Atsushi Okamoto
Transportation Engineering
Kyoto University
606 Kyoto-shi, Sakyo-ku, Yoshida, Japan

Orientation and Construction of Models

Part II: Model Construction Theory with Multiple Photographs*

The model construction problem with multiple photographs is discussed rigorously for the general case and also for the usual case in close-range photogrammetry.

Introduction

Triplet model construction in aerial triangulation is performed based on the model construction theory with multiple photographs. This theory is known in photogrammetry. However, very little has been written that would provide a general and rigorous approach to the model construction problem with multiple photographs. Koelbl[1] investigated the orientation problem of a stereoscopic pair of photographs taken with non-metric cameras and found that one of the six interior orientation parameters can be obtained from the coplanarity condition of conjugate rays in addition to the five exterior orientation elements (the relative ones). With this finding, Koelbl developed a self-calibration method of non-metric cameras by using three convergent photographs taken of the same object. This might be the first approach to the model construction problem with multiple photographs in the case where interior orientation parameters are not given for each photograph. However, the mathematical explanation was not fully made and the general case was not treated in this paper.

Abstract: *The model construction theory with multiple photographs, which can be applied to calibration problems of non-metric cameras without object space controls, is derived. The analysis is made not only for the general case where a photograph has 11 independent parameters, but also for the usual case in close-range photogrammetry, where the geometry of a picture is determined with nine independent elements (six exterior and three interior). Various interesting characteristics of the model construction problem are revealed for the case of three and four photographs. The discussion can be easily extended to the case of more than four pictures. Also, a camera calibration method using metric cameras as controls is presented.*

In this report the general theory for the model construction with multiple photographs is explained based on the mathematical basis[2] of the general orientation problem of a stereoscopic pair of photographs. Also, the model construction theory with multiple photographs is derived for the usual case in close-range photogrammetry, where three interior orientation parameters (three elements for the principal point) are unknown for each picture. Thus, some new theoretical characteristics of the self-calibration problem of non-metric cameras are clarified here.

In actual calibration of non-metric cameras, lens distortion and film deformation must also be considered in addition to central projective parameters (exterior and interior orientation elements) of a picture.

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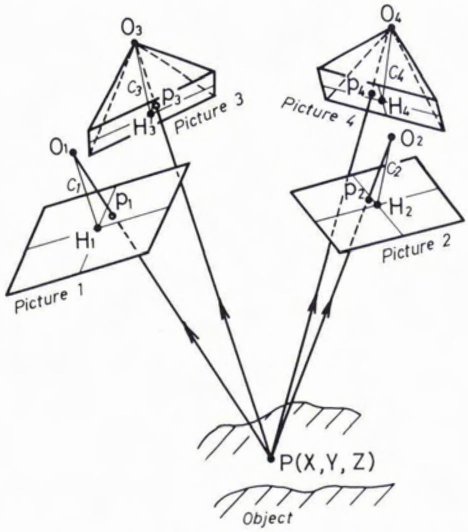

FIG. 10. Four pictures taken of the same object.

These non-central projective parameters, however, can be determined by other geometrical considerations.[3] Thus, they are neglected in the discussion of this paper.

Model Construction Theory with Multiple Photographs for the General Case

THE CASE OF FOUR PHOTOGRAPHS

We will assume that an object space (X,Y,Z) was photographed with four different cameras, as is illustrated in Figure 10. The first stereo model is constructed with the first and second photographs, and the second stereo model with the third and fourth pictures. We will investigate what relationship is

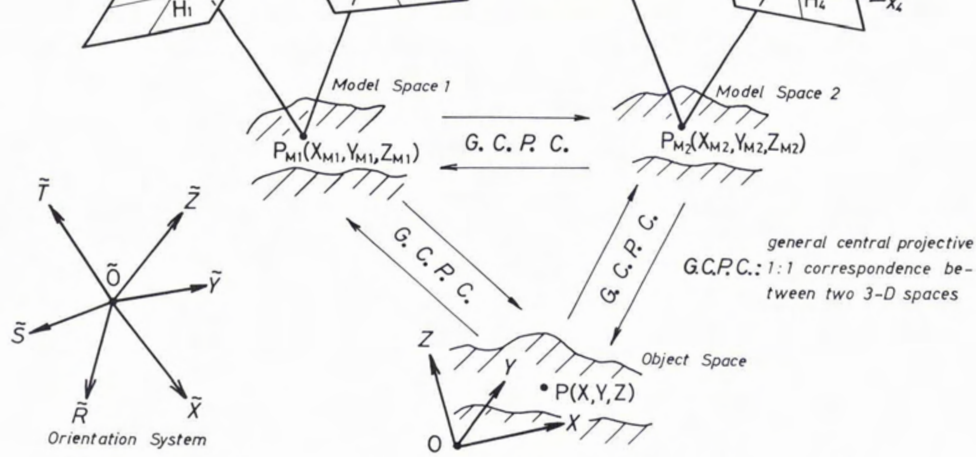

FIG. 11. The 1:1 correspondence between the first and second stereo models.

valid between the first and second stereo models (see Figure 11). The general central projective one-to-one correspondence (Equation 8) is satisfied between the first stereo model space (X_{M1},Y_{M1},Z_{M1}) and the object space (X,Y,Z) in the form

$$
\begin{aligned}
X_{M1} &= \frac{E_1X + E_2Y + E_3Z + E_4}{E_{13}X + E_{14}Y + E_{15}Z + 1} \\
Y_{M1} &= \frac{E_5X + E_6Y + E_7Z + E_8}{E_{13}X + E_{14}Y + E_{15}Z + 1} \\
Z_{M1} &= \frac{E_9X + E_{10}Y + E_{11}Z + E_{12}}{E_{13}X + E_{14}Y + E_{15}Z + 1}
\end{aligned}
\tag{56}
$$

or inversely as

$$
\begin{aligned}
X &= \frac{E'_1X_{M1} + E'_2Y_{M1} + E'_3Z_{M1} + E'_4}{E'_{13}X_{M1} + E'_{14}Y_{M1} + E'_{15}Z_{M1} + 1} \\
Y &= \frac{E'_5X_{M1} + E'_6Y_{M1} + E'_7Z_{M1} + E'_8}{E'_{13}X_{M1} + E'_{14}Y_{M1} + E'_{15}Z_{M1} + 1} \\
Z &= \frac{E'_9X_{M1} + E'_{10}Y_{M1} + E'_{11}Z_{M1} + E'_{12}}{E'_{13}X_{M1} + E'_{14}Y_{M1} + E'_{15}Z_{M1} + 1}
\end{aligned}
\tag{57}
$$

