C. S. Fraser
Division of Surveying Engineering
The University of Calgary
Calgary, Alberta T2N 1N4, Canada

Optimization of Precision in Close-Range Photogrammetry*

Minimal control, minimum-variance estimation, inner adjustment constraints, and the pseudo-inverse are discussed

ABSTRACT: *The optimization of precision in minimally constrained self-calibrating bundle adjustments is discussed. As a "one-number" summary of close-range photogrammetric network strength, the mean positional standard error of the object target points has been adopted. Variations in the magnitudes of both this estimate and relative measures of precision which accompany changes in the type and configuration of the minimal constraint are examined. Imposition of an absolute minimal constraint can be achieved by various methods. In this paper the following approaches are discussed: minimal control, minimum-variance estimation, inner adjustment constraints, and the pseudo-inverse. Features of these methods which are significant in the context of attaining optimum object point coordinate precision are developed. An experiment has been conducted to quantitatively examine some of the factors detailed, and the results of this practical photogrammetric investigation are discussed.*

Introduction

Where close-range photogrammetry is employed as a tool for precise three-dimensional measurements, the parameters of primary interest in a self-calibrating bundle adjustment are usually the coordinates of the object target points. In addition, the variance-covariance matrix of these parameters is sought in order to ascertain the a posteriori precision of both the object point coordinates and functions of these coordinates; e.g., distances, areas, volumes, and coordinate differences. One useful, albeit limited, indicator of the overall accuracy of target point determination is the estimator $\bar{\sigma}_c^2$ which is termed the mean variance of the X, Y, Z object space coordinates.

A commonly used accuracy indicator in aerial triangulation adjustments is the root-mean-square (RMS) error, s_c, of the ground coordinates. Where checkpoint coordinates are available, s_c can be considered to be an unbiased estimate of $\bar{\sigma}_c$, the mean standard error (e.g., Ebner and Grün, 1979). The RMS error, s_c, then indicates the average positional accuracy with respect to the reference coordinate system. However, in high-precision close-range photogrammetry, checkpoint coordinates are invariably unavailable; indeed, "control" is often arbitrarily assigned so as to be just sufficient to define the origin, orientation, and scale of the photogrammetric network. This paper considers only minimally constrained networks, and for such cases $\bar{\sigma}_c$ affords the photogrammetrist a useful one-number summary of photogrammetric network strength. However, the mean positional standard error provides only a limited measure of internal or datum-independent precision because it is a datum-biased estimate, its magnitude being influenced by both the configuration and type (weighted or absolute) of the minimal control.

As has been pointed out by a number of authors (e.g., Pope, 1971; Sprinsky, 1976), geodetic network optimization criteria should include the need to account for off-diagonal covariance information contained in the parameter variance-covariance matrix in addition to the on-diagonal variances from which $\bar{\sigma}_c^2$ is computed. This need has given rise to network analysis based on estimable quantities or invariants which describe the network independent of the arbitrarily assigned datum.

* Presented paper, 48th Annual Meeting of the ASP, Denver, March 14-20, 1982.

0099-1112/82/4804-645$02.25/0

Under general affine transformations of the object space coordinate system, parameters of shape and their a posteriori standard errors remain invariant. For example, under a similarity transformation, both angle and distance-ratio values are unaltered and, if the network has no defect of scale, distances remain invariant. However, in the photogrammetric context distances determined in a scale defective network will not remain invariant with changes in minimal control, a property also shared by other functions of the *X, Y, Z* object point coordinates.

Because a general affine transformation of the reference coordinate system is a distinctly different operation from a change in the minimal constraint configuration (Pope, 1971), only information introduced into the photogrammetric adjustment by the actual observations is preserved without change under a variation of the minimal constraint. In a self-calibrating bundle adjustment the x, y photo coordinates are unbiasedly estimable, and this is generally also true for the additional parameters of the image space. Thus, in order to obtain a measure of object point positional accuracy which is independent of the minimal constraint, one must consider a function of the image coordinate variances; for example, some fraction of the scale number multiplied by the derived mean standard error of the photo coordinates. While such an approach may be practical in aerial triangulation, it is far less applicable in close-range projects which employ convergent imaging configurations and non-uniform scale between the photographs.

In the absence of useful invariant measures of object space coordinate precision, $\bar{\sigma}_c$ again appears as a practical indicator of photogrammetric network strength. For this investigation, the optimization criterion adopted in the initial stages is that the selected minimal constraint should minimize the mean positional standard error, $\bar{\sigma}_c$. However, in the discussion of inner constraints this criterion is extended to include the minimization of $\bar{\sigma}_{c'}$, the mean coordinate standard error of c' selected object target points. The essence of both criteria is that for some "best fitting" reference coordinate system an optimum value of positional accuracy is obtained.

Minimal constraints may be weighted or absolute. Although there is a limited distinction between the nature of "weighted-unknown" and "fixed" minimal control coordinates as far as the solution of a network adjustment is concerned, the magnitude of $\bar{\sigma}_c$ is dependent on the type of constraint imposed. Fraser (1980) has shown that for a minimally constrained photogrammetric bundle adjustment the condition $\bar{\sigma}_c$ = minimum for a particular minimal control pattern can only be achieved when the constraint is absolute. A brief account of weighted constraints is presented in this paper, though for a more detailed discussion the reader is referred to Fraser (1980).

