and third parts of Equation 93:

$$
\begin{aligned}
Y_M &= \frac{E'_4Y' + E'_5Z' + E'_6}{E'_{10}Y' + E'_{11}Z' + 1} \\
Z_M &= \frac{E'_7Y' + E'_8Z' + E'_9}{E'_{10}Y' + E'_{11}Z' + 1} .
\end{aligned} \tag{95}
$$

This one-to-one correspondence has eight independent coefficients, which can be uniquely determined from four points given in the object plane (Y',Z').

The relationship (Equation 94) between the model space $(0,Y_M,Z_M)$ and the object space (X,Y,Z) is essentially the same as Equation 93, but it has another geometrical characteristic in that it can be divided into the similarity condition and the three-dimensional similarity transformation (the three-dimensional central projective one-to-one correspondence) between two three-dimensional spaces, which will be later discussed.

The condition (Equation 90) will be explained in the photogrammetric orientation problem of a stereoscopic pair of one-dimensional central perspective photographs in the following way. As was already discussed, the condition (Equation 90) determines only three independent elements. Also, a stereo model is constructed in a two-dimensional space (see Equation 91 or 93). Thus, we need only set the stereoscopic pair so that they lie on a plane (see Figure 20). Then, we obtain a stereo model, because all corresponding rays intersect on the plane. Also, in this process three independent parameters are determined to be zero for the model coordinate system. These are ϕ_2,κ_2 and B_{x2} among the five exterior orientation parameters $(\phi_2,\omega_2,\kappa_2,B_{x2},B_{z2})$ (relative ones) of the right photograph. It is noted here that the component B_{y2} is equivalent to the model base and can be selected arbitrarily.

(3) *Special case.*

In this paragraph, we will treat the orientation problem of a stereoscopic pair of one-dimensional photographs for the special case where the interior orientation parameters (y_H,c) are known. Thus, a one-dimensinal photograph has only six independent parameters (six exterior orientation elements). As in paragraph (2), which central projective one-to-one correspondence is valid between the model space $(0,Y_M,Z_M)$ and the object space (X,Y,Z) will first be investigated. The general central projective one-to-one correspondence (Equation 95) between the model plane (Y_M,Z_M) and the object plane (Y',Z') is identical to the geometrical relationship between flat terrain and the two-dimensional picture plane taken with a non-metric camera (lens distortion is not considered): i.e.,

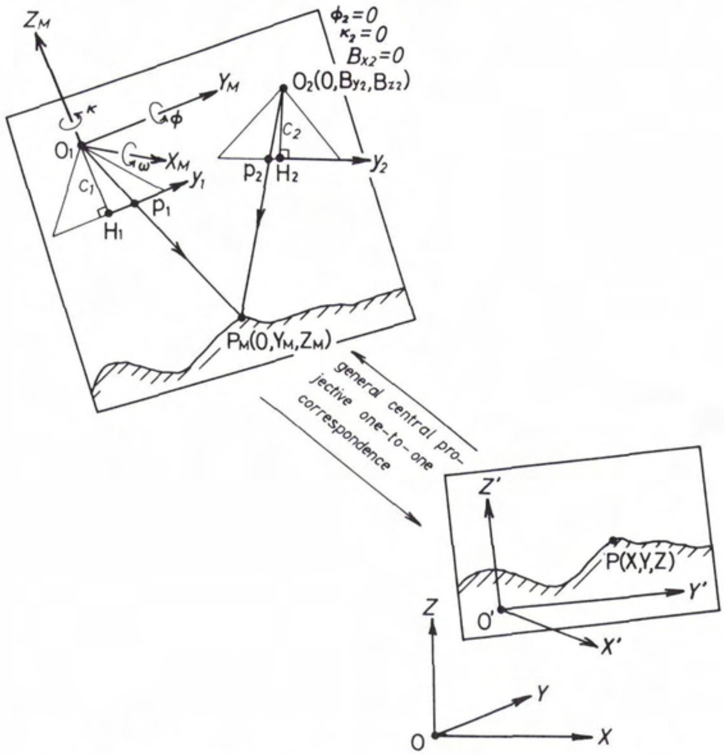

FIG. 20. Model construction and one-to-one correspondence between model and object spaces.

$$x_c = \frac{A_1X + A_2Y + A_3}{A_7X + A_8Y + 1}$$

$$y_c = \frac{A_4X + A_5Y + A_6}{A_7X + A_8Y + 1} . \tag{96}$$

Thus, if the interior orientation parameters of a stereoscopic pair of one-dimensional photographs are given, the central projective one-to-one correspondence between the modal plane (Y_M,Z_M) and the object plane (Y',Z') must be identical to the geometrical relationship between a flat ground surface and the two-dimensional picture plane taken with a metric camera. It follows that this one-to-one correspondence can be uniquely defined with six independent parameters. Also, three points are necessary for the unique determination of the one-to-one correspondence.

