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Orientation and Construction of Models

Part I: The Orientation Problem in Close-Range Photogrammetry*

The orientation problem of a stereoscopic pair of photographs taken with non-metric cameras is investigated both for the general case and for the usual case in close-range photogrammetry.

One of the most important items in close-range photogrammetry is the orientation problem of central perspective photographs taken with non-metric cameras, because their interior orientation elements are usually not given. Much research[1-3] has been carried out on the development of orientation techniques for individual photographs, including non-central projective parameters such as lens distortion (camera calibration method). Little, however, has been done in the way of investigating the orientation problem of a stereoscopic pair of photographs taken with non-metric cameras. Only in a few papers[4,5] has this problem been discussed, but not fully explained. Thus, it will be fundamentally investigated in this report.

In the actual calibration of non-metric cameras, lens distortion and film deformation must be considered in addition to the exterior and interior orientation parameters of the photograph. However, these elements have no mathematical relationship to central projective geometry and can be determined from other geometrical considerations[6]. Therefore, these non-central projective parameters are neglected in the discussion of this report.

Abstract: *The orientation problem of a stereoscopic pair of photographs taken with non-metric cameras is considered, first algebraically and then geometrically. Also, some orientation techniques are presented. The orientation theory given here is quite fundamental and is applicable to various photogrammetric problems such as the calibration problem of non-metric cameras and the orientation problem of stereo-strip imagery. In particular, one promising calibration method can be developed to provide interior orientation parameters for each of a stereoscopic pair of photographs, respectively, by using only distances in the object space as control. To perform the practical test of this calibration method and to find various applications of the proposed orientation theory is my next program.*

The general orientation problem of a stereoscopic pair of photographs is discussed, first algebraically and then geometrically. The concept of a multi-dimensional space having more than three dimensions is introduced into the geometrical approach so as to clarify the photogrammetric meaning of the orientation problem. Also, through geometrical consideration of the orientation problem in the multi-

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dimensional space, the orientation theory can be easily derived for the usual case in close-range photogrammetry, where three interior orientation parameters (three elements for the principal point) are unknown for each picture. Thus, some new theoretical characteristics of the orientation problem of a stereoscopic pair of pictures taken with non-metric cameras are clarified, and useful orientation techniques to calculate directly the traditional photogrammetric parameters are proposed.

Since the orientation theory given here is quite fundamental, it has various application possibilities to

- close-range photogrammetry,
- calibration of non-metric cameras,
- orientation of stereo-strip imageries, and
- independent model construction in aerial triangulation by means of a triplet method with unknown interior orientation.

Mathematical Preparations

Characteristics of General Collinearity Equations

The general collinearity equations relating a photographed object point $P(X,Y,Z)$ and the image point $p(x,y)$ are given in the form

$$x = \frac{A_1X + A_2Y + A_3Z + A_4}{A_9X + A_{10}Y + A_{11}Z + 1}$$
$$y = \frac{A_5X + A_6Y + A_7Z + A_8}{A_9X + A_{10}Y + A_{11}Z + 1} \quad (1)$$

where $A_i (i = 1, \ldots, 11)$ denote independent coefficients. However, this relationship cannot be central-perspectively considered in a three-dimensional space, because a camera has only nine independent parameters (six exterior and three interior orientation elements). In order to grasp the geometrical meaning of Equation 1 in a central perspective way, the concept of a multi-dimensional space having more than three dimensions must be introduced.

Equation 1 can be considered geometrically in still another way based on a central projection and an orthogonal projection in a three-dimensional space. This concept was first found by Abdel-Aziz and Karara[7], and will be discussed as follows. First, the $x_c - y_c$ plane of the comparator coordinate system (x_c,y_c,z_c) is assumed not to be parallel to the picture plane (see Figure 1). The object-space coordinate system (X,Y,Z) is taken as a right-handed, rectangular Cartesian system with its origin at an arbitrary point in the object space. The picture coordinate system (x,y), which is necessary only in the intermediate derivation process, is selected with its origin at the principal point, H, on the picture plane. The projection center, O_A, of the photograph is expressed as X_o,Y_o,Z_o in the object-space coordinate system and as x_{co},y_{co},z_{co} in the comparator coordinate system. Further, c denotes the principal distance. The

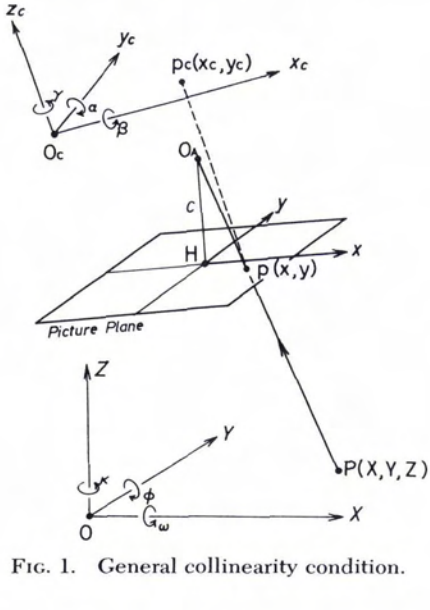

Fig. 1. General collinearity condition.

central projective relationship between an object point $P(X,Y,Z)$ and its image point $p(x,y)$ is described in the form

$$\begin{bmatrix} x \\ y \\ -c \end{bmatrix} = \lambda(\mathbf{D}_\phi \mathbf{D}_\omega \mathbf{D}_\kappa)^t \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix} \tag{2}$$

where $\mathbf{D}_\phi$, $\mathbf{D}_\omega$, and $\mathbf{D}_\kappa$ are rotation matrices of rotation elements ϕ, ω, and κ, respectively, and λ denotes a scale factor. Eliminating λ from Equation 2, we get the conventional collinearity equations

$$\begin{aligned} x &= -c\,\frac{e_{11}(X - X_o) + e_{12}(Y - Y_o) + e_{13}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)} \\ y &= -c\,\frac{e_{21}(X - X_o) + e_{22}(Y - Y_o) + e_{23}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)} \end{aligned} \tag{3}$$

$$(\mathbf{D}_\phi \mathbf{D}_\omega \mathbf{D}_\kappa)^t = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

Equation 3 can also be expressed in a matrix form, i.e.,

$$\begin{bmatrix} x \\ y \\ -c \end{bmatrix} = -c \begin{bmatrix} \dfrac{e_{11}(X - X_o) + e_{12}(Y - Y_o) + e_{13}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)} \\ \dfrac{e_{21}(X - X_o) + e_{22}(Y - Y_o) + e_{23}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)} \\ 1 \end{bmatrix}. \tag{3a}$$

The relationship between a picture point $p(x,y)$ on the picture plane and its measured image point $p_c(x_c,y_c)$ is described as

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \mathbf{D}_\alpha \mathbf{D}_\beta \mathbf{D}_\gamma \begin{bmatrix} x \\ y \\ -c \end{bmatrix} + \begin{bmatrix} x_{co} \\ y_{co} \\ z_{co} \end{bmatrix} \tag{4}$$

where $\mathbf{D}_\alpha$, $\mathbf{D}_\beta$, and $\mathbf{D}_\gamma$ are rotation matrices of rotation parameters α, β, and γ about the comparator coordinate axes y_c, x_c, and z_c, respectively. The coordinate z_c is an unnecessary value for the derivation. From the first and second equations of Equation 4 we get

$$\begin{aligned} x_c - x_{co} &= d_{11}x + d_{12}y - d_{13}c \\ y_c - y_{co} &= d_{21}x + d_{22}y - d_{23}c \end{aligned} \tag{5}$$

where

$$\mathbf{D}_\alpha \mathbf{D}_\beta \mathbf{D}_\gamma = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}.$$

Equation 5 means that the measured image coordinates (x_c,y_c) can be obtained by an orthogonal transformation of a picture point $p(x,y)$ into the $x_c - y_c$ plane of the comparator coordinate system (x_c,y_c,z_c). Further, Equation 5 can also be given in the following matrix form:

$$\begin{bmatrix} x_c - x_{co} \\ y_c - y_{co} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} \begin{bmatrix} x \\ y \\ -c \end{bmatrix}. \tag{5a}$$

By substituting Equation 3a into 5a, we obtain

$$\begin{bmatrix} x_c - x_{co} \\ y_c - y_{co} \end{bmatrix} = -c \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} \begin{bmatrix} \dfrac{e_{11}(X - X_o) + e_{12}(Y - Y_o) + e_{13}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)} \\ \dfrac{e_{21}(X - X_o) + e_{22}(Y - Y_o) + e_{23}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)} \\ 1 \end{bmatrix} \tag{6}$$

which can be reduced to the next form i.e.,

$$x_c = \frac{A_1X + A_2Y + A_3Z + A_4}{A_9X + A_{10}Y + A_{11}Z + 1}$$
$$y_c = \frac{A_5X + A_6Y + A_7Z + A_8}{A_9X + A_{10}Y + A_{11}Z + 1} \quad (7)$$

Because Equation 6 has 12 parameters $(\phi,\omega,\kappa,X_o,Y_o,Z_o,\alpha,\beta,\gamma,x_{co},y_{co},c)$, all coefficients $A_i(i = 1, \ldots, 11)$ in Equation 7 are independent. Thus, Equation 7 is identical to the general collinearity equations (Equation 1). Consequently, it can be understood that the relationship between an object point $P(X,Y,Z)$ and its measured image point $p_c(x_c,y_c)$ has the general central projective characteristics. In the following discussions, we will use Equation 7 as the general collinearity equations instead of Equation 1.