Directly connected with precision is the theory of external reliability (Baarda, 1967; 1968). With regard to photogrammetric adjustment, external reliability measures can be used as tolerances which may in turn offer criteria for expressing the accuracy of both adjusted object space coordinates and functions of these coordinates. However, such sensitivity measures generally require a minimum $\bar{\sigma}_c$ if they are to be optimized for a particular photogrammetric network design. Thus, the pursuit of minimum $\bar{\sigma}_c$ is still warranted for any minimally constrained adjustment. For a full account of the concepts of external reliability as applied to phototriangulation, the reader is referred to Förstner (1979, 1980) and Grün (1978, 1980).

In the following sections aspects of constrained least-squares estimation are reviewed, including minimum-variance estimation and the pseudo-inverse and inner constraint approaches to minimally constrained photogrammetric adjustments. The emphasis is placed on the attainment of minimum mean variance. The influence of variations in minimal constraint configurations on the magnitude of both $\bar{\sigma}_c$ and $\bar{\sigma}_{c'}$ has been quantified using a small close-range photogrammetric network. This test experiment is detailed and the results are discussed.

Mathematical Background

The Linear Model

Consider the linear functional and stochastic model

$$\begin{gathered} \mathbf{V} = \mathbf{AX} - \mathbf{L} \\ E(\mathbf{V}) = \mathbf{O} \rightarrow E(\mathbf{L}) = \mathbf{AX} \\ D(\mathbf{V}) = \mathbf{\Sigma} = \sigma_0^2\, \mathbf{P}^{-1} \end{gathered} \qquad (1)$$

where **A** is the design matrix with dimensions $n \times u$ $(n > u)$ and rank $R(\mathbf{A}) \leq u$; **L** is an $n \times 1$ stochastic vector of observations; **X** is the non-stochastic vector of unknown parameters of dimension u; **V** is a stochastic n-vector of residuals; $\mathbf{\Sigma}$ is the positive-definite variance-covariance matrix of the observations; **P** is the weight matrix; σ_0^2 is the a priori variance factor; and E and D are the expectation and dispersion operators, respectively.

A necessary and sufficient condition for the equation

$$E(\mathbf{L}) = \mathbf{AX} \qquad (2)$$

to be solvable is that (Albert, 1972, p. 30)

$$\mathbf{AA}^{+}E(\mathbf{L}) = E(\mathbf{L}) \qquad (3)$$

where $\mathbf{A}^{+}$ is a chosen member of the family of generalized inverses, termed the pseudo-inverse. Characteristics of the pseudo-inverse will be detailed in a later section. Equation 3 can be viewed as being a consistency condition and if the system, Equation 2, is consistent, then the rank and dimensions of $\mathbf{A}$ determine whether $\mathbf{X}$ is unique, or whether many solutions exist.

If the design matrix $\mathbf{A}$ has no rank defect, i.e. $R(\mathbf{A}) = u$, a unique estimate, the least-squares estimate, $\hat{\mathbf{X}}$,* can be obtained for $\mathbf{X}$; that is,

$$\hat{\mathbf{X}} = \mathbf{N}^{-1}\mathbf{A}^T\mathbf{PL} \text{ where } \mathbf{N} = \mathbf{A}^T\mathbf{PA}. \quad (4)$$

In the full rank case $\mathbf{N}^{-1} = \mathbf{N}^{+}$, thus satisfying the consistency condition and also yielding best linear unbiased estimations, $\hat{\mathbf{X}}$ and $\hat{\sigma}_0^2$, while minimizing $\mathbf{V}^T\mathbf{PV}$. The corresponding variance-covariance matrix, $\boldsymbol{\Sigma}_{\hat{X}}$, of the parameter estimates, $\hat{\mathbf{X}}$, is given by

$$\boldsymbol{\Sigma}_{\hat{X}} = \hat{\sigma}_0^2\mathbf{N}^{-1} \quad (5)$$

where $\hat{\sigma}_0^2$ is the a posteriori variance factor.

The linear model of the self-calibrating bundle adjustment will assume the form of Equation 1 in cases where prior information about the structure and distribution of the parameters, $\mathbf{X}$, is not taken into account. In order to ensure a unique solution for the parameters (exterior orientation elements, object space coordinates, and additional parameters) according to Equation 4, minimal constraints need to be imposed because information required to overcome the network defect of $d = 7$ (three for position, three for orientation, and one for scale) is lacking in the observations $\mathbf{L}$ (photo coordinate measurements).