On the other hand, also in this case, the first part of Equation 93 must be further applied for the unique determination of the central projective one-to-one correspondence between the model space (0,Y_M,Z_M) and the object space (X,Y,Z). This means that the one-to-one correspondence has three more independent elements. Accordingly, we can see that the one-to-one correspondence questioned has nine independent parameters. Also, these elements can be determined with three points given in the object space.

Consequently, three independent parameters must be mathematically obtained from the condition (Equation 90), because a stereoscopic pair of one-dimensional photographs has 12 independent orientation elements to be determined (with regard to the relative orientation of stereoscopic continuous strip imagery, Derenyi[4] noted that only three independent parameters could be determined, because the orientation points formed a line and not a plane).

FOR FLAT TERRAIN

The photographed terrain being flat, the general collinearity equations (Equation 81) of a one-dimensional photograph are reduced to

$$x_c = 0 = \frac{A_1X + A_2Y + A_3}{A_7X + A_8Y + 1}$$

$$y_c = \frac{A_4X + A_5Y + A_6}{A_7X + A_8Y + 1} \quad (97)$$

The first part of Equation 97 indicates the equation of a plane through the photographed object line on the flat ground surface (see Figure 21). Thus, the plane is not uniquely determined and has only two independent parameters. The second part of Equation 97 is the general collinearity condition between an object point $P(Y',0)$ and the measured image point $p_c(y_c)$ on the plane. Thus, it reduces to

$$y_c = \frac{A_1'Y' + A_2'}{A_3'Y' + 1} \quad (98)$$

where the Y'-Z' plane of the new object-space coordinate system (X',Y',Z') lies on the plane and, furthermore, the Y' axis is taken along the ground surface. Thus, Equation 98 has in general three independent elements. Consequently, we can see that the general collinearity equations (Equation 97) of a one-dimensional photograph has five independent elements, if the photographed terrain is flat. Also, this discussion is valid for the general and special cases in the preceding paragraph.

Equation 97 can be central projectively inverse-transformed, which means that a photographed object point $P(X,Y,H)$ (H given and constant) can be uniquely obtained from the measured image coordinates $(0,y_c)$.

As for the general orientation problem of a one-dimensional photograph, Equation 97 is first inverse-transformed as

$$X = \frac{A_1'y_c + A_2'}{A_5'y_c + 1}$$

$$Y = \frac{A_3'y_c + A_4'}{A_5'y_c + 1} \quad (99)$$

Then, independent coefficients A_i' $(i = 1, \ldots, 5)$ are determined with the planimetric coordinates (X,Y) of two points and the X or Y coordinate of one point given in the object space coordinate system (X,Y,Z). By calculating the coefficients A_i' $(i = 1, \ldots, 5)$ in Equation 99, all photographed object points can be obtained by means of Equation 99.

Photogrammetric Orientation Problem of One-Dimensional Central-Perspective Photographs

As we saw in the previous section, a one-dimensional photograph has seven independent parameters in general, if the photographed terrain is hilly. Defining these seven elements in terms of photogrammetric orientation parameters, we will take six exterior orientation elements $(\phi,\omega,\kappa,X_0,Y_0,Z_0)$ of the one-dimensional photograph and an interior orientation element c (the principal distance). Other central projective parameters such as y_H (the y_c coordinate of the principal point) are assumed to be known so as to reconstruct the exposure position and attitude of the one-dimensional photograph correctly, although this assumption is mathematically not necessary.