The same can be described between the second stereo model space (X_{M2},Y_{M2},Z_{M2}) and the object space (X,Y,Z) in the form

$$
\begin{aligned}
X_{M2} &= \frac{G_1X + G_2Y + G_3Z + G_4}{G_{13}X + G_{14}Y + G_{15}Z + 1} \\
Y_{M2} &= \frac{G_5X + G_6Y + G_7Z + G_8}{G_{13}X + G_{14}Y + G_{15}Z + 1} \\
Z_{M2} &= \frac{G_9X + G_{10}Y + G_{11}Z + G_{12}}{G_{13}X + G_{14}Y + G_{15}Z + 1}
\end{aligned}
\tag{58}
$$

or inversely as

$$
\begin{aligned}
X &= \frac{G'_1X_{M2} + G'_2Y_{M2} + G'_3Z_{M2} + G'_4}{G'_{13}X_{M2} + G'_{14}Y_{M2} + G'_{15}Z_{M2} + 1} \\
Y &= \frac{G'_5X_{M2} + G'_6Y_{M2} + G'_7Z_{M2} + G'_8}{G'_{13}X_{M2} + G'_{14}Y_{M2} + G'_{15}Z_{M2} + 1} \\
Z &= \frac{G'_9X_{M2} + G'_{10}Y_{M2} + G'_{11}Z_{M2} + G'_{12}}{G'_{13}X_{M2} + G'_{14}Y_{M2} + G'_{15}Z_{M2} + 1}
\end{aligned}
\tag{59}
$$

By substituting Equation 59 into Equation 56, we get the relationship between the first stereo model space (X_{M1},Y_{M1},Z_{M1}) and the second one (X_{M2},Y_{M2},Z_{M2}) in the form

$$
\begin{aligned}
X_{M1} &= \frac{H_1X_{M2} + H_2Y_{M2} + H_3Z_{M2} + H_4}{H_{13}X_{M2} + H_{14}Y_{M2} + H_{15}Z_{M2} + 1} \\
Y_{M1} &= \frac{H_5X_{M2} + H_6Y_{M2} + H_7Z_{M2} + H_8}{H_{13}X_{M2} + H_{14}Y_{M2} + H_{15}Z_{M2} + 1} \\
Z_{M1} &= \frac{H_9X_{M2} + H_{10}Y_{M2} + H_{11}Z_{M2} + H_{12}}{H_{13}X_{M2} + H_{14}Y_{M2} + H_{15}Z_{M2} + 1}
\end{aligned}
\tag{60}
$$

which coincides with the general central projective one-to-one correspondence (Equation 8) between two three-dimensional spaces. It means that the general central projective one-to-one correspondence is valid between the first and second stereo model spaces.

From the fact mentioned above, it is understood that a photogrammetric model can be constructed with four pictures in the common model coordinate system (X_M,Y_M,Z_M). This procedure will be, precisely outlined, as follows (see Figure 12). The first stereo model is constructed by means of the coplanarity condition of conjugate rays $\mathbf{g}_1$ (l_1,m_1,n_1) and $\mathbf{g}_2$ (l_2,m_2,n_2) in the form

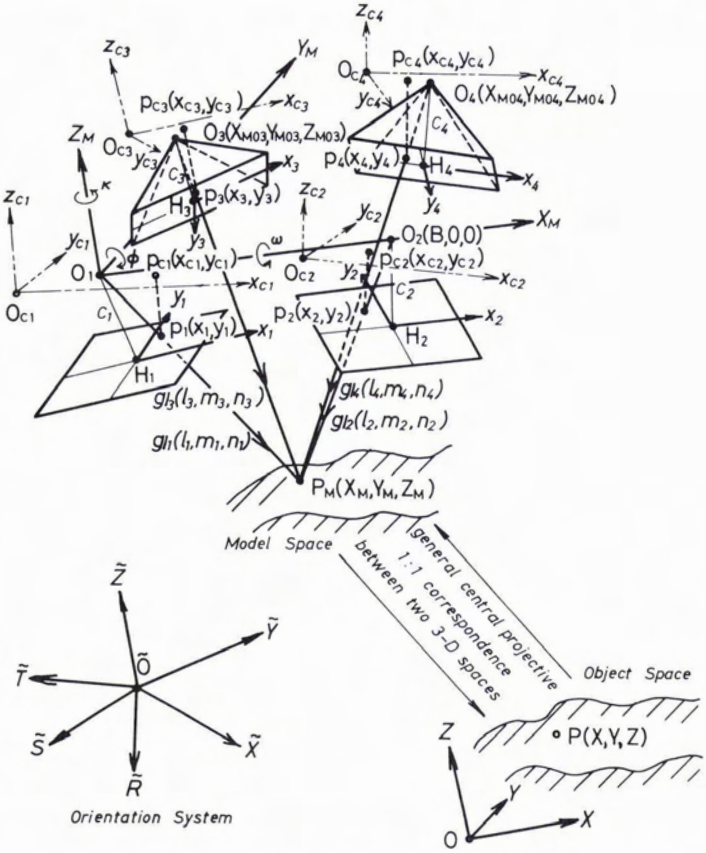

FIG. 12. Model construction with four pictures in the common model coordinate system.