MINIMUM-VARIANCE ESTIMATION

One widely used scheme for imposing minimal constraints to overcome the datum defect inherent in the positive semi-definite matrix, $\mathbf{N}$, is to suppress d appropriate columns from the design matrix, $\mathbf{A}$. An algorithmically more flexible approach, particularly suited to close-range photogrammetric surveys where it is desired to control no more than the origin, orientation, and scale of the object space coordinate system, is to append additional rows to the linear model, Equation 1. A minimum of d rows will be required, and these can be viewed as being additional observation equations, fictitious or real. The linear model then becomes

$$\begin{bmatrix}\mathbf{V}\\ \bar{\mathbf{V}}\end{bmatrix} + \begin{bmatrix}\mathbf{A}\\ -\mathbf{I}\end{bmatrix}\mathbf{X} = \begin{bmatrix}\mathbf{L}\\ \bar{\mathbf{L}}\end{bmatrix};\begin{bmatrix}\mathbf{P} & 0\\ 0 & \mathbf{P}_{X^\circ}\end{bmatrix} \quad (6)$$

where $\mathbf{P}_{X^\circ}$ is a weight matrix indicating the level of prior constraint on the parameters, $\mathbf{X}$. For the following discussion it is convenient to introduce a relative weighting coefficient, k^2, where $\mathbf{P}_{X^\circ} = k^2\mathbf{P}_X$.

The introduction of prior information regarding the mean, $\mathbf{X}^\circ$, and variance, $\boldsymbol{\Sigma}_{X^\circ}$, of what is now the random vector, $\mathbf{X}$, leads to a further estimate, the minimum-variance estimate, $\tilde{\mathbf{X}}$ (e.g., Luenberger, 1969, p. 87), whereby

$$[\mathbf{A}^T\mathbf{PA} + k^2\mathbf{P}_X]\,\tilde{\mathbf{X}} = [\mathbf{A}^T\mathbf{PL} - k^2\mathbf{P}_X\bar{\mathbf{L}}] \quad (7)$$

which minimizes the function

$$\begin{aligned}\phi &= \mathbf{V}^T\mathbf{PV} + k^2\bar{\mathbf{V}}^T\mathbf{P}_X\bar{\mathbf{V}}\\ &= (n + R(\mathbf{P}_X) - R(\mathbf{A}))\tilde{\sigma}_0^2.\end{aligned} \quad (8)$$

The solution vector, $\tilde{\mathbf{X}}$, can only be obtained if the normal matrix is non-singular, and for certain $\mathbf{P}_{X^\circ}$ this non-singularity can always be ensured. For example, if $\mathbf{P}_X = \mathbf{I}$, $(\mathbf{N} + k^2\mathbf{P}_X)$ is non-singular for non-zero k (Albert, 1972, p. 12). The minimum-variance estimate, $\tilde{\mathbf{X}}$, has the structure of a multivariate weighted average between the standard least-squares estimate, $\hat{\mathbf{X}}$, and the prior mean, $\mathbf{X}^\circ$, in accordance with the magnitude of $\mathbf{P}_{X^\circ}$.

Equation 6 can also be considered as a Bayesian formulation (e.g., Lindley and Smith, 1972; Bossler, 1972), although under this approach the degrees of freedom differs marginally from the value given in Equation 8. Such a variation can be considered inconsequential for most photogrammetric adjustments, thus making the Bayesian and minimum-variance estimation approaches effectively equivalent.

The mean variance or, more correctly, the mean square error of the point estimates, $\tilde{\mathbf{X}}$, is given by

$$\bar{\sigma}_{\tilde{X}}^2 = 1/u \operatorname{tr}\boldsymbol{\Sigma}_{\tilde{X}} \text{ where } \boldsymbol{\Sigma}_{\tilde{X}} = \tilde{\sigma}_o^2\,[\mathbf{N} + k^2\mathbf{P}_X]^{-1} \quad (9)$$

The mean square error comprises two components: variance and bias (Marquardt, 1970), and its magnitude is inversely proportional to that of the smallest eigenvalue of $\mathbf{N}$. Thus, $\bar{\sigma}_{\tilde{X}}^2$ is seriously inflated when $(\mathbf{N} + k^2\mathbf{P}_X)$ is near singular. According to two theorems of ridge regression (e.g., Draper and van Nostrand, 1979), ϕ is a monotone increasing function of k while $\bar{\sigma}_{\tilde{X}}^2$ is a continuous monotone decreasing function of k. In a network adjustment the aim is to assign values to $k^2\mathbf{P}_X$ such that $\bar{\sigma}_{\tilde{X}} << \bar{\sigma}_{\hat{X}}$ while not inflating ϕ to any serious degree.

THE PSEUDO-INVERSE

As a first approach in seeking the optimization of $\bar{\sigma}_c$, the minimal constraint yielding $\bar{\sigma}_{\hat{X}}$ = minimum can be considered. In accordance with free network adjustment theory (e.g., Pope, 1971; Welsch, 1979), the solution

$$\hat{\mathbf{X}} = [\mathbf{P}^{1/2}\mathbf{A}]^{+}\,\mathbf{P}^{1/2}\mathbf{L} \quad (10)$$

will yield a minimum value of $\bar{\sigma}_{\hat{X}}$, the minimal constraint condition being $\|\hat{\mathbf{X}}\|$ = minimum. Geometrically, this condition can be interpreted as fixing the coordinate system origin at the center of gravity of the photogrammetric network (as defined by the provisional coordinates), maintaining average network orientation and holding the mean scale as fixed. In the sense of minimizing the mean

* The cap ($\hat{\mathbf{X}}$) indicates a least-squares estimate from a minimal constraint adjustment, whereas the tilde ($\tilde{\mathbf{X}}$) is used to identify the minimum-variance estimate.

variance of the parameters, Equation 10 yields the optimum solution of the overdetermined system, Equation 1.