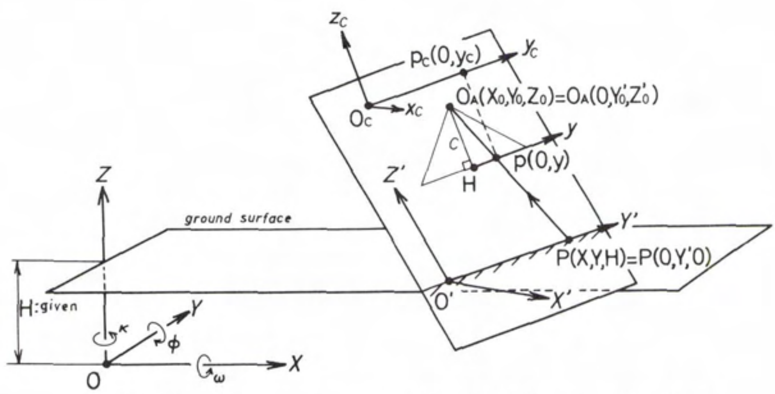

FIG. 21. Central projection on a plane for a flat ground surface.

For a flat ground surface, the general collinearity equation has only five independent parameters. Thus, we will select five exterior orientation elements $(\omega,\kappa,X_0,Y_0,Z_0)$ of a one-dimensional photograph as these five independent elements. The remaining one exterior orientation parameter ϕ (rotation element about the Y axis of the object-space coordinate system (X,Y,Z)) is considered to be zero, since it has the same geometrical effect as X_0.

ORIENTATION PROBLEM OF A STEREOSCOPIC PAIR OF ONE-DIMENSIONAL CENTRAL-PERSPECTIVE PHOTOGRAPHS

(1) *General case.*

The general photogrammetric orientation procedure for a stereoscopic pair of one-dimensional photographs will be discussed as follows (see Figure 22). A stereo model is constructed in an arbitrary plane. For this purpose, a stereoscopic pair of one-dimensional photographs are set to lie on a plane. Taking the Y_M-Z_M plane of the model coordinate-system (X_M,Y_M,Z_M) on the plane, we can assume, for the exterior orientation parameters of the stereo pair in the model coordinate system, that

$$\phi_1 = \omega_1 = \kappa_1 = 0, \qquad {}_MX_{01} = {}_MY_{01} = {}_MZ_{01} = 0 \tag{100}$$

for the left photograph, and

$$\phi_2 = \kappa_2 = 0, \qquad B_{x2} = 0 \tag{101}$$

for the right photograph, respectively, where

$$B_{x2} = {}_MX_{02} - {}_MX_{01},\ B_{y2} = {}_MY_{02} - {}_MY_{01},\ B_{z2} = {}_MZ_{02} - {}_MZ_{01}.$$

Furthermore, B_{y2} is equivalent to the model base and can be selected as an arbitrary value. The two remaining exterior orientation elements (ω_2,B_{z2}) for the model coordinate system and the interior orientation parameters (c_1,c_2) (the principal distance of the stereoscopic pair of one-dimensional photographs) can be essentially obtained from the general central projective one-to-one correspondence (Equation 94) between the model space $(0,Y_M,Z_M)$ and the object space (X,Y,Z) and can be taken mathematically as arbitrary values during the phase of model construction. Also, other interior orientation elements such as y_{H1},y_{H2} (the measured y_c-coordinates of the principal points) cannot be determined in the general photogrammetric orientation problem of a stereoscopic pair of one-dimensional photographs and, thus,