$$\begin{vmatrix} {}_{M1}\bar{Y}_{p1} & {}_{M1}\bar{Y}_{p2} \\ {}_{M1}\bar{Z}_{p1} & {}_{M1}\bar{Z}_{p2} \end{vmatrix} = 0 \tag{61}$$

where $({}_{M1}\bar{X}_{p1}, {}_{M1}\bar{Y}_{p1}, {}_{M1}\bar{Z}_{p1})$ and $({}_{M1}\bar{X}_{p2}, {}_{M1}\bar{Y}_{p2}, {}_{M1}\bar{Z}_{p2})$ denote the reduced transformed picture coordinates of a picture point in the common model coordinate-system in the form

$$\begin{bmatrix} {}_{M1}\bar{X}_{p1} \\ {}_{M1}\bar{Y}_{p1} \\ {}_{M1}\bar{Z}_{p1} \end{bmatrix} = \mathbf{D}_{\phi_1}\mathbf{D}_{\kappa_1} \begin{bmatrix} x_1 \\ y_1 \\ -c_1 \end{bmatrix} \tag{62}$$

for the first photograph, and

$$\begin{bmatrix} {}_{M1}\bar{X}_{p2} \\ {}_{M1}\bar{Y}_{p2} \\ {}_{M1}\bar{Z}_{p2} \end{bmatrix} = \mathbf{D}_{\phi_2}\mathbf{D}_{\omega_2}\mathbf{D}_{\kappa_2} \begin{bmatrix} x_2 \\ y_2 \\ -c_2 \end{bmatrix} \tag{63}$$

for the second one, respectively. Conjugate rays $\mathbf{g}_3(l_3,m_3,n_3)$ and $\mathbf{g}_4(l_4,m_4n_4)$ are described for the second stereo model as

$$\mathbf{g}_3\colon \frac{X_M - X_{M03}}{l_3} = \frac{Y_M - Y_{M03}}{m_3} = \frac{Z_M - Z_{M03}}{n_3} = \rho_3 \tag{64}$$

$$\mathbf{g}_4\colon \frac{X_M - X_{M04}}{l_4} = \frac{Y_M - Y_{M04}}{m_4} = \frac{Z_M - Z_{M04}}{n_4} = \rho_4 \tag{65}$$

in which $(X_{M03},Y_{M03},Z_{M03})$ and $(X_{M04},Y_{M04},Z_{M04})$ indicate the space coordinates of the projection center for the third and fourth photographs in the common model coordinate system, respectively. The reduced transformed picture coordinates $({}_{M2}\bar{X}_{p3},{}_{M2}\bar{Y}_{p3},{}_{M2}\bar{Z}_{p3})$ and $({}_{M2}\bar{X}_{p4},{}_{M2}\bar{Y}_{p4},{}_{M2}\bar{Z}_{p4})$ are

$$\begin{bmatrix} {}_{M2}\bar{X}_{p3} \\ {}_{M2}\bar{Y}_{p3} \\ {}_{M2}\bar{Z}_{p3} \end{bmatrix} = \mathbf{D}_{\phi_3}\mathbf{D}_{\omega_3}\mathbf{D}_{\kappa_3}\begin{bmatrix} x_3 \\ y_3 \\ -c_3 \end{bmatrix} \tag{66}$$

for the third photograph, and

$$\begin{bmatrix} {}_{M2}\bar{X}_{p4} \\ {}_{M2}\bar{Y}_{p4} \\ {}_{M2}\bar{Z}_{p4} \end{bmatrix} = \mathbf{D}_{\phi_4}\mathbf{D}_{\omega_4}\mathbf{D}_{\kappa_4}\begin{bmatrix} x_4 \\ y_4 \\ -c_4 \end{bmatrix} \tag{67}$$

for the fourth one, respectively. The coplanarity condition of conjugate rays $\mathbf{g}_3$ and $\mathbf{g}_4$ is derived from Equations 64, 65, 66, and 67 as

$$\begin{vmatrix} X_{M04} - X_{M03} & {}_{M2}\bar{X}_{p3} & {}_{M2}\bar{X}_{p4} \\ Y_{M04} - Y_{M03} & {}_{M2}\bar{Y}_{p3} & {}_{M2}\bar{Y}_{p4} \\ Z_{M04} - Z_{M03} & {}_{M2}\bar{Z}_{p3} & {}_{M2}\bar{Z}_{p4} \end{vmatrix} = 0 \tag{68}$$

and it yields

$$\begin{vmatrix} {}_2B_X & {}_{M2}\bar{X}_{p3} & {}_{M2}\bar{X}_{p4} \\ {}_2B_Y & {}_{M2}\bar{Y}_{p3} & {}_{M2}\bar{Y}_{p4} \\ {}_2B_Z & {}_{M2}\bar{Z}_{p3} & {}_{M2}\bar{Z}_{p4} \end{vmatrix} = 0 \tag{69}$$

in which

$${}_2B_X = X_{M04} - X_{M03}, \quad {}_2B_Y = Y_{M04} - Y_{M03}, \quad {}_2B_Z = Z_{M04} - Z_{M03}.$$

The space coordinates of model points are given as

$$\begin{aligned} X_{M1} &= \frac{{}_{M1}\bar{X}_{p1}\,{}_{M1}\bar{Y}_{p2}B}{{}_{M1}\bar{X}_{p1}\,{}_{M1}\bar{Y}_{p2} - {}_{M1}\bar{Y}_{p1}\,{}_{M1}\bar{X}_{p2}} \\ Y_{M1} &= \frac{{}_{M1}\bar{Y}_{p1}\,{}_{M1}\bar{Y}_{p2}\,B}{{}_{M1}\bar{X}_{p1}\,{}_{M1}\bar{Y}_{p2} - {}_{M1}\bar{Y}_{p1}\,{}_{M1}\bar{X}_{p2}} \\ Z_{M1} &= \frac{{}_{M1}\bar{Z}_{p1}\,{}_{M1}\bar{Y}_{p2}\,B}{{}_{M1}\bar{X}_{p1}\,{}_{M1}\bar{Y}_{p2} - {}_{M1}\bar{Y}_{p1}\,{}_{M1}\bar{X}_{p2}} \end{aligned} \tag{70}$$

for the first stereo model, and

$$\begin{aligned} X_{M2} &= X_{M03} + \frac{{}_{M2}\bar{X}_{p3}\,({}_{M2}\bar{Y}_{p4}\,{}_2B_X - {}_{M2}\bar{X}_{p4}\,{}_2B_Y)}{{}_{M2}\bar{X}_{p3}\,{}_{M2}\bar{Y}_{p4} - {}_{M2}\bar{Y}_{p3}\,{}_{M2}\bar{X}_{p4}} \\ Y_{M2} &= Y_{M03} + \frac{{}_{M2}\bar{Y}_{p3}\,({}_{M2}\bar{Y}_{p4}\,{}_2B_X - {}_{M2}\bar{X}_{p4}\,{}_2B_Y)}{{}_{M2}\bar{X}_{p3}\,{}_{M2}\bar{Y}_{p4} - {}_{M2}\bar{Y}_{p3}\,{}_{M2}\bar{X}_{p4}} \\ Z_{M2} &= Z_{M03} + \frac{{}_{M2}\bar{Z}_{p3}\,({}_{M2}\bar{Y}_{p4}\,{}_2B_X - {}_{M2}\bar{X}_{p4}\,{}_2B_Y)}{{}_{M2}\bar{X}_{p3}\,{}_{M2}\bar{Y}_{p4} - {}_{M2}\bar{Y}_{p3}\,{}_{M2}\bar{X}_{p4}} \end{aligned} \tag{71}$$