The pseudo-inverse of the matrix $[\mathbf{P}^{1/2}\mathbf{A}]$ has the following property (Albert, 1972, p. 26):

$$\begin{aligned}\left[\mathbf{P}^{1/2}\mathbf{A}\right]^{+} &= \left[\left[\mathbf{P}^{1/2}\,\mathbf{A}\right]^{\mathrm{T}}\left[\mathbf{P}^{1/2}\,\mathbf{A}\right]\right]^{+}\ \left[\mathbf{P}^{1/2}\mathbf{A}\right]^{\mathrm{T}} \\ &= \left[\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right]^{+}\ \mathbf{A}^{\mathrm{T}}\mathbf{P}^{1/2}\end{aligned} \qquad (11)$$

Further, $[\mathbf{P}^{1/2}\mathbf{A}]^{+}$ may be expressed by the limit definition (Pope and Hanson, 1972; Albert, 1972, p. 19)

$$\left[\mathbf{P}^{1/2}\mathbf{A}\right]^{+} = \lim_{k^2\to 0}\left[\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} + k^2\mathbf{I}\right]^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}^{1/2}. \quad (12)$$

Post-multiplication of Equations 11 and 12 by $\mathbf{P}^{1/2}\mathbf{L}$ yields

$$\hat{\mathbf{X}} = \mathbf{N}^{+}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} = \lim_{k^2\to 0}\left[\mathbf{N} + k^2\mathbf{I}\right]^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} \quad (13)$$

From Equation 13 it is apparent that for $\mathbf{P}_{\mathrm{X}} = \mathbf{I}$ and k "small" the minimum-variance estimate, $\tilde{\mathbf{X}}$, is equivalent to $\hat{\mathbf{X}}$. Fraser (1980) has shown for a close-range photogrammetric network that the implied equality $\tilde{\mathbf{X}} \simeq \hat{\mathbf{X}}$ may hold for a large range of k values. As a general rule the minimum-variance estimate, $\tilde{\mathbf{X}}$, obtained with $\mathbf{P}_{\mathrm{X}^{\mathrm{o}}} = k^2\mathbf{I}$ can be expected to approach the pseudo-inverse estimate, $\hat{\mathbf{X}}$, when $\mathbf{V}^{\mathrm{T}}\mathbf{P}\mathbf{V} >> k^2\tilde{\mathbf{V}}^{\mathrm{T}}\tilde{\mathbf{V}} \simeq 0$; i.e., when the weighted constraint, $k^2\mathbf{I}$, on the prior means, $\mathbf{X}^{\mathrm{o}}$, is numerically equivalent to the absolute minimal constraint $\|\hat{\mathbf{X}}\|$ = minimum. The fact that weighted constraints act as absolute constraints as far as the solution is concerned is well recognized (e.g., Pope, 1971). In the case of Equation 13 the numerical restriction on k^2 is that firstly it must be of sufficient magnitude to ensure that $[\mathbf{N} + k^2\mathbf{I}]$ is numerically regular, and secondly it must not be large enough to numerically overconstrain the solution.

When it is recalled that the pseudo-inverse yields a minimum mean variance estimate, $\bar{\sigma}_{\hat{\mathrm{X}}}^2$, for the parameters, and that an equivalent solution to this minimal constraint approach can be obtained by assigning a loose a priori variance, $1/k^2$, to all parameters, one question that arises is whether $[\mathbf{N} + k^2\mathbf{I}]^{-1}$ is a minimum-trace variance-covariance matrix. Should the equality $[\mathbf{N} + k^2\mathbf{I}]^{-1} \simeq \mathbf{N}^{+}$ be valid over a broad range of k values, as was Equation 13, then an optimum estimate of the mean variance of the parameters would be obtained by means of the minimum-variance approach (Equation 7) with $\mathbf{P}_{\mathrm{X}^{\mathrm{o}}} = k^2\mathbf{I}$.

The matrix $[\mathbf{N} + k^2\mathbf{I}]$ can be expressed in terms of the pseudo-inverse by the series expansion (e.g., Pope and Hanson, 1972):

$$\begin{aligned}[\mathbf{N} + k^2\mathbf{I}]^{-1} = [\mathbf{I} - \mathbf{N}\mathbf{N}^{+}]\,k^{-2} + \mathbf{N}^{+} - k^2\mathbf{N}^{+^2} \\ + k^4\mathbf{N}^{+^3} - \ldots\end{aligned} \qquad (14)$$

For sufficiently small k the right-hand side of Equation 14 will only approach $\mathbf{N}^{+}$ when the projection matrix $[\mathbf{I} - \mathbf{N}\mathbf{N}^{+}]$ of rank d is close to a null matrix. But, for rank defective $\mathbf{N}$, $\mathbf{N}\mathbf{N}^{+} \neq \mathbf{I}$ and k^{-2} operates as an inflation factor on the projection matrix. Thus, for minimal constraint adjustments, the inverse $[\mathbf{N} + k^2\mathbf{I}]^{-1}$ is an unsatisfactory, typically seriously inflated variance-covariance matrix of the parameters.