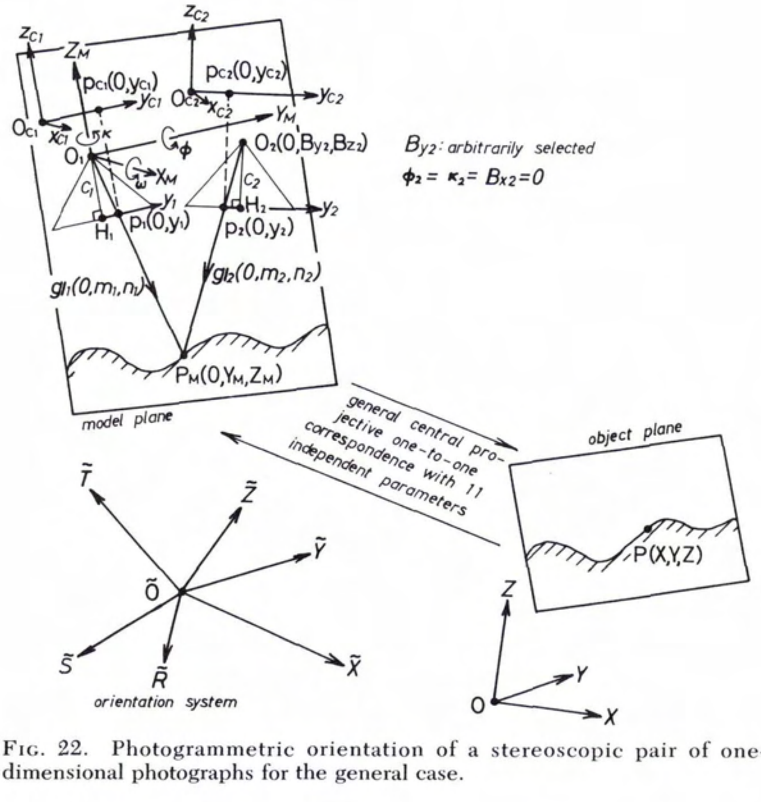

FIG. 22. Photogrammetric orientation of a stereoscopic pair of one-dimensional photographs for the general case.

may be assumed to be given. It must be noted that in the model construction process three independent orientation parameters are determined to be

$$\phi_2 = \kappa_2 = B_{x2} = 0$$

and need not be calculated.

A model point $P_M(0,Y_M,Z_M)$ is obtained as follows. Corresponding rays $\mathbf{g}_1$ and $\mathbf{g}_2$, which intersect automatically on the model plane, are described in the form

$$\mathbf{g}_1: \frac{Y_M}{m_1} = \frac{Z_M}{n_1} = \rho_1 \tag{102}$$

$$\mathbf{g}_2: \frac{Y_M - B_{y2}}{m_2} = \frac{Z_M - B_{z2}}{n_2} = \rho_2 \tag{103}$$

where $(0,m_1,n_1)$ and $(0,m_2,n_2)$ are direction cosines of corresponding rays about the model coordinate system (X_M,Y_M,Z_M). They are derived in the following way. The relationship between a picture point $p(0,y)$ and its measured image point $p_c(0,y_c)$ is

$$\begin{aligned} y_1 &= y_{c1} - y_{H1} \text{ (assumed as known)} \\ y_2 &= y_{c2} - y_{H2} \text{ (assumed as known).} \end{aligned} \tag{104}$$

The transformed picture coordinates $(0,{}_MY_p,{}_MZ_p)$ of a picture point, which are the space coordinates of the picture point in the model coordinate system (X_M,Y_M,Z_M), are described in the form

$$\begin{bmatrix} 0 \\ {}_MY_{p1} \\ {}_MZ_{p1} \end{bmatrix} = \begin{bmatrix} 0 \\ y_{c1} - y_{H1} \\ -c_1 \end{bmatrix} \tag{105}$$

for the left one-dimensional photograph, and

$$\begin{bmatrix} 0 \\ {}_MY_{p2} \\ {}_MZ_{p2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & \sin\omega_2 & \cos\omega_2 \end{bmatrix} \begin{bmatrix} 0 \\ y_{c2} - y_{H2} \\ -c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{y2} \\ B_{z2} \end{bmatrix} \tag{106}$$

for the right one, respectively. Thus, the direction cosines are expressed by means of Equations 105 and 106 in the form

$$\begin{aligned} l_1 &= 0, \quad m_1 = {}_MY_{p1}/L_1, \quad n_1 = {}_MZ_{p1}/L_1 \\ l_2 &= 0, \quad m_2 = ({}_MY_{p2} - B_{y2})/L_2, \quad n_2 = ({}_MZ_{p2} - B_{z2})/L_2 \end{aligned} \tag{107}$$

where

$$L_1 = \sqrt{{}_MY_{p1}^2 + {}_MZ_{p1}^2}, \qquad L_2 = \sqrt{({}_MY_{p2} - B_{y2})^2 + ({}_MZ_{p2} - B_{z2})^2}.$$

A model point $P_M(0,Y_M,Z_M)$ is calculated from Equations 102 and 103 as

$$\begin{aligned} \rho_1 &= \frac{n_2B_{y2} - m_2B_{z2}}{m_1n_2 - n_1m_2} \\ X_M &= 0 \\ Y_M &= \rho_1 m_1 = \frac{m_1n_2B_{y2} - m_1m_2B_{z2}}{m_1n_2 - n_1m_2} \\ Z_M &= \rho_1 n_1 = \frac{n_1n_2B_{y2} - n_1m_2B_{z2}}{m_1n_2 - n_1m_2}. \end{aligned} \tag{108}$$