Because the number of independent coefficients in Equation 7 is 11, there is one dependent parameter among 12 photogrammetric orientation elements $(\phi,\omega,\kappa,X_o,Y_o,Z_o,\alpha,\beta,\gamma,x_{co},y_{co},c)$ in Equation 6. It follows that we can select one of the 12 parameters in Equation 6 arbitrarily. Therefore, let us assume that γ is equal to zero. Omitting γ from Equation 6, we have the general collinearity equations of a photograph with respect to the 11 general photogrammetric orientation parameters $(\phi,\omega,\kappa,X_o,Y_o,Z_o,\alpha,\beta,x_{co},y_{co},c)$.

CHARACTERISTICS OF GENERAL CENTRAL PROJECTIVE ONE-TO-ONE CORRESPONDENCE BETWEEN TWO THREE-DIMENSIONAL SPACES

The general central projective one-to-one correspondence is expressed between two three-dimensional spaces (X_1,Y_1,Z_1) and (X_2,Y_2,Z_2) in the form[8]

$$X_1 = \frac{B_1X_2 + B_2Y_2 + B_3Z_2 + B_4}{B_{13}X_2 + B_{14}Y_2 + B_{15}Z_2 + 1}$$
$$Y_1 = \frac{B_5X_2 + B_6Y_2 + B_7Z_2 + B_8}{B_{13}X_2 + B_{14}Y_2 + B_{15}Z_2 + 1} \quad (8)$$
$$Z_1 = \frac{B_9X_2 + B_{10}Y_2 + B_{11}Z_2 + B_{12}}{B_{13}X_2 + B_{14}Y_2 + B_{15}Z_2 + 1}$$

in which the coefficients $B_i(i = 1, \ldots, 15)$ are all independent. In a geometrical sence, Equation 8 can also be considered to express the general central projective relationship between two three-dimensional spaces in a multi-dimensional space having more than four dimensions. However, we will consider the general central projective one-to-one correspondence (Equation 8) between two three-dimensional spaces according to the concept based on a central projection and an orthogonal projection, as in the previous paragraph. This procedure will be described in a four-dimensional space as follows (see Figure 2). The object-space coordinate system (X,Y,Z,T), the fictitious three-dimensional picture coordinate system (x,y,z), and the fictitious comparator coordinate system (x_c,y_c,z_c,t_c) are selected as is demonstrated in Figure 2. The projection center O_A of a fictitious three-dimensional photograph is expressed as $(x_{co},y_{co},z_{co},t_{co})$ in the comparator coordinate system and as (X_o,Y_o,Z_o,T_o) in the object-space coordinate system. Further, c denotes the principal distance of the fictitious photograph. Also, the object space is assumed to be a three-dimensional plane (X,Y,Z).

The central projective relationship between an object point $P(X,Y,Z,0)$ and its three-dimensional image point $p(x,y,z)$ is given in the form

$$\begin{bmatrix} x \\ y \\ z \\ -c \end{bmatrix} = \lambda(\mathbf{D}_\phi\mathbf{D}_\omega\mathbf{D}_\kappa\mathbf{D}_\mu)^t \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \\ 0 - T_o \end{bmatrix} \quad (9)$$

in which $\mathbf{D}_\phi$, $\mathbf{D}_\omega$, $\mathbf{D}_\kappa$, and $\mathbf{D}_\mu$ are rotation matrices of rotation elements ϕ, ω, κ, and μ about the object-space coordinate axes Y, X, Z, and T, respectively, and λ denotes a scale factor. Eliminating λ from Equation 9, we have

$$x = -c\frac{e_{11}(X - X_o) + e_{12}(Y - Y_o) + e_{13}(Z - Z_o) + e_{14}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)}$$
$$y = -c\frac{e_{21}(X - X_o) + e_{22}(Y - Y_o) + e_{23}(Z - Z_o) + e_{24}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)} \quad (10)$$
$$z = -c\frac{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o) + e_{34}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)}$$

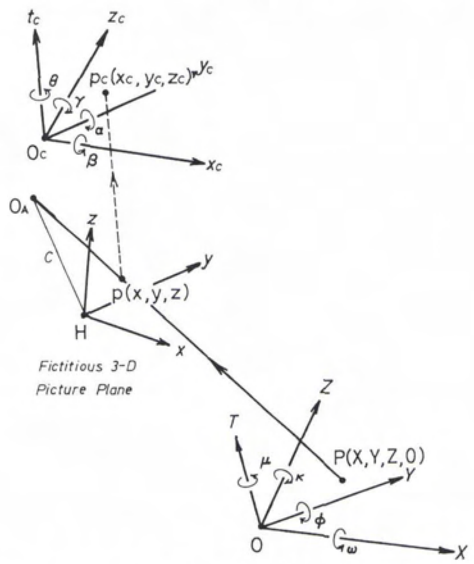

FIG. 2. Four-dimensional transformations.

where

$$(\mathbf{D}_\phi \mathbf{D}_\omega \mathbf{D}_\kappa \mathbf{D}_\mu)^t = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}.$$

Equation 10 is also expressed in matrix form as

$$\begin{bmatrix} x \\ y \\ z \\ -c \end{bmatrix} = -c \begin{bmatrix} \xi \\ \eta \\ \zeta \\ 1 \end{bmatrix} \tag{11}$$

where

$$\xi = \frac{e_{11}(X - X_o) + e_{12}(Y - Y_o) + e_{13}(Z - Z_o) + e_{14}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)}$$

$$\eta = \frac{e_{21}(X - X_o) + e_{22}(Y - Y_o) + e_{23}(Z - Z_o) + e_{24}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)}$$

$$\zeta = \frac{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o) + e_{34}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)}.$$

On the other hand, the relationship between a picture point $p(x,y,z)$ and its measured image point $p_c(x_c,y_c,z_c)$ is described as

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ t_c \end{bmatrix} = \mathbf{D}_\alpha \mathbf{D}_\beta \mathbf{D}_\gamma \mathbf{D}_\theta \begin{bmatrix} x \\ y \\ z \\ -c \end{bmatrix} + \begin{bmatrix} x_{co} \\ y_{co} \\ z_{co} \\ t_{co} \end{bmatrix} \tag{12}$$

Where $\mathbf{D}_\alpha$, $\mathbf{D}_\beta$, $\mathbf{D}_\gamma$, $\mathbf{D}_\theta$ are rotation matrices of rotation elements α, β, γ, and θ about the fictitious comparator coordinate axes y_c, x_c, z_c, and t_c, respectively. The first, second, and third equations of Equation 12 are described in the form

$$\begin{aligned} x_c - x_{co} &= d_{11}x + d_{12}y + d_{13}z - d_{14}c \\ y_c - y_{co} &= d_{21}x + d_{22}y + d_{23}z - d_{24}c \\ z_c - z_{co} &= d_{31}x + d_{32}y + d_{33}z - d_{34}c \end{aligned} \tag{13}$$

in which

$$\mathbf{D}_\alpha \mathbf{D}_\beta \mathbf{D}_\gamma \mathbf{D}_\theta = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix}.$$

Equation 13 indicates that the measured image coordinates (x_c,y_c,z_c) can be obtained by an orthogonal transformation of the picture point $p(x,y,z)$ into the three-dimensional plane (x_c,y_c,z_c) of the fictitious comparator coordinate system (x_c,y_c,z_c,t_c). Also, Equation 13 can be expressed in a matrix form as

$$\begin{bmatrix} x_c - x_{co} \\ y_c - y_{co} \\ z_c - z_{co} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ -c \end{bmatrix}. \tag{14}$$

By substituting Equation 11 into Equation 14, we have

$$\begin{bmatrix} x_c - x_{co} \\ y_c - y_{co} \\ z_c - z_{co} \end{bmatrix} = -c \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \\ 1 \end{bmatrix} \tag{14a}$$

which reduces, simply, to

$$\begin{aligned} x_c &= \frac{A_1X + A_2Y + A_3Z + A_4}{A_{13}X + A_{14}Y + A_{15}Z + 1} \\ y_c &= \frac{A_5X + A_6Y + A_7Z + A_8}{A_{13}X + A_{14}Y + A_{15}Z + 1} \\ z_c &= \frac{A_9X + A_{10}Y + A_{11}Z + A_{12}}{A_{13}X + A_{14}Y + A_{15}Z + 1}. \end{aligned} \tag{15}$$

Because Equation 14a has 16 parameters $(\phi,\omega,\kappa,\mu,X_o,Y_o,Z_o,T_o,x_{co},y_{co},z_{co},c,\alpha,\beta,\gamma,\theta)$, all coefficients $A_i(i = 1, \ldots, 15)$ are independent. This means that the relationship (Equation 15) between an object point $P(X,Y,Z)$ and its measured image point $p_c(x_c,y_c,z_c)$ is identical to the general central projective one-to-one correspondence (Equation 8) between two three-dimensional spaces. It follows that the general central projective one-to-one correspondence between two three-dimensional spaces can also be composed of two transformations in a four-dimensional space: central projective transformation and orthogonal transformation.