In order to overcome the rank defect in the photogrammetric network, $\mathbf{P}_{\mathrm{X}}$ can be structured as a diagonal matrix comprising seven appropriate non-zero elements, each of unit value. The imposition of this weighted constraint removes the singularity of the normal equation matrix, but the magnitude of $\mathrm{tr}\boldsymbol{\Sigma}_{\tilde{\mathrm{X}}}$ is still dependent on the value of k^2. This can be illustrated by referring to the singular value decomposition of $[\mathbf{N} + k^2\mathbf{P}_{\mathrm{X}}]$. Minimum $\bar{\sigma}_{\tilde{\mathrm{X}}}$ is obtained when k^2 is selected such that the smallest eigenvalue λ_u of $[\mathbf{N} + k^2\mathbf{P}_{\mathrm{X}}]$ attains its maximum magnitude. It is intuitively apparent that λ_u becomes a maximum as $k^2 \to \infty$ (e.g., see Fraser, 1980), then the weighted minimal constraint $k^2\mathbf{P}_{\mathrm{X}}$ becomes numerically equivalent to the absolute constraint imposed by suppressing seven appropriate columns from the design matrix $\mathbf{A}$. Thus, it can be stated that for a selected minimal constraint the minimum value of $\bar{\sigma}_{\tilde{\mathrm{X}}}$ is obtained when the constraint is absolute and of all possible absolute minimal constraints, the condition $\|\hat{\mathbf{X}}\|$ = minimum will yield the optimum value of the mean standard error of the parameters.

Whereas the pseudo-inverse can be used to yield minimum $\bar{\sigma}_{\hat{\mathrm{X}}}$, no such assurance can be made regarding the mean positional standard error of the object space coordinates, $\bar{\sigma}_c$, in a self-calibrating bundle adjustment. The trace of $\mathbf{N}^{+}$ can be expressed as the sum of three component traces:

$$\hat{\sigma}_{\mathrm{o}}^2\ \mathrm{tr}\mathbf{N}^{+} = \mathrm{tr}\boldsymbol{\Sigma}_{\mathrm{o}} + \mathrm{tr}\boldsymbol{\Sigma}_c + \mathrm{tr}\boldsymbol{\Sigma}_a \qquad (15)$$

where $\boldsymbol{\Sigma}_{\mathrm{o}}$, $\boldsymbol{\Sigma}_c$, and $\boldsymbol{\Sigma}_a$ are a posteriori variance-covariance matrices of the exterior orientation elements, object space coordinates, and additional parameters, respectively. Minimum $\bar{\sigma}_{\hat{\mathrm{X}}}$ need not imply minimum $\mathrm{tr}\boldsymbol{\Sigma}_c$. For example, consider the case of very long focal length photography where, typically, $\mathrm{tr}\boldsymbol{\Sigma}_{\mathrm{o}} > \mathrm{tr}\boldsymbol{\Sigma}_c$ if the minimal constraint is imposed by fixing seven coordinates of three object target points (say $2 \times XYZ$ and $1 \times Z$). The influence of the pseudo-inverse in providing an optimized presentation of $\bar{\sigma}_{\hat{\mathrm{X}}}$ in such an adjustment is to achieve a reduction in the magnitude of $\mathrm{tr}\boldsymbol{\Sigma}_{\mathrm{o}}$ at the expense of an increase in the magnitude of $\mathrm{tr}\boldsymbol{\Sigma}_c$. Nevertheless, $\mathbf{N}^{+}$ may effectively minimize $\bar{\sigma}_c$.

The mean variance, $\bar{\sigma}_c^2$, obtained via the pseudo-inverse, $\mathbf{N}^{+}$, may be interpreted as a measure of the error implied in object space coordinates due solely to observational uncertainties (see Sprinsky, 1976, p. 50). However, it must be remembered that N^+ refers to an artificial datum or statistical origin and, as pointed out by Bossler (1973), comparisons between the estimate $\bar{\sigma}_c^2$ obtained via the pseudo-inverse and the mean variance of object space coordinates obtained from the variance-covariance matrix, $\Sigma_{\hat{X}}$, are therefore inequitable. However abstract the concept, N^+ provides the photogrammetrist with a minimum-trace variance-covariance matrix with which to assess the precision of both the object space coordinates of a minimally constrained network and desired functions of those coordinates. Although some may question the validity of the pseudo-inverse as a variance-covariance matrix, N^+ can be employed to compute meaningful precision estimates for certain linear combinations of **X** (e.g., Bossler, 1973; Sprinsky, 1976).