The general central projective one-to-one correspondence between the model space $(0,Y_M,Z_M)$ and the object space (X,Y,Z) is given by Equation 94. However, Equation 94 has an inconvenient form for calculating the photogrammetric orientation parameters of a stereoscopic pair of one-dimensional photographs. Therefore, we will take another approach to determine these parameters directly (see Figure 23). This procedure will be precisely outlined as follows. The model and the object are considered not to exist in the same three-dimensional space, because they are generally not similar. In order to make them similar, we must introduce the similarity condition between the model and object spaces into the model construction process in the same three-dimensional space as the object space. Also, the

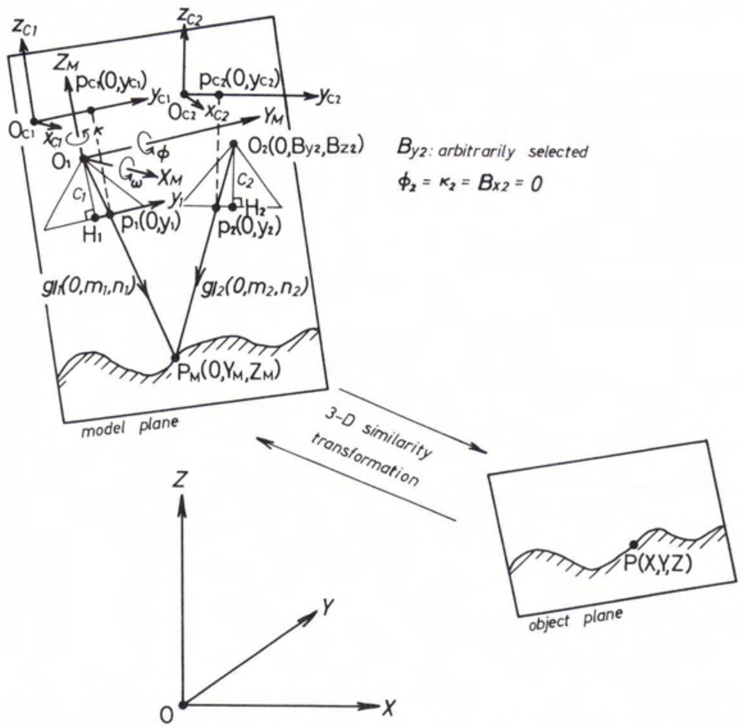

FIG. 23. Model construction in the same three-dimensional space as that for the object for the general case.

similarity condition must be constructed by means of four points, which are mathematically necessary for the unique determination of the general central projective one-to-one correspondence (Equation 94) between the model and object spaces. Furthermore, because the degrees of freedom of four points on a plane is five, the similarity condition must be constructed by setting up the following expression:

$$\frac{L_{Mi}\ \text{(line segment in the model space)}}{L_{oi}\ \text{(line segment in the object space)}} = m\ \text{(constant)}$$

for five independent line segments (see Figure 24). In the above expression, m denotes the scale factor between the model and the object. Thus, we get four independent equations as the similarity condition between the model and object spaces. The expression for the similarity condition is actually described in the form

$$\sqrt{\frac{(Y_{Mi} - Y_{Mk})^2 + (Z_{Mi} - Z_{Mk})^2}{(X_i - X_k)^2 + (Y_i - Y_k)^2 + (Z_i - Z_k)^2}} = m. \tag{109}$$

By solving the similarity condition with respect to four unknowns $(\omega_2, B_{z2}, c_1, c_2)$ during the phase of the model construction in the same three-dimensional space as the object space, we can calculate a model

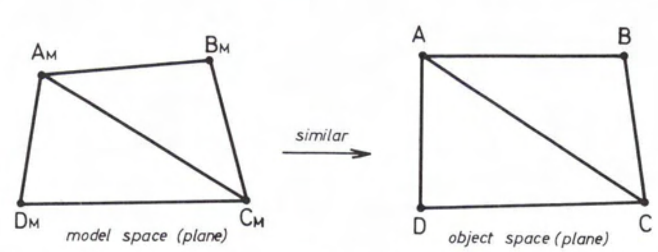

FIG. 24. Similarity condition for the general case.

point $P_M(0,Y_M,Z_M)$ by means of Equation 108. Then, all model points can be transformed into the object space (X,Y,Z) by the three-dimensional similarity transformation;

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = m \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega & -\sin\Omega \\ 0 & \sin\Omega & \cos\Omega \end{bmatrix} \begin{bmatrix} \cos K & -\sin K & 0 \\ \sin K & \cos K & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_M \\ Y_M \\ Z_M \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}. \quad (110)$$

From the discussion mentioned above, we can see that the general central projective one-to-one correspondence (Equation 94) between the model and object spaces, points on which are, however, given three-dimensionally, can be divided into the similarity condition and the three-dimensional similarity transformation (the three-dimensional central projective one-to-one correspondence).