General Orientation Problem of a Stereoscopic Pair of Photographs

ALGEBRAIC CONSIDERATION

The fundamental equations for the general orientation problem of central perspective photographs are the general collinearity equations (Equation 7) between a photographed object point $P(X,Y,Z)$ and its measured image point $p_c(x_c,y_c)$. Since Equation 7 cannot be central-projectively inverse-transformed, it is necessary to employ a stereoscopic pair of photographs for the unique determination of an object point from the measured image coordinates, if the object space is three-dimensional. The orientation problem for a stereoscopic pair of photographs will be discussed by introducing an algebraic approach as follows:

The general collinearity equations are described in the form

$$\begin{aligned} x_{c1} &= \frac{{}_1A_1X + {}_1A_2Y + {}_1A_3Z + {}_1A_4}{{}_1A_9X + {}_1A_{10}Y + {}_1A_{11}Z + 1} \\ y_{c1} &= \frac{{}_1A_5X + {}_1A_6Y + {}_1A_7Z + {}_1A_8}{{}_1A_9X + {}_1A_{10}Y + {}_1A_{11}Z + 1} \end{aligned} \tag{16}$$

for the left photograph, and

$$\begin{aligned} x_{c2} &= \frac{{}_2A_1X + {}_2A_2Y + {}_2A_3Z + {}_2A_4}{{}_2A_9X + {}_2A_{10}Y + {}_2A_{11}Z + 1} \\ y_{c2} &= \frac{{}_2A_5X + {}_2A_6Y + {}_2A_7Z + {}_2A_8}{{}_2A_9X + {}_2A_{10}Y + {}_2A_{11}Z + 1} \end{aligned} \tag{17}$$

for the right one, respectively. Equations 16 and 17 can also be rewritten in linear form with respect to the object space coordinates (X,Y,Z) as

$$\begin{aligned}
&(x_{c1}\cdot{}_1A_9 - {}_1A_1)X + (x_{c1}\cdot{}_1A_{10} - {}_1A_2)Y + (x_{c1}\cdot{}_1A_{11} - {}_1A_3)Z + (x_{c1} - {}_1A_4) = 0\\
&(y_{c1}\cdot{}_1A_9 - {}_1A_5)X + (y_{c1}\cdot{}_1A_{10} - {}_1A_6)Y + (y_{c1}\cdot{}_1A_{11} - {}_1A_7)Z + (y_{c1} - {}_1A_8) = 0\\
&(x_{c2}\cdot{}_2A_9 - {}_2A_1)X + (x_{c2}\cdot{}_2A_{10} - {}_2A_2)Y + (x_{c2}\cdot{}_2A_{11} - {}_2A_3)Z + (x_{c2} - {}_2A_4) = 0\\
&(y_{c2}\cdot{}_2A_9 - {}_2A_5)X + (y_{c2}\cdot{}_2A_{10} - {}_2A_6)Y + (y_{c2}\cdot{}_2A_{11} - {}_2A_7)Z + (y_{c2} - {}_2A_8) = 0\ .
\end{aligned} \tag{18}$$

The condition that Equation 18 is satisfied for an arbitrary object point $P(X,Y,Z)$ can be given in the following determinant form

$$\begin{vmatrix}
x_{c1}\cdot{}_1A_9 - {}_1A_1 & x_{c1}\cdot{}_1A_{10} - {}_1A_2 & x_{c1}\cdot{}_1A_{11} - {}_1A_3 & x_{c1} - {}_1A_4\\
y_{c1}\cdot{}_1A_9 - {}_1A_5 & y_{c1}\cdot{}_1A_{10} - {}_1A_6 & y_{c1}\cdot{}_1A_{11} - {}_1A_7 & y_{c1} - {}_1A_8\\
x_{c2}\cdot{}_2A_9 - {}_2A_1 & x_{c2}\cdot{}_2A_{10} - {}_2A_2 & x_{c2}\cdot{}_2A_{11} - {}_2A_3 & x_{c2} - {}_2A_4\\
y_{c2}\cdot{}_2A_9 - {}_2A_5 & y_{c2}\cdot{}_2A_{10} - {}_2A_6 & y_{c2}\cdot{}_2A_{11} - {}_2A_7 & y_{c2} - {}_2A_8
\end{vmatrix} = 0. \tag{19}$$

On the other hand, the space coordinates (X,Y,Z) of an object point can be uniquely determined by means of the first, second, and third equations of Equation 18, if the condition (Equation 19) is valid for all photographed object points. Thus, (x_{c1},y_{c1},x_{c2}) (the x_c and y_c coordinates of a left measured image point and the x_c or y_c coordinate of the corresponding right one) and (X,Y,Z) (the space coordinates of the object point) must be central-projectively transformed from each other under the condition of Equation 19. It means that the following three equations with respect to (X,Y,Z), i.e.,

$$\begin{aligned}
x_{c1} &= \frac{{}_1A_1X + {}_1A_2Y + {}_1A_3Z + {}_1A_4}{{}_1A_9X + {}_1A_{10}Y + {}_1A_{11}Z + 1}\\
y_{c1} &= \frac{{}_1A_5X + {}_1A_6Y + {}_1A_7Z + {}_1A_8}{{}_1A_9X + {}_1A_{10}Y + {}_1A_{11}Z + 1}\\
x_{c2} &= \frac{{}_2A_1X + {}_2A_2Y + {}_2A_3Z + {}_2A_4}{{}_2A_9X + {}_2A_{10}Y + {}_2A_{11}Z + 1}
\end{aligned} \tag{20}$$

must have the solution in the form

$$\begin{aligned}
X &= \frac{B_1x_{c1} + B_2y_{c1} + B_3x_{c2} + B_4}{B_{13}x_{c1} + B_{14}y_{c1} + B_{15}x_{c2} + 1}\\
Y &= \frac{B_5x_{c1} + B_6y_{c1} + B_7x_{c2} + B_8}{B_{13}x_{c1} + B_{14}y_{c1} + B_{15}x_{c2} + 1}\\
Z &= \frac{B_9x_{c1} + B_{10}y_{c1} + B_{11}x_{c2} + B_{12}}{B_{13}x_{c1} + B_{14}y_{c1} + B_{15}x_{c2} + 1}
\end{aligned} \tag{21}$$

in which the coefficients $B_i(i = 1, \ldots, 15)$ are all independent. Accordingly, Equation 20 can also be rewritten as

$$\begin{aligned}
x_{c1} &= \frac{B_1'X + B_2'Y + B_3'Z + B_4'}{B_{13}'X + B_{14}'Y + B_{15}'Z + 1}\\
y_{c1} &= \frac{B_5'X + B_6'Y + B_7'Z + B_8'}{B_{13}'X + B_{14}'Y + B_{15}'Z + 1}\\
x_{c2} &= \frac{B_9'X + B_{10}'Y + B_{11}'Z + B_{12}'}{B_{13}'X + B_{14}'Y + B_{15}'Z + 1}\ ,
\end{aligned} \tag{22}$$

if the condition (Equation 19) is satisfied for an arbitrary object point.

The discussion mentioned above revealed the following facts: The 15 independent parameters can be mathematically determined by means of Equation 21, if five points are given in the object space. Then, seven independent elements must be mathematically obtained from the condition (Equation 19), since 22 independent parameters must become known for the unique determination of all photographed object points from their measured image coordinates.