INNER CONSTRAINTS

For this investigation, the pseudo-inverse was computed by means of a singular value decomposition algorithm. Such a method of computation is rather expensive in terms of both computer time and storage, yet it does have the advantage of directly yielding the eigenvalues of **N** from which precision and reliability bounds can be formulated (e.g., Pelzer, 1979). A mathematically equivalent method of computing the estimates $\hat{X} = N^+A^TPL$ is by means of the use of inner constraints. Meissl (1969) has shown that a free network solution can be obtained by subjecting a given network to a Helmert transformation. That is, the solution conditions V^TPV = minimum, $\|\hat{X}\|$ = minimum, and $tr\Sigma_{\hat{X}}$ = minimum can be satisfied by subjecting the photogrammetric network to three rotations, a scale change, and three translations, all differentially small. This approach, which is fully developed in Blaha (1971), is subject to the same comments regarding the validity of the variance-covariance matrix as were made for N^+. Nevertheless, the method is perhaps the most straightforward of a number of techniques for implementing an inner constraint solution (Blaha (1980) discusses five mathematically equivalent approaches) as it involves simply "bordering" the singular normal matrix **N** with a transformation matrix **G** which satisfies the condition **AG** = 0. The solution vector $\hat{X}$ is then obtained as

$$\begin{bmatrix} \hat{X} \\ K_d \end{bmatrix} = \begin{bmatrix} N & G \\ G^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^TPL \\ 0 \end{bmatrix} \equiv \begin{bmatrix} N^+ & G(G^TG)^{-1} \\ (G_TG)^{-1}\,G^T & 0 \end{bmatrix} \begin{bmatrix} A^TPL \\ 0 \end{bmatrix} \quad (16)$$

where K_d is a vector of Lagrangian multipliers of length d.

With the relatively small networks encountered in close-range photogrammetry, a direct solution can be considered for the bordered normal equation system. Such an algorithm is the block Gaussian elimination for symmetric indefinite matrices developed by Bunch and Parlett (1971). In situations where the size of the network is such that a direct solution to Equation 16 is no longer feasible, the method of recursive partitioning offers an advantageous alternative approach (Blaha, 1980; Brown, 1980).

In the context of a photogrammetric network having defects of translation, orientation, and scale, **G** can be partitioned into the form

$$G^T = (G_o^T \;\; G_c{}^T) \quad (17)$$

where, for the exterior orientation parameters of photo i (Granshaw, 1980),

$$G_{0_i} = \begin{bmatrix} 1 & 0 & 0 & 0 & Z_i^c & -Y_i^c & X_i^c \\ 0 & 1 & 0 & & 0 & X_i^c & Y_i^c \\ 0 & 0 & 1 & Y_i^c & -X_i^c & 0 & Z_i^c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (18)$$

and, for the object sapce coordinates of point j,

$$G_{c_j} = \begin{bmatrix} 1 & 0 & 0 & & & -Y_j & X_j \\ 0 & 1 & 0 & -Z_j & & X_j & Y_j \\ 0 & 0 & 1 & Y_j & -X_j & 0 & Z_j \end{bmatrix} \quad (19)$$

If the network defect is less than seven, appropriate columns are suppressed from G_{0_i} and G_{c_j}. For example, if a priori information regarding the scale of the network has been incorporated in the bundle adjustment, column seven of **G** could be suppressed.

As a conceptually equivalent method to the pseudo-inverse approach, the imposition of inner constraints yields minimum $\bar{\sigma}_{\hat{X}}$ but not necessarily minimum $\bar{\sigma}_c$. In seeking to circumvent this problem, consider the case where $G_o = 0$. Here $tr\Sigma_c$ is minimized explicitly and thus the desired aim of achieving an optimum value of $\bar{\sigma}_c$ is realized. This approach has been taken one step further by Papo and Perelmuter (1980) who have reported a method whereby inner constraints are assigned only to a selected subset of the object space coordinates, thus yielding a mean standard error, $\bar{\sigma}_{c'}$, which is a minimum for this chosen coordinate subset. The authors state that such an approach provides a superior means for filtering out measurement errors. However, it is less than clear as to why this should be so, especially considering that the adjusted observables (photo coordinate measurements) are invariant with changes in the minimal constraint configuration.

In comparison to the normally adopted procedure of "fixing" seven coordinate values a priori, the approach suggested by Papo and Perelmuter (1980) does display advantages in the context of optimizing both the mean variance of object point coordinates and the precision of functions of these coordinates. Rather than "hanging" the network on three points, mean origin, scale, and orientation are determined from the provisional coordinates of c' selected points. An example of where such an approach may be applicable is in a close-range photogrammetric network comprising a closely grouped cluster of points of interest, plus outlying points which have simply been included to enhance the geometric strength of the network. The small test network discussed in the following section includes such a configuration.

Rather than adopting the more complex algorithmic formulation of Papo and Perelmuter (1980), it appears more generally practical to implement the inner constraint condition $\|\hat{\mathbf{X}}_{c'}\|$ = minimum for the chosen c' object points via the "bordering" technique described above. For a self-calibrating bundle adjustment, this would lead to an inner constraint matrix of the form

$$\mathbf{G}^{\mathrm{T}} = \underset{7\times 6m}{(\ 0}\ \ \underset{7\times 3c'}{\mathbf{G}^{\mathrm{T}}_{c'}}\ \ \underset{7\times 3(c-c')}{0}\ \ \underset{7\times a}{0\ )} \tag{20}$$

where m is the number of photos, c the number of object points and a the number of additional parameters. The form of **G** given by Equation 20 has been applied in an experiment conducted for this investigation.