for the second one, respectively. The general central projective transformation of the second stereo model space (X_{M2},Y_{M2},Z_{M2}) into the first one (X_{M1},Y_{M1},Z_{M1}) is mathematically equivalent to the condition that five points in the second stereo model space coincide with the five corresponding points in the first one, respectively. This condition is expressed in the form

$$\begin{aligned} {}_iX_{M1} &= {}_iX_{M2} \\ {}_iY_{M1} &= {}_iY_{M2} \\ {}_iZ_{M1} &= {}_iZ_{M2} \\ (i &= 1, \ldots, 5) \end{aligned} \tag{72}$$

The coplanarity condition (Equation 61) of conjugate rays $\mathbf{g}_1$ and $\mathbf{g}_2$ has 15 orientation parameters $(\phi_1,\omega_1,x_{c01},y_{c01},c_1,\alpha_1,\beta_1,\phi_2,\omega_2,\kappa_2,x_{c02},y_{c02},c_2,\alpha_2,\beta_2)$, while the coplanarity condition (Equation 68) of $\mathbf{g}_3$ and $\mathbf{g}_4$ includes 22 orientation elements $(\phi_3,\omega_3,\kappa_3,X_{M03},Y_{M03},Z_{M03},x_{c03},y_{c03},c_3,\alpha_3,\beta_3,\phi_4,\omega_4,\kappa_4,X_{M04},Y_{M04},Z_{M04},x_{c04},y_{c04},$

c_4, α_4, β_4). Furthermore, in the process of the first stereo model construction, seven independent orientation parameters of the 15 ones in Equation 61 are mathematically determined by means of the coplanarity condition (Equation 61). Also, during the phase of the second model construction, seven independent orientation elements of the 22 ones in Equation 68 are mathematically obtained from the coplanarity condition (Equation 68). The condtion (Equation 72) provides mathematically 15 independent orientation parameters.

By solving Equations 61, 68, and 72 with respect to 29 independent orientation unknowns among 37 parameters included in these equations, we can construct a photogrammetric model with four pictures in the common model coordinate system (X_M,Y_M,Z_M). However, the constructed model is not similar to the object, because the general central projective one-to-one correspondence is also valid between the photogrammetric model and object spaces. The 15 parameters determined by the transformation of the model space (X_M,Y_M,Z_M) into the object one (X,Y,Z) are divided into seven exterior orientation parameters and eight interior orientation parameters which remain undetermined in the model construction process with four pictures. The number of necessary object points is five.

The discussion about the photogrammetric model construction with four photographs clarified the following facts. The coplanarity condition of conjugate rays has seven independent orientation elements to be determined, and these seven parameters are divided into five exterior orientation parameters (relative) and two interior orientation parameters. Accordingly, ten exterior orientation parameters and four interior orientation parameters are obtained in the process of the first and second stereo model construction. On the other hand, the general central projective transformation of the second stereo model space into the first one provides 15 independent orientation parameters. Also, these 15 elements are considered to be divided into seven exterior orientation parameters and eight interior orientation parameters. Therefore, the photogrammetric model construction with four pictures has the potential to determine 12 interior orientation elements in addition to 17 exterior orientation elements. This potential can be used for self-calibration of non-metric cameras, though eight interior orientation parameters cannot be obtained among the 20 parameters of four pictures. In order to get the desired interior orientation elements, these eight elements must be assumed to be known. It means that the photogrammetric model constructed with four pictures must be similar to the object for the purpose of camera calibration. In other words, the constructed model must exist in the same three-dimensional space as the object space.

THE CASE OF THREE PHOTOGRAPHS

In this paragraph, the photogrammetric model construction will be analyzed for the case where an object space (X,Y,Z) was photographed with three different cameras (see Figure 13). This can be treated as the special case for the previous one, where the second and third photographs have the same orientation parameters ($\phi,\omega,\kappa,X_0,Y_0,Z_0,x_{c0},y_{c0},c,\alpha,\beta$). Thus, the case of three pictures is equivalent to the case of four photographs with 11 constraints. The photogrammetric model construction will be described as follows (see Figure 14). The first stereo model is constructed by means of the coplanarity condition (Equation 61) of conjugate rays $\mathbf{g}_1$ and $\mathbf{g}_2$, which includes 15 orientation parameters ($\phi_1,\kappa_1,x_{c01},y_{c01},c_1,\alpha_1,$

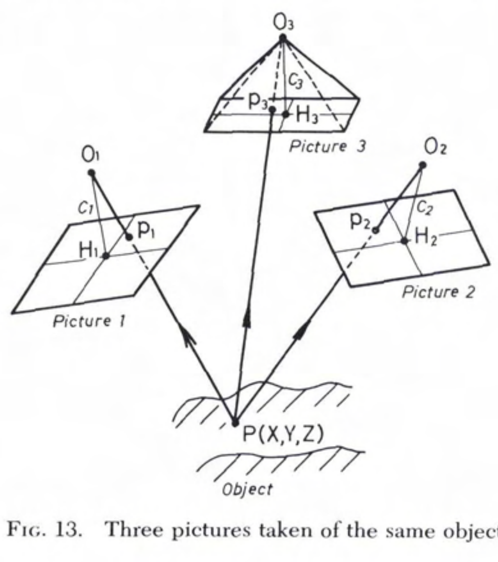

FIG. 13. Three pictures taken of the same object.