(2) *Special case.*

This paragraph treats the photogrammetric orientation problem of a stereoscopic pair of one-dimensional photographs for the special case where the interior orientation parameters are all given. Thus, the photogrammetric orientation parameters of a one-dimensional photograph reduce to only the six exterior orientation parameters $(\phi,\omega,\kappa,X_0,Y_0,Z_0)$. The orientation procedure is almost the same as in Paragraph (1). Then, it will be briefly described as follows.

A stereo model is also constructed on an arbitrary plane, and in this process three relative orientation elements (ϕ_2,κ_2,B_{x2}) are determined to be zero for the model coordinate system (X_M,Y_M,Z_M). The central projective one-to-one correspondence between the model space $(0,Y_M,Z_M)$ and the object space (X,Y,Z) can be described with nine independent parameters in the same form as Equation 94. Furthermore, these nine elements are uniquely determined by means of three points given in the object space. This means that the stereo model is not similar to the object, or in other words that the stereo model does not exist in the same three-dimensional space as the object space. Accordingly, the similarity condition between the model and the object must be introduced so as to make them similar. Also, since the degrees of freedom for three points on a plane is three, we can construct the similarity condition by setting up Equation 109 for three independent line segments (see Figure 25). Then, we get two independent equations as the condition for the model construction in the same three-dimensional space as the object space.

By solving the similarity condition with respect to two independent orientation unknowns (ω_2,B_{z2}) during the phase of model construction in the same three-dimensional space as the object space, we can calculate a model point $P_M(0,Y_M,Z_M)$ by means of Equation 108. Then, all model points can be transformed into the object space (X,Y,Z) by the three-dimensional similarity transformation (Equation 110). (Derenyi[4] noted that two independent orientation parameters for the model construction must either be known or assumed to be zero. However, these two parameters can be determined by means of the similarity condition between the model and object spaces.).

ORIENTATION PROBLEM OF INDIVIDUAL ONE-DIMENSIONAL PHOTOGRAPHS

If the photographed ground surface is flat, only five exterior orientation parameters $(\omega,\kappa,X_0,Y_0,Z_0)$ can be determined in the orientation problem of one-dimensional photograph. Thus, the interior orientation parameters (c,y_H) are considered to be known and, furthermore, one rotation parameter, ϕ, of the one-dimensional photograph is assumed to be zero. The five exterior orientation elements will be calculated as follows (see Figure 26).

First, the transformed picture coordinates (X_p,Y_p,Z_p) of a picture point $p(0,y)$ are expressed in the object-space coordinate system (X,Y,Z) in the form

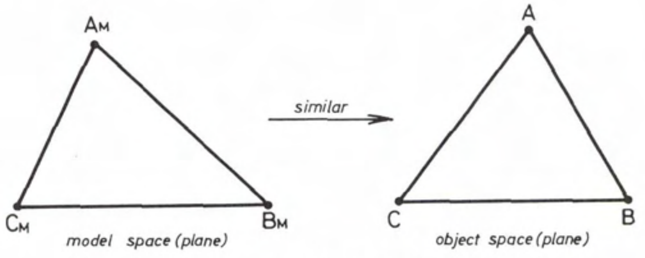

FIG. 25. Similarity condition for the special case.

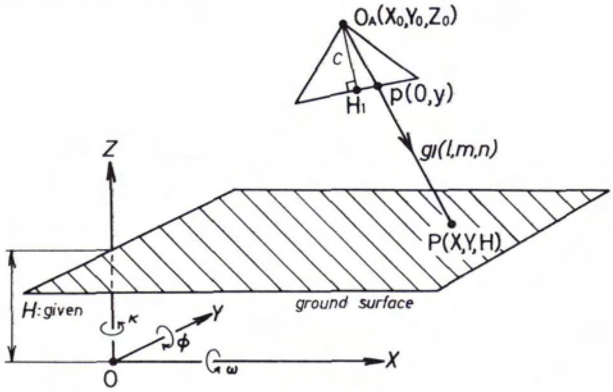

FIG. 26. Photogrammetric orientation of individual one-dimensional photographs.