Experimental Application

Description

The principal aim of the experiment conducted was to examine the extent of variations in the mean positional standard error of the object target point coordinates which accompany changes in the minimal constraint configuration in a small photogrammetric network adjustment. Both self-calibration and the standard bundle adjustment were to be considered. For the present investigation only absolute minimal constraints are considered because an analysis of the influence on the estimate $\bar{\sigma}_c$ of changes in both k^2 and the minimal control pattern has been previously reported in Fraser (1980).

A total of 17 photogrammetric adjustments of the same network were computed. A different minimal constraint was applied to each, the constraint being one of three distinct formulations:

- $\mathbf{P}_{X^o} = k^2\mathbf{P}_X$, with $\mathbf{P}_X$ being a diagonal matrix comprising seven non-zero elements, each of unit value, and $1/k$ = 0.0001 (i.e., standard error = 1 μm). This is equivalent to "fixing" seven coordinates, chosen here to be 13 combinations of 2 × X,Y,Z and 1 × Z, and one case of fixed exposure station exterior orientation parameters plus a scale constraint.
- Imposition of the condition $\|\hat{\mathbf{X}}\|$ = minimum. This approach includes the pseudo-inverse technique and the equivalent method of full-column inner constraints, Equation 16.
- Inner constraints applied only to the object space coordinates, i.e., $\|\hat{\mathbf{X}}_c\|$ = minimum; and imposition of the condition $\|\hat{\mathbf{X}}_{c'}\|$ = minimum for the subset of coordinates of c' object target points.

The experimental test network adopted is not atypical of an imaging geometry that might be employed in an engineering or industrial photogrammetric survey. Figure 1 illustrates the geometry of the network which comprised four photos and 12 object target points. Eight of the object points lie in the XY plane (Z = 0), with the remaining four being at a "height" Z of 36 cm. Four exposure stations were used, with the kappa rotation at each being 90° or 180° different from the values at the remaining three. All camera axes were directed toward the center of the target array, point number 14, and this gave rise to a convergence angle of 90° between camera axes 1 and 2. The camera axis at exposure station 3 was approximately normal to the XY plane. A Hasselblad MK70 camera with Biogon 60-mm lens was used for the photography, the focal setting being 200 cm. Preliminary photogrammetric data reduction involved the two-dimensional image transformation using eight reseaux as control, single photo resections to determine the refined approximate exterior orientation element values, and space intersections to determine the approximate object point coordinates to an accuracy of 0.1 to 0.2 cm.

In the self-calibrating bundle adjustments computed for the experiment, three inner cone parameters were carried as weighted unknowns: the principal distance parameter, dc, and the principal point coordinates, x_0 and y_0. These particular three additional parameters were carried not in the anticipation of significant accuracy improvements (the camera had previously been self-calibrated) but more to enable their influence on the magni-

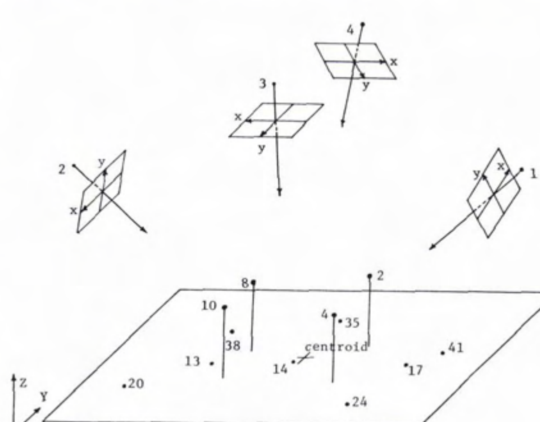

Fig. 1. Object target array and camera station configuration.

tude of $\bar{\sigma}_c$ to be evaluated. For all but four minimal control patterns, a standard bundle adjustment was also computed.

RESULTS

Image coordinates and additional parameters. As they represent the observables of the photogrammetric network, the adjusted photo coordinates and their residuals are invariant with respect to changes in the type and pattern of absolute minimal constraint. In the adjustments without additional parameters the RMS value of the image coordinate residuals was $s_x = s_y = \pm 2.3$ μm, whereas for the self-calibration adjustments $s_x = \pm 2.1$ μm and $s_y = \pm 2.3$ μm.

Estimates of the inner core parameters dc, x_0, and y_0, along with their respective standard errors, were also found to be invariant under changes in the minimal constraint. However, the correlation coefficients expressing the degree of coupling between the additional parameters and both the exterior orientation elements and the object point coordinates were subject to dramatic variation, as will be detailed in the following paragraphs.

Variations in minimal control. Table 1 lists the magnitude of the mean positional standard error, $\bar{\sigma}_c$, computed for each minimal constraint configuration, for both the self-calibration and standard bundle adjustments. Serials 1 to 14 of the table illustrate how the magnitude of $\bar{\sigma}_c$ is closely related to which minimal control point pattern is selected, i.e., to which seven object space parameters are held fixed.