In the general photogrammetric orientation problem of a stereoscopic pair of photographs, Equation 19 corresponds to the condition for model construction (the coplanarity condition of corresponding rays). Also, the stereo model is constructed in a three-dimensional space. On the other hand, Equation 21 or 22 is equivalent to the general central projective one-to-one correspondence between the model space (X_M,Y_M,Z_M) and the object space (X,Y,Z), i.e.,

$$X_M = \frac{E_1X + E_2Y + E_3Z + E_4}{E_{13}X + E_{14}Y + E_{15}Z + 1}$$

$$Y_M = \frac{E_5X + E_6Y + E_7Z + E_8}{E_{13}X + E_{14}Y + E_{15}Z + 1} \tag{23}$$

$$Z_M = \frac{E_9X + E_{10}Y + E_{11}Z + E_{12}}{E_{13}X + E_{14}Y + E_{15}Z + 1} ,$$

or inversely in the form

$$X = \frac{E'_1X_M + E'_2Y_M + E'_3Z_M + E'_4}{E'_{13}X_M + E'_{14}Y_M + E'_{15}Z_M + 1}$$

$$Y = \frac{E'_5X_M + E'_6Y_M + E'_7Z_M + E'_8}{E'_{13}X_M + E'_{14}Y_M + E'_{15}Z_M + 1} \tag{24}$$

$$Z = \frac{E'_9X_M + E'_{10}Y_M + E'_{11}Z_M + E'_{12}}{E'_{13}X_M + E'_{14}Y_M + E'_{15}Z_M + 1} .$$

The condition (Equation 19) and the general central projective one-to-one correspondence (Equation 23 or 24) between the model space (X_M,Y_M,Z_M) and the object space (X,Y,Z) are fundamentally employed in order to perform the general orientation of a stereoscopic pair of pictures in the algebraic way. However, these two types of equations are very inconvenient for this purpose, because seven independent parameters in Equation 19 are difficult to define. Thus, we will take another approach to calculate directly the 22 independent coefficients ${}_iA_j(i = 1,2; j = 1, \ldots, 11)$ in Equation 16 and 17. Mathematically, the condition (Equation 19) can be set up for seven corresponding left and right measured image points, and Equation 20 for five given object points. We will then get 22 independent equations for the determination of the 22 independent coefficients ${}_iA_j(i = 1,2; j = 1, \ldots, 11)$. By solving these 22 equations with respect to the 22 unknown coefficients simultaneously, we can obtain the 22 parameters necessary for the unique determination of all photographed object points.

GEOMETRICAL CONSIDERATION

We can also consider the general orientation problem of a stereoscopic pair of photographs geometrically, using the fact that the relationship (Equation 7) between an object point $P(X,Y,Z)$ and its measured image point $p_c(x_c,y_c)$ has the general central projective property in the case where the $x_c - y_c$ plane of the comparator coordinate system (x_c,y_c,z_c) is not parallel to the picture plane. This orientation procedure will be precisely outlined as follows (see Figure 3). The general photogrammetric orientation parameters are defined as $\phi_1, \omega_1, \kappa_1, X_{o1}, Y_{o1}, Z_{o1}, x_{co1}, y_{co1}, c_1, \alpha_1$, and β_1 for the left picture and as $\phi_2, \omega_2, \kappa_2, X_{o2}, Y_{o2}, Z_{o2}, x_{co2}, y_{co2}, c_2, \alpha_2$, and β_2 for the right one, respectively. First, we will discuss about the stereo model construction in a three-dimensional space. The experior orientation elements (relative ones in conventional photogrammetry) are assumed to be five rotation parameters $(\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2)$. Corresponding rays $\mathbf{g}_1(l_1,m_1,n_1)$ and $\mathbf{g}_2(l_2,m_2,n_2)$ are expressed in the model coordinate system (X_M,Y_M,Z_M) as

$$\mathbf{g}_1\text{:}\ \frac{X_M}{l_1} = \frac{Y_M}{m_1} = \frac{Z_M}{n_1} = \rho_1 \tag{25}$$

$$\mathbf{g}_2\text{:}\ \frac{X_M - B}{l_2} = \frac{Y_M}{m_2} = \frac{Z_M}{n_2} = \rho_2 \tag{26}$$

where (l_1,m_1,n_1) and (l_2,m_2,n_2) denote direction cosines of the corresponding rays $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively, and B is the model base. The direction cosines will be derived in the following way. We will, at first, find space coordinates $({}_MX_p,{}_MY_p,{}_MZ_p)$ of a picture point $p(x,y)$ in the model coordinate system (X_M,Y_M,Z_M), which are termed the transformed picture coordinates. They are given in the form

$$\begin{bmatrix} {}_MX_{p1} \\ {}_MY_{p1} \\ {}_MZ_{p1} \end{bmatrix} = \mathbf{D}_{\phi_1}\mathbf{D}_{\kappa_1} \begin{bmatrix} x_1 \\ y_1 \\ -c_1 \end{bmatrix} \tag{27}$$

for the left picture, and

$$\begin{bmatrix} {}_MX_{p2} \\ {}_MY_{p2} \\ {}_MZ_{p2} \end{bmatrix} = \mathbf{D}_{\phi_2}\mathbf{D}_{\omega_2}\mathbf{D}_{\kappa_2} \begin{bmatrix} x_2 \\ y_2 \\ -c_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

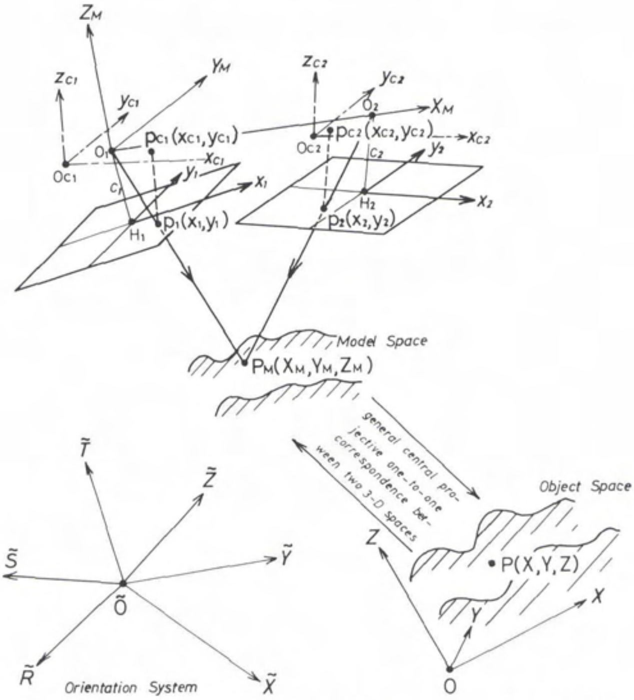

FIG. 3. General orientation problem of a stereoscopic pair of photographs.

for the right one, respectively. Further, the next expression will be adopted, i.e.,

$$\begin{bmatrix} {}_M\bar{X}_p \\ {}_M\bar{Y}_p \\ {}_M\bar{Z}_p \end{bmatrix} = \mathbf{D}_\phi \mathbf{D}_\omega \mathbf{D}_\kappa \begin{bmatrix} x \\ y \\ -c \end{bmatrix} \tag{29}$$

which denotes the reduced transformed picture coordinates. The direction cosines can be described by means of Equation 29 in the form

$$l = {}_M\bar{X}_p/A,\ m = {}_M\bar{Y}_p/A,\ n = {}_M\bar{Z}_p/A \tag{30}$$

where

$$A = \sqrt{{}_M\bar{X}_p^2 + {}_M\bar{Y}_p^2 + {}_M\bar{Z}_p^2}.$$

The relationship between a picture point $p(x,y)$ and its measured image point $p_c(x_c,y_c)$ is obtained from Equation 5 as

$$\begin{aligned} x_c - x_{co} + d_{13}c &= d_{11}x + d_{12}y \\ y_c - y_{co} + d_{23}c &= d_{21}x + d_{22}y\ . \end{aligned} \tag{31}$$

Equation 31 can also be expressed in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1} \begin{bmatrix} x_c - x_{co} + d_{13}c \\ y_c - y_{co} + d_{23}c \end{bmatrix} \tag{32}$$

where d_{ij} is a matrix element with respect to α and β about the comparator coordinate axes y_c and x_c. By substituting Equation 32 into Equation 29, we get the final expression for the reduced transformed picture coordinates, i.e.,

$$\begin{bmatrix} {}_M\bar{X}_p \\ {}_M\bar{Y}_p \\ {}_M\bar{Z}_p \end{bmatrix} = \mathbf{D}_\phi \mathbf{D}_\omega \mathbf{D}_\kappa \begin{bmatrix} h_{11}(x_c - x_{co} + d_{13}c) + h_{12}(y_c - y_{co} + d_{23}c) \\ h_{21}(x_c - x_{co} + d_{13}c) + h_{22}(y_c - y_{co} + d_{23}c) \\ -c \end{bmatrix} \tag{33}$$

in which

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}^{-1}$$

The coplanarity condition of corresponding rays $\mathbf{g}_1$ and $\mathbf{g}_2$ can be derived from Equations 25 and 26 in the form:

$$\begin{vmatrix} B & l_1 & l_2 \\ 0 & m_1 & m_2 \\ 0 & n_1 & n_2 \end{vmatrix} = 0. \tag{34}$$

The condition (Equation 34) reduces, simply, to

$$\begin{vmatrix} {}_M\bar{Y}_{p1} & {}_M\bar{Y}_{p2} \\ {}_M\bar{Z}_{p1} & {}_M\bar{Z}_{p2} \end{vmatrix} = 0. \tag{35}$$

From the coplanarity condition (Equation 35) of corresponding rays, we can determine mathematically seven independent orientation parameters among 15 photogrammetric orientation elements ($\phi_1,\kappa_1,x_{co1},y_{co1},c_1,\alpha_1,\beta_1,\phi_2,\omega_2,\kappa_2,x_{co2},y_{co2},c_2,\alpha_2,\beta_2$) included in Equation 35. Then, a stereo model is constructed in a three-dimensional space.