$$\begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} = \mathbf{D}_\omega \mathbf{D}_\kappa \begin{bmatrix} 0 \\ y \\ -c \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} \qquad (111)$$

$$y = y_c - y_H$$

where $\mathbf{D}_\omega$ and $\mathbf{D}_\kappa$ denote rotation matrices of rotation elements ω and κ, respectively. A ray **g** through the projection center $O_A(X_0,Y_0,Z_0)$ and the transformed picture point $p_p(X_p,Y_p,Z_p)$ is described as

$$\mathbf{g}: \frac{X - X_0}{l} = \frac{Y - Y_0}{m} = \frac{Z - Z_0}{n}, \qquad (112)$$

in which (l,m,n) are direction cosines of the ray **g** and take the form:

$$\begin{aligned} & l = \overline{X}_p/L, \qquad m = \overline{Y}_p/L, \qquad n = \overline{Z}_p/L \\ & L = \sqrt{\overline{X}_p^2 + \overline{Y}_p^2 + \overline{Z}_p^2} \\ & \overline{X}_p = X_p - X_0, \qquad \overline{Y}_p = Y_p - Y_0, \qquad \overline{Z}_p = Z_p - Z_0 \quad . \end{aligned} \qquad (113)$$

The five exterior orientation parameters are determined so that the ray **g** may travel through the object point $P(X,Y,Z)$ corresponding to $p(0,y)$. Using this technique, we can construct the (determination) equations for the five orientation unknowns in the form

$$\begin{aligned} X &= X_0 + \frac{1}{n}(H - Z_0) = X_0 + \frac{d_{12}y - d_{13}c}{d_{32}y - d_{33}c}(H - Z_0) \\ Y &= Y_0 + \frac{m}{n}(H - Z_0) = Y_0 + \frac{d_{22}y - d_{23}c}{d_{32}y - d_{33}c}(H - Z_0) \end{aligned} \qquad (114)$$

where

$$\mathbf{D}_\omega \mathbf{D}_\kappa = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}.$$

By setting up Equation 114 for three object points (rigorously, the X- *and* Y-coordinates of two points and the X- or Y-coordinate of one point) and solving them with respect to $(\omega,\kappa,X_0,Y_0,Z_0)$, we can calculate the planimetric coordinates of all photographed object points in the object-space coordinate system (X,Y,Z) by means of Equation 114.

Also, in the case of a hilly terrain, we can determine the seven photogrammetric orientation parameters $(\phi,\omega,\kappa,X_0,Y_0,Z_0,c)$ of an one-dimensional photograph using the same orientation technique as described above. However, object points cannot be uniquely obtained from the measured image point $p_c(0,y_c)$.

Practical Discussion for the Orientation Problem of One-Dimensional Central Perspective Photographs

Strip imagery is taken with a central projection in the across-track direction and with an orthogonal projection in the along-track direction. Thus, the across-track coordinate (y-coordinate) of an image point has the central projective characteristics, while the along-track coordinate (x-coordinate) is proportional to the ground track distance, if the aircraft travels at a constant speed. This is also valid for MSS imagery under the assumption that exterior orientation parameters of an MSS are constant along a scan line.

Extracting one line image from a strip photograph or an MSS image (see Figure 27), the collinear relationship (Equation 81) is maintained between an object point and the measured image point. Thus, in principle, the orientation theories discussed in the previous chapters are applicable to strip and MSS imagery. However, the presented orientation techniques themselves may be of little practical use due to the facts that

- Ground control is necessary for every line-image, and
- The central projection for a stereoscopic pair of line images is seldom generated on the same plane if the photographed terrain is hilly. In the usual case[4], all points selected on the left photograph are recorded at the same instant, while the recording of the corresponding points on the right photograph are spread over a certain time period, as is demonstrated in Figure 28.

In the practical orientation of strip and MSS imagery, we assume[3,7] that

> the changes of exterior orientation parameters of imaging devices along the flight path can be modeled with some functional form such as polynomials and Fourier's series; thus, the functional forms are determined for the flight path instead of calculating the exterior orientation elements for each line-image.