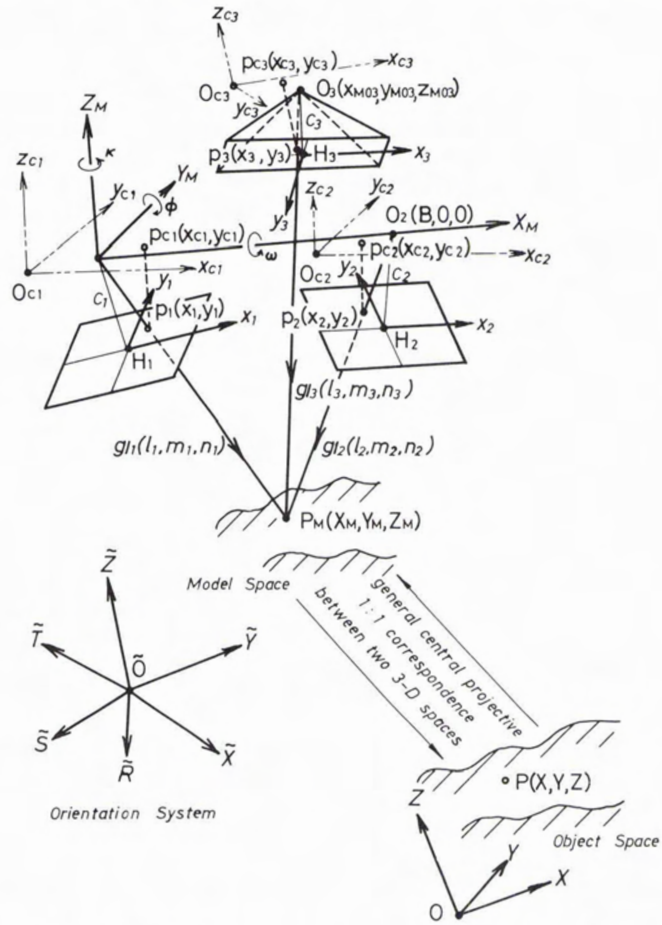

FIG. 14. Model construction with three pictures in the common model coordinate system.

$\beta_1,\phi_2,\omega_2,\kappa_2,x_{c02},y_{c02},c_2,\alpha_2,\beta_2)$. Also, the second stereo model is constructed based on the coplanarity condition of conjugate rays $\mathbf{g}_2$ and $\mathbf{g}_3$ in the form

$$\begin{vmatrix} X_{M03} - B & {}_{M2}\bar{X}_{p2} & {}_{M2}\bar{X}_{p3} \\ Y_{M03} & {}_{M2}\bar{Y}_{p2} & {}_{M2}\bar{Y}_{p3} \\ Z_{M03} & {}_{M2}\bar{Z}_{p2} & {}_{M2}\bar{Z}_{p3} \end{vmatrix} = 0 \tag{73}$$

where

$${}_{M2}\bar{X}_{p2} = {}_{M1}\bar{X}_{p2},\ {}_{M2}\bar{Y}_{p2} = {}_{M1}\bar{Y}_{p2},\ {}_{M2}\bar{Z}_{p2} = {}_{M1}\bar{Z}_{p2}.$$

The condition (Equation 73) has 19 orientation elements ($\phi_2,\omega_2,\kappa_2,x_{c02},y_{c02},c_2,\alpha_2,\beta_2,\phi_3,\omega_3,\kappa_3,X_{M03},Y_{M03},Z_{M03},$ $x_{c03},y_{c03},c_3,\alpha_3,\beta_3$).

The space coordinates of model points can be obtained by means of Equation (70) for the first stereo model, and those for the second one are given in the form:

$$\begin{aligned} X_{M2} &= B + \frac{{}_{M2}\bar{X}_{p2}({}_{M2}\bar{Y}_{p3}(X_{M03} - B) - {}_{M2}\bar{X}_{p3}Y_{M03})}{{}_{M2}\bar{X}_{p2}\,{}_{M2}\bar{Y}_{p3} - {}_{M2}\bar{Y}_{p2}\,{}_{M2}\bar{X}_{p3}} \\ Y_{M2} &= \frac{{}_{M2}\bar{Y}_{p2}({}_{M2}\bar{Y}_{p3}(X_{M03} - B) - {}_{M2}\bar{X}_{p3}Y_{M03})}{{}_{M2}\bar{X}_{p2}\,{}_{M2}\bar{Y}_{p3} - {}_{M2}\bar{Y}_{p2}\,{}_{M2}\bar{X}_{p3}} \\ Z_{M2} &= \frac{{}_{M2}\bar{Z}_{p2}({}_{M2}\bar{Y}_{p3}(X_{M03} - B) - {}_{M2}\bar{X}_{p3}Y_{M03})}{{}_{M2}\bar{X}_{p2}\,{}_{M2}\bar{Y}_{p3} - {}_{M2}\bar{Y}_{p2}\,{}_{M2}\bar{X}_{p3}} \end{aligned} \tag{74}$$

The condition to make the second stereo model coincide with the first one has, in principle, 15 independent orientation parameters to be determined. However, we must consider the 11 constraints which represent the special case for the standard case with four pictures. These 11 constraints must be introduced into the transformation process of the second stereo model space (X_{M2},Y_{M2},Z_{M2}) into the first one (X_{M1},Y_{M1},Z_{M1}). Thus, we have only four independent orientation parameters to be determined in this process.

By solving Equations (61), (72), and (73) with respect to 18 independent orientation unknowns among 26 elements included in these equations, we can construct a photogrammetric model with three pictures in the common model coordinate system (X_M,Y_M,Z_M). Also, the constructed model does not exist in the same three-dimensional space as the object one, because they are not similar. Furthermore, if five points are given in the object space, 15 independent orientation parameters are determined by the general central projective transformation of the model space (X_M,Y_M,Z_M) into the object space (X,Y,Z). These 15 parameters are also divided into seven exterior orientation elements and eight interior ones which remain undetermined in the process of the photogrammetric model construction with three pictures.

The discussion mentioned above revealed the following facts. The coplanarity condition of conjugate rays provides mathematically seven independent orientation parameters, and these seven elements are considered to be divided into five exterior orientation parameters (relative ones) and two interior orientation elements. Thus, ten exterior orientation parameters and four interior orientation elements are mathematically obtained during the phase of the first- and second stereo model construction. Further, the transformation of the second stereo model space into the first provides only four independent orientation parameters, which are considered to be divided into one exterior orientation element (a scale factor) and three interior ones. Therefore, the photogrammetric model construction with three pictures has the potential to provide seven interior orientation elements in addition to 11 exterior ones.