On the other hand, the general central projective one-to-one correspondence (Equation 23) must be valid between the model space (X_M,Y_M,Z_M) and the object space (X,Y,Z). Also, 15 independent orientation parameters can be mathematically obtained from the one-to-one correspondence (Equation 23), if five points are given in the object space. This means geometrically that the constructed stereo model and the object exist in different three-dimensional spaces in the multi-dimensional space having more than three dimensions, because the three-dimensional similarity transformation (the three-dimensional central projective transformation) must be satisfied between the model and object, if they lie in the same three-dimensional space.

The coplanarity condition (Equation 35) of corresponding rays and the general central projective one-to-one correspondence (Equation 23) between the model and object spaces have very inconvenient forms for calculating directly the 22 traditional photogrammetric orientation parameters of a stereoscopic pair of photographs, first because seven independent photogrammetric orientation elements for the model construction must be defined, and second because it is difficult to express the general central projective one-to-one correspondence (Equation 23) in terms of 15 independent photogrammetric orientation elements. Therefore, we will take another approach to construct a stereo model in the same three-dimensional space as the object one, as is usually done in conventional photogrammetry. Also, for this end the second condition for the model construction must be introduced, which can be derived from the object space information. This orientation procedure will be described as follows (see Figure 4).

We know in conventional photogrammetry that the model is similar to the object, i.e., if they exist in the same three-dimensionl space. As the second condition for the model construction we use this similarity condition, because it is also valid between two three-dimensional spaces in the multi-dimensional space having more than three dimensions. On the other hand, the model and object spaces have the general central projective one-to-one correspondence uniquely determined with five given points. It follows that we must apply the similarity condition to the five points given in the two three-dimensional spaces so as to make them similar (see Figure 5). Also, since the degrees of freedom of five points in a three-dimensional space are nine, we can construct the similarity condition between two three-dimensional spaces by setting up the following expression:

$$\frac{L_{Mi}\ \text{(line segment in the model space)}}{L_{Oi}\ \text{(line segment in the object space)}} = m\ \text{(constant)}$$

for nine independent line segments. In the above expression, m denotes the scale factor between the model and the object. Therefore, we get eight independent equations from the second condition to construct a stereo model in the same three-dimensional space as the object one. Consequently, we obtain 15 independent equations (seven coplanarity equations plus eight equations based on the similarity condition) for the determination of the 15 unknowns during the phase of the model construction in the same three-dimensional space as the object one. The expression for the similarity condition is actually derived from Equations 25 and 26 in the form

$$\rho_1 = \frac{m_2 B}{l_1 m_2 - m_1 l_2}$$

$$\begin{aligned} X_M &= l_1\rho_1 = \frac{l_1 m_2 B}{l_1 m_2 - m_1 l_2} = \frac{{}_M\bar{X}_{p1}\,{}_M\bar{Y}_{p2} B}{{}_M\bar{X}_{p1}\,{}_M\bar{Y}_{p2} - {}_M\bar{Y}_{p1}\,{}_M\bar{X}_{p2}} \\ Y_M &= m_1\rho_1 = \frac{m_1 m_2 B}{l_1 m_2 - m_1 l_2} = \frac{{}_M\bar{Y}_{p1}\,{}_M\bar{Y}_{p2} B}{{}_M\bar{X}_{p1}\,{}_M\bar{Y}_{p2} - {}_M\bar{Y}_{p1}\,{}_M\bar{X}_{p2}} \\ Z_M &= n_1\rho_1 = \frac{n_1 m_2 B}{l_1 m_2 - m_1 l_2} = \frac{{}_M\bar{Z}_{p1}\,{}_M\bar{Y}_{p2} B}{{}_M\bar{X}_{p1}\,{}_M\bar{Y}_{p2} - {}_M\bar{Y}_{p1}\,{}_M\bar{X}_{p2}} \end{aligned} \tag{36}$$

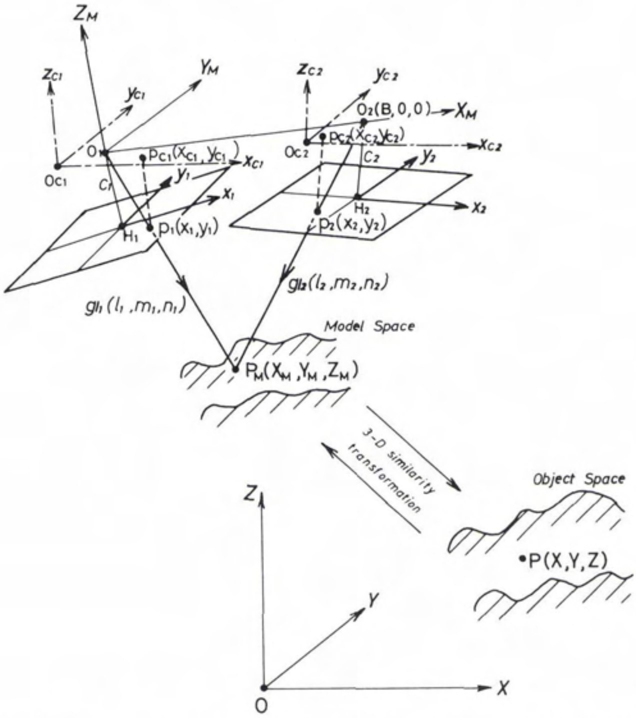

FIG. 4. Model construction in the same three-dimensional space as the object one for the general case.

$$\sqrt{\frac{(X_{Mi} - X_{Mk})^2 + (Y_{Mi} - Y_{Mk})^2 + (Z_{Mi} - Z_{Mk})^2}{(X_i - X_k)^2 + (Y_i - Y_k)^2 + (Z_i - Z_k)^2}} = m. \tag{37}$$

Solving the coplanarity condition (Equation 35) and the similarity condition (Equation 37) with respect to ϕ_1, κ_1, x_{co1}, y_{co1}, c_1, α_1, β_1, ϕ_2, ω_2, κ_2, x_{co2}, y_{co2}, c_2, α_2, and β_2 simultaneously, we can calculate a model point $P_M(X_M,Y_M,Z_M)$ by means of Equation 36 in the same three-dimensional space as the object one. Then, all model points can be transformed into the object space by the three-dimensional similarity transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = m \begin{bmatrix} \cos\Phi & 0 & \sin\Phi \\ 0 & 1 & 0 \\ -\sin\Phi & 0 & \cos\Phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega & -\sin\Omega \\ 0 & \sin\Omega & \cos\Omega \end{bmatrix} \begin{bmatrix} \cos K & -\sin K & 0 \\ \sin K & \cos K & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_M \\ Y_M \\ Z_M \end{bmatrix} + \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix}. \tag{38}$$

Through the consideration of the general orientation problem of a stereoscopic pair of photographs discussed above, we can see that the general central projective one-to-one correspondence (Equation 8) between two three-dimensional spaces is divided into the similarity condition and the three-dimensional similarity transformation (the three-dimensional central projective transformation).

SIMULTANEOUS DETERMINATION OF ALL ORIENTATION UNKNOWNS FOR THE GENERAL CASE

This chapter treats the simultaneous determination of all orientation unknowns of a stereoscopic pair of photographs geometrically for the general case where a photograph has 11 independent parameters. In this orientation technique the 22 photogrammetric orientation elements are calculated directly and

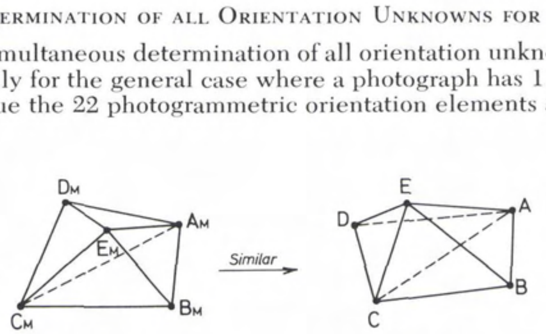

FIG. 5. Similarity condition for the general case.

simultaneously. Thus, we must consider the general orientation problem of a stereoscopic pair of photographs in the same three-dimensional space as the object one, as is demonstrated in Figure 6.