The orientation theory of individual one-dimensional photographs is flexible for such difficulties and, thus, many practical orientation methods[3] have been developed for the case where the terrain is flat or for the case where the elevations of all photographed object points are given. Little[4,6,8], however, has been done in the way of investigating the practical orientation problem for stereo strip photographs and stereo MSS data coverage. This is because such imagery has no central perspective characteristics for a plane. An orientation technique based on the simultaneous determination of all exterior orientation parameters for a stereoscopic pair of conventional pictures may be easily applicable to stereo strip photographs and stereo MSS data coverage, because this orientation method is instead based on the orientation theory of individual photographs. It is precisely discussed by Okamoto[9] and is here omitted.

Concluding Remarks

The orientation problem of one-dimensional central perspective photographs has been theoretically considered in this paper. The discussion was not directed to practical use, but to a comparison with the orientation problem of conventional (two-dimensional) photographs. Thus, the discussions given here are very instructive for understanding the orientation problem of central perspective photographs.

First, mathematical considerations were made regarding the orientation problem of individual one-dimensional photographs and also of the stereoscopic pair by introducing an algebraic approach. Various interesting characteristics for the latter have been revealed. They are as follows:

- A stereo model is constructed in a two-dimensional space (plane) with three independent parameters for the general case;
- The general central projective one-to-one correspondence between the model and object spaces has 11 independent parameters to be determined; and
- In the case where interior orientation parameters of one-dimensional photographs are given, the condition for stereo model construction also has three independent parameters. However, the central projective one-to-one correspondence between the model and object spaces can be determined with nine independent elements.

Next, the orientation problem of a stereoscopic pair of one-dimensional photographs was discussed geometrically in order to obtain directly the traditional orientation parameters. For this purpose, the

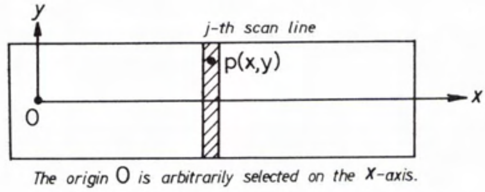

Fig. 27. Strip imagery.

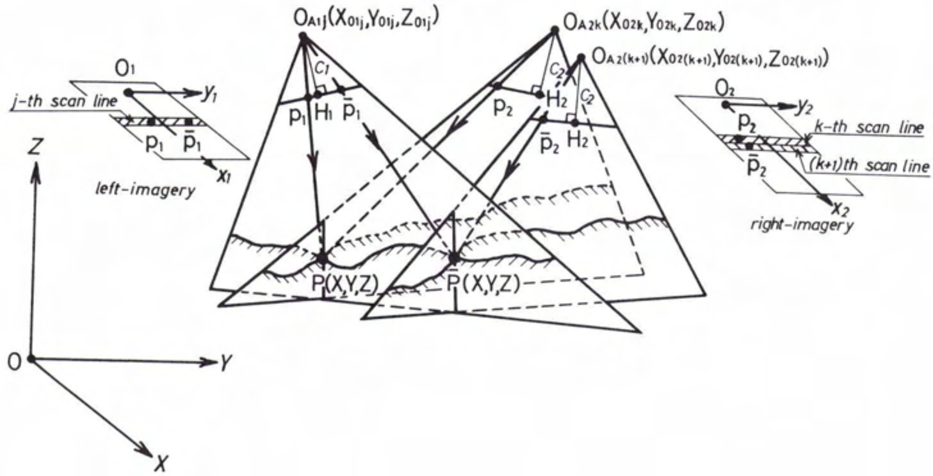

FIG. 28. Central projection in the case of sidelapping flight for a real ground surface.

concept of a multidimensional space having more than three dimensions was introduced. The geometrical considerations of the orientation problem in the multidimensional space clarified the following:

- For stereo model construction we need only set a stereoscopic pair of one-dimensional photographs so that they lie in a plane. In this procedure three independent photogrammetric orientation parameters (ϕ_2,κ_2,B_{x2}) are determined to be zero for the model coordinate system. It is unnecessary to calculate the three elements from the coplanarity condition of corresponding rays as is usually done in conventional photogrammetry.
- The constructed stereo model is not similar to the object, because the three-dimensional similarity transformation between the model and object spaces is not satisfied. This means that the stereo model cannot be constructed in the same three-dimensional space as that of the object.
- The central projective one-to-one correspondence can be divided into the similarity condition and the three-dimensional similarity transformation between the model and object spaces. From this fact, a stereo model can be constructed in the same three-dimensional space as that of the object by introducing the similarity condition between the model and the object during the phase of model construction on a plane.

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