Model Construction Theory with Multiple Pictures for the Usual Case in Close-Range Photogrammetry

The Case of Four Photographs

A photogrammetric model can also be constructed with four photographs in the common model coordinate-system(X_M,Y_M,Z_M) for the usual case in close-range photogrammetry (see Figure 15). The first stereo model is constructed by means of the coplanarity condition (Equation 61) of conjugate rays $\mathbf{g}_1(l_1,m_1,n_1)$ and $\mathbf{g}_2(l_2,m_2,n_2)$. However, the reduced transformed picture coordinates are simplified to

$$\begin{bmatrix} {}_{M1}\bar{X}_{p1} \\ {}_{M1}\bar{Y}_{p1} \\ {}_{M1}\bar{Z}_{p1} \end{bmatrix} = \mathbf{D}_{\phi_1}\mathbf{D}_{\kappa_1}\begin{bmatrix} x_{c1} - x_{H1} \\ y_{c1} - y_{H1} \\ -c_1 \end{bmatrix} \quad (75)$$

for the first photograph, and

$$\begin{bmatrix} {}_{M1}\bar{X}_{p2} \\ {}_{M1}\bar{Y}_{p2} \\ {}_{M1}\bar{Z}_{p2} \end{bmatrix} = \mathbf{D}_{\phi_2}\mathbf{D}_{\omega_2}\mathbf{D}_{\kappa_2}\begin{bmatrix} x_{c2} - x_{H2} \\ y_{c2} - y_{H2} \\ -c_2 \end{bmatrix} \quad (76)$$

for the second one, respectively. In order to construct the second stereo model, the coplanarity condition (Equation 68) of conjugate rays $\mathbf{g}_3(l_3,m_3,n_3)$ and $\mathbf{g}_4(l_4,m_4,n_4)$ is employed. The reduced transformed picture coordinates in (Equation 68) yield in this case

$$\begin{bmatrix} {}_{M2}\bar{X}_{p3} \\ {}_{M2}\bar{Y}_{p3} \\ {}_{M2}\bar{Z}_{p3} \end{bmatrix} = \mathbf{D}_{\phi_3}\mathbf{D}_{\omega_3}\mathbf{D}_{\kappa_3}\begin{bmatrix} x_{c3} - y_{H3} \\ y_{c3} - y_{H3} \\ -c_3 \end{bmatrix} \quad (77)$$

for the third picture, and

$$\begin{bmatrix} {}_{M2}\bar{X}_{p4} \\ {}_{M2}\bar{Y}_{p4} \\ {}_{M2}\bar{Z}_{p4} \end{bmatrix} = \mathbf{D}_{\phi_4}\mathbf{D}_{\omega_4}\mathbf{D}_{\kappa_4}\begin{bmatrix} x_{c4} - x_{H4} \\ y_{c4} - y_{H4} \\ -c_4 \end{bmatrix} \quad (78)$$

for the fourth picture, respectively. On the other hand, the four-dimensional central projective one-to-one correspondence is mathematically valid between the first and second stereo models, and it can be uniquely determined with four corresponding points in both stereo model spaces. The condition that the four points in the first stereo model space must coincide with the four corresponding points in the second one, respectively, is also described by means of Equation 72.

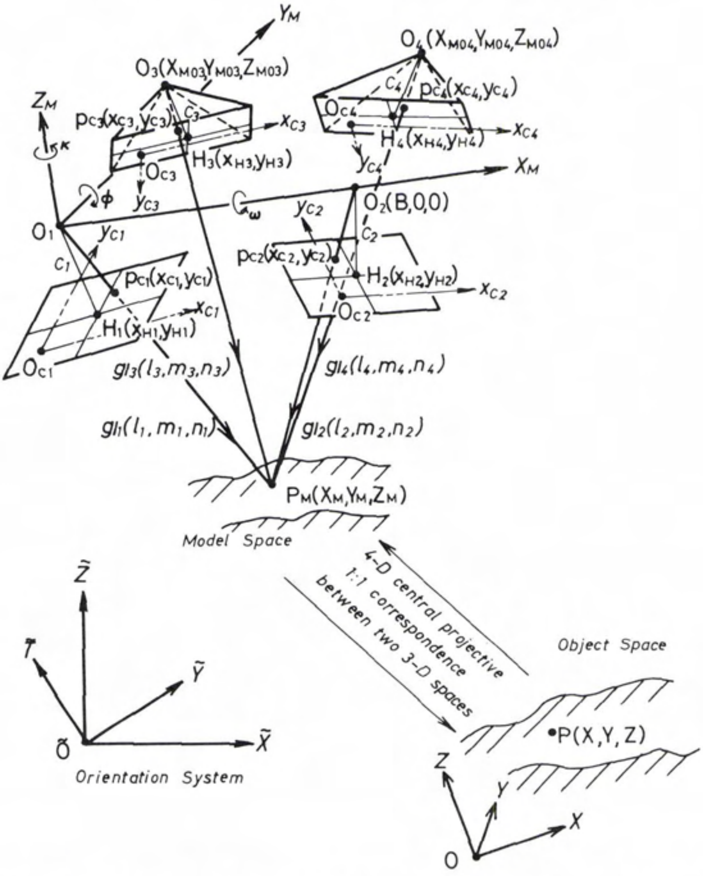

FIG. 15. Model construction with four photographs in the usual case in close-range photogrammetry.

By solving Equations 61, 68, and 72 with respect to 24 independent orientation unknowns among 29 parameters in these equations, we can construct a photogrammetric model with four pictures in the common model coordinate sysem (X_M,Y_M,Z_M). However, the constructed model does not exist in the same three-dimensional space as the object one, since the four-dimensional central projective one-to-one correspondence with 12 independent elements is satisfied between these two spaces. In other words, the photogrammetric model is not similar to the object. The parameters obtained by the transformation of the photogrammetric model space into the object one are considreed to be seven exterior orientation elements and five interior orientation elements which remain undetermined in the process of the photogrammetric model construction with four pictures.