Corresponding rays $\mathbf{g}_1$ and $\mathbf{g}_2$ have the following expressions in the object-space coordinate system (X,Y,Z):

$$\mathbf{g}_1: \frac{X - X_{o1}}{l_1} = \frac{Y - Y_{o1}}{m_1} = \frac{Z - Z_{o1}}{n_1} = \rho_1 \tag{39}$$

$$\mathbf{g}_2: \frac{X - X_{o2}}{l_2} = \frac{Y - Y_{o2}}{m_2} = \frac{Z - Z_{o2}}{n_2} = \rho_2 \tag{40}$$

in which (l_1,m_1,n_1) and (l_2,m_2,n_2) are direction cosines of $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively, and (X_{o1},Y_{o1},Z_{o1}) and (X_{o2},Y_{o2},Z_{o2}) indicate the projection centers of the left and right photographs in the object-space coordinate system. The direction cosines can be obtained in the same manner as in the previous chapter. Also, the transformed picture coordinates (X_p,Y_p,Z_p) and the reduced ones $(\bar{X}_p,\bar{Y}_p,\bar{Z}_p)$ of a picture point $p(x,y)$ can be constructed in the same way as before.

The coplanarity condition of corresponding rays is derived from Equations 39 and 40 in the form

$$\begin{vmatrix} X_{o2} - X_{o1} & l_1 & l_2 \\ Y_{o2} - Y_{o1} & m_1 & m_2 \\ Z_{o2} - Z_{o1} & n_1 & n_2 \end{vmatrix} = 0 \tag{41}$$

and it yields

$$\begin{vmatrix} B_X & \bar{X}_{p1} & \bar{X}_{p2} \\ B_Y & \bar{Y}_{p1} & \bar{Y}_{p2} \\ B_Z & \bar{Z}_{p1} & \bar{Z}_{p2} \end{vmatrix} = 0 \tag{42}$$

in which

$$B_X = X_{o6} - X_{o1}, B_Y = Y_{o2} - Y_{o1}, B_Z = Z_{o2} - Z_{o1}.$$

Further, an object point $P(X,Y,Z)$ can also be given from Equations 39 and 40 as

$$\rho_1 = \frac{m_2 B_X - l_2 B_Y}{l_1 m_2 - m_1 l_2}$$

$$\begin{aligned} X &= \rho_1 l_1 + X_{o1} = \frac{l_1(m_2 B_X - l_2 B_Y)}{l_1 m_2 - m_1 l_2} + X_{o1} = \frac{\bar{X}_{p1}(\bar{Y}_{p2} B_X - \bar{X}_{p2} B_Y)}{\bar{X}_{p1}\bar{Y}_{p2} - \bar{Y}_{p1}\bar{X}_{p2}} + X_{o1} \\ Y &= \rho_1 m_1 + Y_{o1} = \frac{m_1(m_2 B_X - l_2 B_Y)}{l_1 m_2 - m_1 l_2} + Y_{o1} = \frac{\bar{Y}_{p1}(\bar{Y}_{p2} B_X - \bar{X}_{p2} B_Y)}{\bar{X}_{p1}\bar{Y}_{p2} - \bar{Y}_{p1}\bar{X}_{p2}} + Y_{o1} \\ Z &= \rho_1 n_1 + Z_{o1} = \frac{n_1(m_2 B_X - l_2 B_Y)}{l_1 m_2 - m_1 l_2} + Z_{o1} = \frac{\bar{Z}_{p1}(\bar{Y}_{p2} B_X - \bar{X}_{p2} B_Y)}{\bar{X}_{p1}\bar{Y}_{p2} - \bar{Y}_{p1}\bar{X}_{p2}} + Z_{o1} \end{aligned} \tag{43}$$

The coplanarity condition (Equation 42) is mathematically valid for seven sets of corresponding measured image coordinates (x_{c1},y_{c1}) and (x_{c2},y_{c2}), because it has seven independent orientation parameters. On the other hand, we can set up Equation 43 for five points given in the object space, since Equation 43 is equivalent to the general central projective one-to-one correspondence between the model and object spaces. Then, we have 22 independent equations for the unique determination of the 22 general photogrammetric orientation elements of the left and right photographs. Solving these 22 equations with respect to the 22 parameters simultaneously, we can calculate space coordinates of all photographed points by means of Equation 43 in the object-space coordinate system.

In conventional photogrammetry, the simultaneous determination of all unknown orientation parameters of a stereoscopic pair of photographs is constructed in another way[9]. The fundamental consideration is, however, based on Equations 39 and 40. In this orientation approach, ground control points are, at first, classified into the following four types: (1) ground control points with the space coordinates (X,Y,Z) given; (2) ground control points with only the planimetric coordinates (X,Y) known; (3) ground control points with only the height (the Z-coordinate) given; and (4) ground control points without the coordinate information. Then, (determination) equations for the unknown general photogrammetric orientation parameters are derived for each type of ground control points respectively.

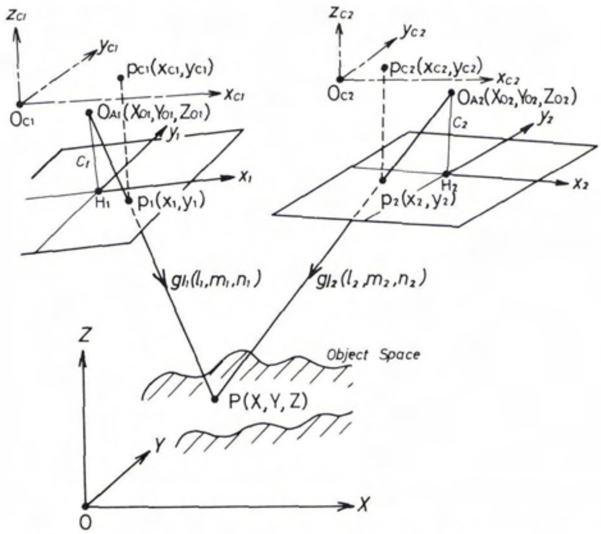

FIG. 6. Analytical geometric simultaneous determination.

(1) The first type.

The (determination) equations for the first type are derived from Equations 39 and 40 in the form

$$X = X_{o1} + \frac{l_1}{n_1}(Z - Z_{o1}) = X_{o1} + \frac{\bar{X}_{p1}}{\bar{Z}_{p1}}(Z - Z_{o1})$$

$$Y = Y_{o1} + \frac{m_1}{n_1}(Z - Z_{o1}) = Y_{o1} + \frac{\bar{Y}_{p1}}{\bar{Z}_{p1}}(Z - Z_{o1}) \tag{44}$$

for the left photograph, and

$$X = X_{o2} + \frac{l_2}{n_2}(Z - Z_{o2}) = X_{o1} + B_X + \frac{\bar{X}_{p2}}{\bar{Z}_{p2}}(Z - Z_{o1} - B_Z)$$

$$Y = Y_{o2} + \frac{m_2}{n_2}(Z - Z_{o2}) = Y_{o1} + B_Y + \frac{\bar{Y}_{p2}}{\bar{Z}_{p2}}(Z - Z_{o1} - B_Z) \tag{45}$$

for the right one. Instead of Equations 44 and 45, we can use the general collinearity equations (expressed in terms of the general photogrammetric orientation parameters) for the left and right photographs, which is essential in photogrammetry.

(2) The second type.

The (determination) equations in this case are expressed as

$$X = X_{o1} + \frac{l_1}{m_1}(Y - Y_{o1}) = X_{o1} + \frac{\bar{X}_{p1}}{\bar{Y}_{p1}}(Y - Y_{o1})$$

$$X = X_{o2} + \frac{l_2}{m_2}(Y - Y_{o2}) = X_{o1} + B_X + \frac{\bar{X}_{p2}}{\bar{Y}_{p2}}(Y - Y_{o1} - B_Y) \tag{46}$$

$$\frac{\bar{Z}_{p1}}{\bar{X}_{p1}}(X - X_{o1}) = B_Z + \frac{\bar{Z}_{p2}}{\bar{X}_{p2}}(X - X_{o1} - B_X)$$

The first and second equations can be directly obtained from Equations 39 and 40, if the planimetric coordinates (X,Y) of a ground control point are known. The third equation indicates the condition that the Z-coordinate of this point calculated from Equation 39 must be equal to that from Equation 40.

(3) The third type.