The coplanarity condition of conjugate rays provides mathematically six independent orientation parameters in the usual case considered here. These six elements are one interior orientation parameter and five exterior ones (relative ones). Thus, ten exterior orientation elements and two interior ones are obtained during the phase of the first and second stereo model construction. On the other hand, 12 independent orientation parameters are mathematically determined by the four-dimensional central projective transformation of the second stereo model space into the first one. Also, these 12 parameters are divided into seven exterior orientation elements and five interior ones. It follows that the photogrammetric model construction with four pictures has the potential to provide seven interior orientation elements besides 17 exterior ones. The self-calibration of non-metric cameras employs this potential, though five interior orientation parameters can not be obtained among the 12 interior orientation parameters of four pictures. These five parameters must be known in order to get the desired values of the seven interior orientation elements to be determined in the self-calibration, because the photogrammetric model must be similar to the object for this end.

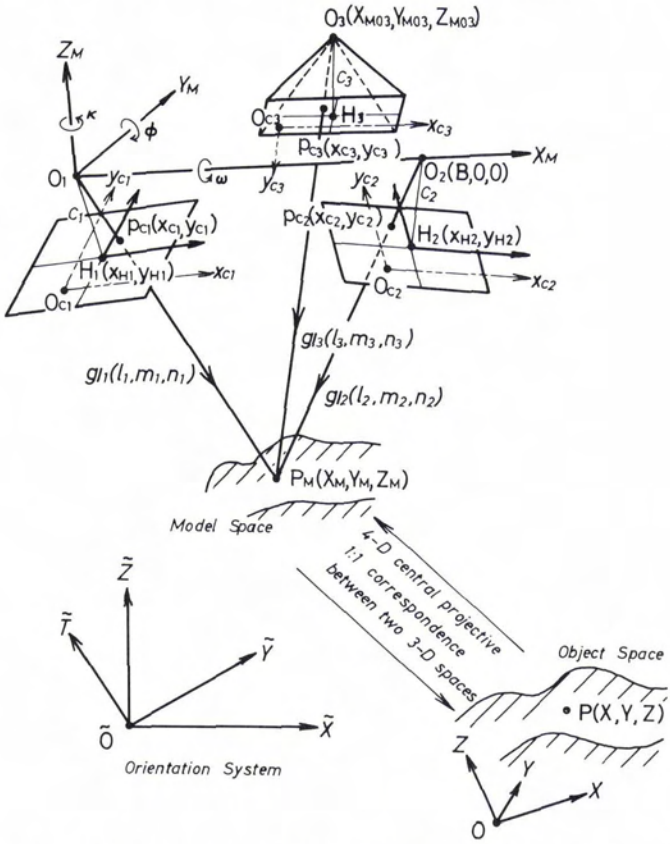

FIG. 16. Model construction with three photographs in the usual case in close-range photogrammetry.

THE CASE OF THREE PHOTOGRAPHS

The photogrammetric model construction with three pictures can be analyzed as the special case for the standard one with four pictures (see Figure 16). The constraints to be considered are that the second and third pictures in the standard case with four photographs have the same orientation parameters $(\phi,\omega,\kappa,X_0,Y_0,Z_0,x_H,y_H,c)$ in the common model coordinate system (X_M,Y_M,Z_M). Thus, the nine constraints must be introduced into the photogrammetric model construction process with three pictures.

The first stereo model is constructed with the first and second pictures based on the coplanarity condition (Equation 61) of conjugate rays $\mathbf{g}_1(l_1,m_1,n_1)$ and $\mathbf{g}_2(l_2,m_2,n_2)$. Also, it includes 11 orientation parameters $(\phi_1,\kappa_1,x_{H1},y_{H1},c_1,\phi_2,\omega_2,\kappa_2,x_{H2},y_{H2},c_2)$. The second stereo model is constructed from the second and third pictures by using the coplanarity condition (Equation 73) of conjugate rays $\mathbf{g}_2(l_2,m_2,n_2)$ and $\mathbf{g}_3(l_3,m_3,n_3)$. The reduced transformed picture coordinates in Equation 73 are described for the usual case with three photographs in the form

$$\begin{bmatrix} {}_{M2}\bar{X}_{p2} \\ {}_{M2}\bar{Y}_{p2} \\ {}_{M2}\bar{Z}_{p2} \end{bmatrix} = \mathbf{D}_{\phi_2}\mathbf{D}_{\omega_2}\mathbf{D}_{\kappa_2}\begin{bmatrix} x_{c2}-x_{H2} \\ y_{c2}-y_{H2} \\ -c_2 \end{bmatrix} \tag{79}$$

for the second picture, and

$$\begin{bmatrix} {}_{M2}\bar{X}_{p3} \\ {}_{M2}\bar{Y}_{p3} \\ {}_{M2}\bar{Z}_{p3} \end{bmatrix} = \mathbf{D}_{\phi_3}\mathbf{D}_{\omega_3}\mathbf{D}_{\kappa_3}\begin{bmatrix} x_{c3}-x_{H3} \\ y_{c3}-y_{H3} \\ -c_3 \end{bmatrix} \tag{80}$$

for the third one, respectively. Also, the coplanarity condition (Equation 73) in this case has 15 orientation elements $(\phi_2,\omega_2,\kappa_2,x_{H2},y_{H2},c_2,\phi_3,\omega_3,\kappa_3,X_{M03},Y_{M03},Z_{M03},x_{H3},y_{H3},c_3)$.

The space coordinates of model points are expressed by means of Equations 70 and 74 for the first and second stereo models, respectively. The condition that the second stereo model must coincide with the first one is also given by means of Equation 72, which has, in principle, 12 independent parameters

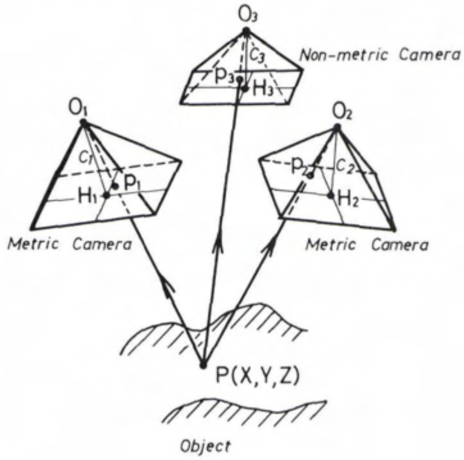

FIG. 17. A calibration method of non-metric cameras using metric cameras as controls.

in the usual case with four photographs. However, the nine constraints must be introduced into the photogrammetric model construction with three pictures. Accordingly, only three independent orientation elements can be obtained from the condition (Equation 72) in this case.