For ground control points with only the height given, we have the conditions that the X and Y coordinates calculated from Equation 39 must coincide with those from Equation 40. This condition can be formulated in the next form, i.e.,

$$X_{o1} + \frac{l_1}{n_1}(Z - Z_{o1}) = X_{o2} + \frac{l_2}{n_2}(Z - Z_{o2})$$
$$Y_{o1} + \frac{m_1}{n_1}(Z - Z_{o1}) = Y_{o2} + \frac{m_2}{n_2}(Z - Z_{o2}) \qquad (47)$$

which is rewritten as

$$\frac{\bar{X}_{p1}}{\bar{Z}_{p1}}(Z - Z_{o1}) = B_X + \frac{\bar{X}_{p2}}{\bar{Z}_{p2}}(Z - Z_{o1} - B_Z)$$
$$\frac{\bar{Y}_{p1}}{\bar{Z}_{p1}}(Z - Z_{o1}) = B_Y + \frac{\bar{Y}_{p2}}{\bar{Z}_{p2}}(Z - Z_{o1} - B_Z)\,. \qquad (48)$$

(4) The fourth type.

The (determination) equation for this type is the coplanarity condition (Equation 42) of corresponding rays $\mathbf{g}_1$ and $\mathbf{g}_2$.

In this conventional approach, the (determination) Equations 42, 44, 45, 46, and 48 are solved by a least-squares adjustment with respect to ϕ_1, ω_1, κ_1, X_{o1}, Y_{o1}, Z_{o1}, x_{co1}, y_{co1}, c_1, α_1, and β_1 for the left picture and ϕ_2, ω_2, κ_2, X_{o2}, Y_{o2}, Z_{o2}, x_{co2}, y_{co2}, c_2, α_2, and β_2 for the right one simultaneously. However, it will be noted that the conventional approach must be essentially equivalent to the first one. Accordingly, we must be careful that the (determination) equations for the first, second, and third types of ground control points include one equation which is equivalent to the coplanarity condition of corresponding rays, respectively.

The Usual Case in Close-Range Photogrammetry

In this chapter, the orientation problem of a stereoscopic pair of photographs will be discussed for the usual case in close-range photogrammetry, where the $x_c - y_c$ plane of the comparator coordinate system (x_c,y_c,z_c) is parallel to the picture plane (See Figure 1). Thus, the relationship between a picture point $p(x,y)$ and its measured image point $p_c(x_c,y_c)$ reduces, simply, to

$$x_c - x_{co} = x$$
$$y_c - y_{co} = y \quad . \qquad (49)$$

Equation 49 means that the planimetric coordinates (x_{co},y_{co}) of the projection center of a photograph in the comparator coordinate system are identical to the principal point coordinates (x_H,y_H). Accordingly, we have

$$x_c - x_H = x$$
$$y_c - y_H = y \quad . \qquad (50)$$

Also the photogrammetric orientation parameters of a picture are reduced to nine, i.e., ϕ, ω, κ, X_o, Y_o, Z_o, x_H, y_H, and c.

Orientation Problem of a Stereoscopic Pair of Photographs

This paragraph treats the orientation problem of a stereoscopic pair of photographs, based on a geometrical consideration (see Figure 7). A stereo model is constructed by means of the coplanarity condition of corresponding rays $\mathbf{g}_1(l_1,m_1,n_1)$ and $\mathbf{g}_2(l_2,m_2,n_2)$:

$$\begin{vmatrix} {}_M\bar{Y}_{p1} & {}_M\bar{Y}_{p2} \\ {}_M\bar{Z}_{p1} & {}_M\bar{Z}_{p2} \end{vmatrix} = 0 \qquad (51)$$

where the reduced transformed picture coordinates are described in the form

$$\begin{bmatrix} {}_M\bar{X}_{p1} \\ {}_M\bar{Y}_{p1} \\ {}_M\bar{Z}_{p1} \end{bmatrix} = \mathbf{D}_{\phi_1}\mathbf{D}_{\kappa_1}\begin{bmatrix} x_{c1} - x_{H1} \\ y_{c1} - y_{H1} \\ -c_1 \end{bmatrix} \qquad (52)$$

for the left picture, and

$$\begin{bmatrix} {}_M\bar{X}_{p2} \\ {}_M\bar{Y}_{p2} \\ {}_M\bar{Z}_{p2} \end{bmatrix} = \mathbf{D}_{\phi_2}\mathbf{D}_{\omega_2}\mathbf{D}_{\kappa_2}\begin{bmatrix} x_{c2} - x_{H2} \\ y_{c2} - y_{H2} \\ -c_2 \end{bmatrix} \qquad (53)$$

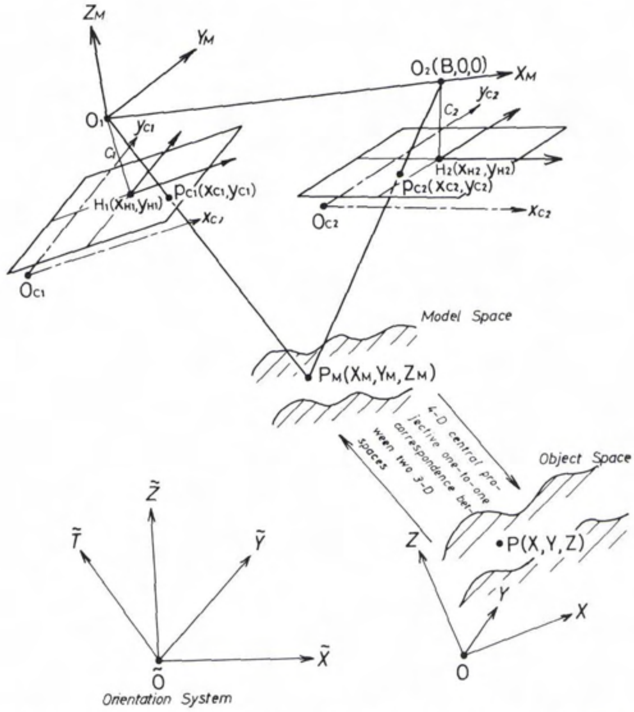

FIG. 7. Orientation problem of a stereoscopic pair of pictures taken with non-metric cameras.

for the right one, respectively. First, we will investigate what relationship is valid between the model space (X_M,Y_M,Z_M) and the object space (X,Y,Z). According to the characteristics of the general central projective one-to-one correspondence (Equation 8) between two three-dimensional spaces discussed previously, it can be understood that in the case where the three-dimensional plane (x_c,y_c,z_c) of the fictitious comparator coordinate system (x_c,y_c,z_c,t_c) is parallel to the three-dimensional fictitious picture plane (x,y,z), the four-dimensional central projective one-to-one correspondence is satisfied between the two three-dimensional spaces (x_c,y_c,z_c) and (X,Y,Z) in the form

$$x_c - x_{co} = -c\frac{e_{11}(X-X_o)+e_{12}(Y-Y_o)+e_{13}(Z-Z_o)+e_{14}(0-T_o)}{e_{41}(X-X_o)+e_{42}(Y-Y_o)+e_{43}(Z-Z_o)+e_{44}(0-T_o)}$$

$$y_c - y_{co} = -c\frac{e_{21}(X-X_o)+e_{22}(Y-Y_o)+e_{23}(Z-Z_o)+e_{24}(0-T_o)}{e_{41}(X-X_o)+e_{42}(Y-Y_o)+e_{43}(Z-Z_o)+e_{44}(0-T_o)} \qquad (54)$$

$$z_c - x_{co} = -c\frac{e_{31}(X-X_o)+e_{32}(Y-Y_o)+e_{33}(Z-Z_o)+e_{34}(0-T_o)}{e_{41}(X-X_o)+e_{42}(Y-Y_o)+e_{43}(Z-Z_o)+e_{44}(0-T_o)}\,.$$

By indicating the principal point coordinates of the fictitious three-dimensional photograph as (x_H,y_H,z_H), the four-dimensional central projective one-to-one correspondence (Equation 54) can also be described between (x_c,y_c,z_c) and (X,Y,Z) as

$$x_c - x_H = -c\frac{e_{11}(X-X_o)+e_{12}(Y-Y_o)+e_{13}(Z-Z_o)+e_{14}(0-T_o)}{e_{41}(X-X_o)+e_{42}(Y-Y_o)+e_{43}(Z-Z_o)+e_{44}(0-T_o)}$$

$$y_c - y_H = -c\frac{e_{21}(X-X_o)+e_{22}(Y-Y_o)+e_{23}(Z-Z_o)+e_{24}(0-T_o)}{e_{41}(X-X_o)+e_{42}(Y-Y_o)+e_{43}(Z-Z_o)+e_{44}(0-T_o)} \qquad (55)$$

$$z_c - z_H = -c\frac{e_{31}(X-X_o)+e_{32}(Y-Y_o)+e_{33}(Z-Z_o)+e_{34}(0-T_o)}{e_{41}(X-X_o)+e_{42}(Y-Y_o)+e_{43}(Z-Z_o)+e_{44}(0-T_o)}\,,$$

which has 12 independent parameters $(\phi,\omega,\kappa,\mu,X_o,Y_o,Z_o,T_o,x_H,y_H,z_H,c)$. Further, four points are mathematically necessary in the object space so as to determine the one-to-one correspondence.