By solving Equations 61, 72, and 73 with respect to 15 independent orientation unknowns among 20 elements in these equations, a photogrammetric model can be constructed with three pictures in the common model coordinate system (X_M, Y_M, Z_M). Also, the constructed model is not similar to the object, because the four-dimensional central projective one-to-one correspondence is satisfied between the photogrammetric model space (X_M, Y_M, Z_M) and the object space (X, Y, Z). By this transformation, 12 independent orientation parameters are mathematically obtained, if four points are given in the object space. Also these 12 elements are considered to be divided into seven exterior orientation parameters (those in absolute orientation) and five interior orientation parameters which remain undetermined during the phase of the photogrammetric model construction with three pictures.

The above discussion revealed the following facts. The coplanarity condition of conjugate rays provides six independent orientation parameters, and these six elements are divided into one interior orientation parameter and five exterior orientation parameters (relative). Thus, 10 exterior orientation elements and two interior orientation elements can be mathematically obtained in the process of the first and second stereo model construction. On the other hand, the phase to make the second stereo model coincide with the first one provides mathematically only three independent orientation parameters, and these three elements can be divided into one exterior orientation parameter (a scale factor) and two interior orientation parameters. Consequently, the photogrammetric model construction with three pictures has the potential to determine four interior orientation elements in addition to 11 exterior orientation parameters. This potential is used for the self-calibration of non-metric cameras, though five interior orientation parameters can not be obtained among the nine interior orientation parameters of three pictures. Further, these five interior orientation parameters must be known for the purpose of camera calibration (Koelbl employed the coplanarity condition of conjugate rays $\mathbf{g}_1(l_1, m_1, n_1)$ and $\mathbf{g}_3(l_3, m_3, n_3)$ instead of the condition (Equation 72) in his self-calibration method).

PRACTICAL DISCUSSION FOR THE MODEL CONSTRUCTION PROBLEM WITH MULTIPLE PHOTOGRAPHS

Model construction with multiple photographs has the potential to provide some interior orientation parameters. This potential was first applied by Koelbl[1] to self-calibration of non-metric cameras, which did not require object space controls. However, in Koelbl's self-calibration method, the interior orientation must be assumed as unchanged between photographs, because the interior orientation parameters cannot be obtained for each picture separately from the model construction theory with multiple photographs. Whether this assumption can be reasonable must be further discussed. Faig[4] indicated that the interior stability of non-metric cameras might be rather weak. In order to calculate the interior orientation elements for each picture separately, positions or at least distances[2] must be used as controls in the object space.

The model construction theory with multiple photographs can be further applicable to many other photogrammetric problems such as camera calibration using metric cameras as controls and triplet model construction in aerial triangulation. The first example is described as follows (see Figure 17). It is considered that an object space was photographed with three different cameras, two of which were, however, metric cameras. The first stereo model is constructed with the first and second pictures taken with the metric cameras 1 and 2, respectively. In this process, five exterior orientation elements ($\phi_1,\kappa_1\phi_2,\omega_2,\kappa_2$) are determined from the coplanarity condition (Equation 61) of conjugate rays $\mathbf{g}_1$ and $\mathbf{g}_2$, because the interior orientation parameters (x_H,y_H,c) are given for metric cameras 1 and 2. The second stereo model is constructed with the second picture taken with metric camera 2 and the third picture taken with non-metric camera 3. For this purpose, the coplanarity condition (Equation 73) of conjugate rays $\mathbf{g}_2$ and $\mathbf{g}_3$ is employed that includes 12 unknown orientation parameters ($\phi_2,\omega_2,\kappa_2,\phi_3,\omega_3,\kappa_3,X_{M03},Y_{M03},Z_{M03},x_{H3},y_{H3},c_3$). Also, the coplanarity condition (Equation 73) provides six independent orientation elements. Further, the condition (Equation 72) to make the second stereo model coincide with the first one determines three independent orientation parameters.

The above discussion clarified that the photogrammetric model construction with three pictures has the potential to provide three interior orientation elements in this case, because one interior orientation element is considered to be given, which can be mathematically determined from the model construction theory with three photographs in the previous chapter. By solving Equations 61, 72, and 73 with respect to 14 independent orientation unknowns ($\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2,\phi_3,\omega_3,\kappa_3,X_{M03},Y_{M03},Z_{M03},x_{H3},y_{H3},c_3$), we can construct a photogrammetric model similar to the object in the common model coordinate system (X_M,Y_M,Z_M). This fact shows that calibration of a non-metric camera can be performed by using metric cameras as controls.

Concluding Remarks

The model construction problem with multiple photographs has been theoretically discussed in this paper. First, the general case was treated where a photograph has 11 independent parameters. The consideration was made geometrically by introducing the concept of multi-dimensional space having more than three dimensions, and various interesting characteristics of the model construction theory with multiple photographs were clarified. They are

- The case of four photographs is the standard one for analysis. The photogrammetric model construction with four pictures has the potential to provide 12 of the 20 interior orientation elements.
- The case of three photographs can be treated as the special case for the standard one with four pictures. Seven of the 15 interior orientation parameters can be obtained by the potential of three photographs.
- Without object space controls, eight interior orientation elements cannot be mathematically determined for any case.

Next, the photogrammetric model construction problem was also investigated for the usual case in close-range photogrammetry, where a picture has nine independent orientation parameters (six exterior and three interior). The geometrical approach to this problem revealed the following facts:

- The photogrammetric model construction with four pictures has the potential to determine seven of the 12 interior orientation elements.
- With three pictures, four of the nine interior orientation parameters can be obtained from the model construction theory.
- In any case, five interior orientation parameters cannot be mathematically provided, if positions or distances are not used in object space.

According to Hallert[3], non-central projective parameters such as lens distortion and film deformation can be determined by means of the coplanarity condition of conjugate rays. Thus, adding these non-central projective deformations to measured image coordinates, we can calibrate non-metric cameras based on the photogrammetric model construction theory with multiple photographs, although it is necessary to assume that some interior orientation elements are known. This technique is well-known as the self-calibration method of non-metric cameras. Also, the photogrammetric model construction theory with multiple photographs can be applied to a camera calibration method using metric cameras as controls.

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