From the discussion mentioned above, it follows that the four-dimensional central projective one-to-one correspondence must be valid between the model space (X_M,Y_M,Z_M) and the object space (X,Y,Z), if the $x_c - y_c$ plane (actual) of the comparator coordinate system (x_c,y_c,z_c) is parallel to the picture plane. Also, the one-to-one correspondence includes 12 orientation parameters. Consequently, the coplanarity condition of corresponding rays provides mathematically six independent orientation parameters, since 18 independent orientation elements must be determined in the orientation problem of a stereoscopic pair of pictures for the usual case in close-range photogrammetry considered here (Koelbl[4] found empirically that one interior orientation parameter in addition to the five exterior ones can be calculated from the coplanarity condition of corresponding rays by means of convergent photographs).

By means of the coplanarity condition of corresponding rays itself we can determine six independent orientation parameters. Also, a stereo model can be constructed in a three-dimensional space of the four-dimensional one. However, the stereo model must be constructed in the same three-dimensional space as the object one in order to calculate directly the 18 independent orientation elements of the left and right pictures (see Figure 8). Therefore, the second condition for the model construction will be introduced. This is the similarity condition between the model space (X_M,Y_M,Z_M) and the object one (X,Y,Z). In order to make them similar, we must apply the similarity condition to four points which are necessary for the unique determination of the four-dimensional central projective one-to-one correspondence between two three-dimensional spaces (see Figure 9). Also, as the degree of freedom of four points in a three-dimensional space is six, we can construct the similarity condition by setting up the next expression

$$\frac{L_{Mi}\ \text{(line segment in the model space)}}{L_{0i}\ \text{(line segment in the object space)}} = m\ \text{(constant)}$$

for six independent line segments. Thus, five independent equations can be obtained as the second condition for the model construction in the same three-dimensional space as the object one. Consequently, we have 11 independent equations (six coplanarity equations plus five equations based on the similarity condition) for the determination of 11 orientation parameters $(\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2,x_{H1},y_{H1},c_1,x_{H2},y_{H2},c_2)$ included in Equation 51.

By solving the 11 independent equations with respect to the 11 orientation unknowns in Equation 51, we can calculate a model point $P_M(X_M,Y_M,Z_M)$ in the same three-dimensional space as the object one. Finally, all model points are transformed into the object space by the three-dimensional similarity transformation.

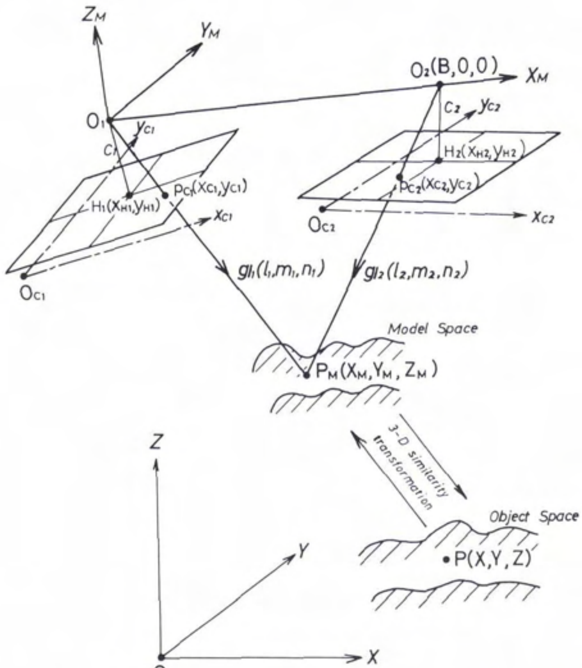

FIG. 8. Model construction in the same three-dimensional space as the object one for the special case.

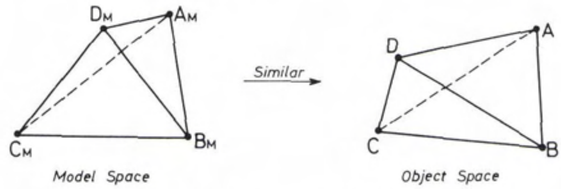

FIG. 9. Similaity condition for the special case.

From the discussion given here, we can precisely see that a stereo model can be constructed only by means of the coplanarity condition of conjugate rays with five exterior orientation elements (ϕ_1,κ_1,ϕ_2,ω_2, κ_2) in the same three-dimensional space as the object one, if all interior orientation parameters of a stereoscopic pair of pictures are given, as is for metric cameras.

SIMULTANEOUS DETERMINATION OF ALL ORIENTATION UNKNOWNS FOR USUAL CASE

The orientation procedure based on the simultaneous determination of all orientation unknowns for a stereoscopic pair of pictures is almost the same as was discussed in the previous chapter. In this case, however, we must note the following three remarks:

- a picture has nine independent orientation parameters,
- the coplanarity condition of corresponding rays has six independent orientation elements, and
- four points are mathematically necessary for the determination of all orientation parameters of a stereoscopic pair of pictures.

CONCLUDING REMARKS

The analytical orientation problem of a stereoscopic pair of photographs taken with non-metric cameras has been theoretically investigated in this paper. Non-central projective parameters such as lens distortion and film deformation were neglected in the discussion given here, since these are not parameters related to central projective geometry and can be determined from other geometric considerations. The orientation problem was analyzed algebraically as well as geometrically for the general case where a photograph has 11 independent orientation elements, and the important aspects have been clarified that:

- The stereo model is constructed in a three-dimensional space with seven independent orientation parameters;
- The general central projective one-to-one correspondence must be valid between the model and object spaces, which is uniquely determined with five points; and
- The general central projective one-to-one correspondence can be divided into the similarity condition and the three-dimensional similarity transformation (the three-dimensional central projective transformation).

Thus, a practically useful orientation method to calculate, directly, the general photogrammetric orientation parameters of a stereoscopic pair of pictures was developed by applying the similarity condition between the model and object spaces for the construction of the stereo model in the same three-dimensional space as the object one. Further, simultaneous determination of all orientation unknowns was also considered for the general case.

The case where the $x_c - y_c$ plane of the comparator coordinate system is parallel to the picture plane is the most important one in close-range photogrammetry. Careful geometrical considerations of the orientation problem have revealed the following important facts:

- The coplanarity condition of conjugate rays has six independent orientation elements;
- The four-dimensional central projective one-to-one correspondence is satisfied between the model and object spaces, which can be uniquely determined with four points; and
- The four-dimensional central projective one-to-one correspondence can also be divided into the similarity condition and the three-dimensional similarity transformation.

The orientation theory discussed in this paper is applicable to the following problems:

- Orientation problem of stereo-strip imageries. It is essentially analyzed on the same geometrical basis,
- Calibration problem of non-metric cameras. According to Hallert[6], the non-central projective parameters such as lens distortion can be determined from the coplanarity condition of corresponding rays. Thus, by adding lens distortion to measured image coordinates, we can calibrate non-metric cameras using the proposed orientation methods. The control requirement remains only as distance measurement which is comparatively easily performed. Further, it is possible to formulate Koelbl's method (self-calibration method of non-metric cameras) in another form.

- Triplet model can be constructed even for the case where the interior orientation parameters are not given, under the assumption that the interior stability of aerial cameras is maintained.

References

1. Wong, K. W., Mathematical Formulation and Digital Analysis in Close-Range Photogrammetry, *Photogrammetric Engineering and Remote Sensing*, Vol. 41, No. 11, November 1975, pp. 1355-1373.
2. Brown, D. C., Close-Range Camera Calibration, *Photogrammetric Engineering*, Vol. 37, No. 8, August 1971, pp. 855-866.
3. Jacobi, O., Kalibrieren gewoehnlicher Photoapparate und deren Verwendung als Messkammern, *Bildmessung und Luftbildwesen*, 2/1968, pp. 59-70.
4. Koelbl, O., Selbstkalibrierung von Aufnahmekammern, *Bildmessung und Luftbildwesen*, 1/1972, pp. 31-37.
5. Faig, W., Calibration of Close-Range Photogrammetric Systems: Mathematical Formulation, *Photogrammetric Engineering and Remote Sensing*, Vol. 41, No. 12, December 1975, pp. 1479-1486.
6. Hallert, B., Ueber die Bestimmung der radialen Verzeichnung von Luftaufnahmen, *Zeitschrift für Vermessungswesen*, No. 4/1956, pp. 139-142.
7. Abdel-Aziz, Y. I., and H. M. Karara, Direct Linear Transformation into Object Space Coordinates in Close-Range Photogrammetry, in *Proceedings of the Symposium on Close-Range Photogrammetry*, University of Illinois at Urbana-Champaign, Urbana, Illinois, January 26-29, 1971, pp. 1-18.
8. Ghosh, S. K., *Analytical Photogrammetry*, Pergamon Press, New York, 1979.
9. Finsterwalder, R., and W. Hofmann, *Photogrammetrie*, Walter de Gruyter and Co. Berlin, 1968.

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