The wide range of values of $\bar{\sigma}_c$ appearing in the table highlights the need for a careful choice of the minimal control pattern if one is to achieve an optimal value of the mean standard error of object point coordinates. No attempt is made here to establish more than a qualitative link between the magnitude of $\bar{\sigma}_c$ and the geometry of the three points which form the minimal control, as in view of the uniqueness of most close-range photogrammetric projects such an endeavor may prove futile. However, two observations which have been made regarding the relationship between the distribution of the minimal control and the magnitude of mean standard errors are that as the centroid of the control point triangle approaches the centroid of the object target array, $\bar{\sigma}_c$ tends to decrease in magnitude, and this decrease is also apparent as the area of the triangle increases.

For minimal control patterns Serials 1, 2, and 3, which appear to be "optimum" configurations, the a posteriori coordinate standard error of non-control points ranged in magnitude from 40 μm to 195 μm. Point 14 exhibited the highest precision with standard error estimates of 48, 40, and 80 μm for the X, Y, and Z coordinates, respectively, whereas for point 20 the corresponding values were 113, 75, and 195 μm. The latter point, which lay farthest from the centroid and was only imaged on three exposures, displayed the lowest positional accuracy.

For Serials 1 to 13 in Table 1 the magnitude of $\bar{\sigma}_c$ listed for each of the self-calibration adjustments is 2 to 5 μm greater than the value obtained

TABLE 1. ESTIMATES OF THE MEAN POSITIONAL STANDARD ERRORS $\bar{\sigma}_c$ AND $\bar{\sigma}_{c'}$ OF OBJECT POINT COORDINATES

Serial	Minimal control point configuration		a. With additional parameters	b. Without additional parameters
	"fixed" X,Y	"fixed" Z	$\lvert\bar{\sigma}_c\rvert$ (μm)	$\lvert\bar{\sigma}_c\rvert$ (μm)
1	13, 24	35	75	73
2	38, 41	24	75	71
3	13, 17	35	75	72
4	13, 17	24	79	77
5	38, 24	20	80	76
6	20, 41	35	83	—
7	35, 10	17	83	78
8	35, 24	41	88	—
9	4, 8	2	90	86
10	20, 24	14	102	99
11	13, 14	8	147	145
12	20, 38	35	161	—
13	41, 17	14	197	—
14	"fixed" $\omega, \phi, \kappa, X^c, Y^c, Z^c$, of camera station 3, and fixed scale		272	78
15	$\lVert\hat{\mathbf{X}}\rVert$ = minimum		106	57
16	$\lVert\hat{\mathbf{X}}_c\rVert$ = minimum		50	49
			$\lvert\bar{\sigma}_{c'}\rvert$	$\lvert\bar{\sigma}_{c'}\rvert$
17	$\lVert\hat{\mathbf{X}}_{c'}\rVert$ = minimum		40	39

in the corresponding standard bundle adjustment. This slight loss of mean precision is thought to be related to the determinancy of scale being less precise in the photogrammetric adjustments which incorporated the Gaussian focal length as a weighted unknown. Typically, in each of the self-calibration adjustments the correlation coefficient between the additional parameters and the X, Y, Z coordinates ranged in magnitude from 0.01 to 0.4. But, the coupling between the parameters dc and Z^c for exposure stations 3 and 4 was very significant, the correlation coefficient being about 0.97.

Whereas the standard bundle method yields a marginal improvement in mean positional standard error for Serials 1 to 13, the reduction in the magnitude of $\bar{\sigma}_c$ is significant in Serial 14. In this adjustment, in which the exterior orientation elements at camera station 3 were held fixed in addition to the distance from the exposure station to point 14, the standard error improved by a factor of three. One plausible explanation for this is that the minimally constrained photogrammetric network, Serial 14a, was numerically scale defective. The normal equation matrix corresponding to Serial 14a was not numerically positive-definite with λ_u and λ_{u-1}, the two smallest eigenvalues, both having values less than zero. It is worthy of note that in this case there was only limited coupling between the additional parameters and the exterior orientation elements, yet the correlation between the inner cone parameters and the object point coordinates ranged up to 0.98, a marked increase over the corresponding values for Serials 1 to 13.

Pseudo-inverse and full-column inner constraints. As can be seen in Table 1, the pseudo-inverse solution and the mathematically equivalent full-column inner constraint method (Equation 16) yield mean standard errors of magnitude 106 and 57 μm, respectively, for the self-calibration and standard bundle adjustments. In a similar manner to that just described, for Serial 15a the imposition of the inner constraint condition $\|\hat{\mathbf{X}}\|$ = minimum tends to significantly increase the correlation between the principal distance parameter, dc, and the X, Y, Z object point coordinates, thus inflating the magnitude of $\bar{\sigma}_c$. Correlation coefficient values reached 0.96 for numerous object space coordinates.

As regards the efficiency of computing $\mathbf{N}^+$ versus "bordering" the normal equation matrix, the latter approach seems to be superior in that both significantly less computation time and less storage are required.

Inner Constraint Condition $\|\hat{\mathbf{X}}_c\|$ = *Minimum.* From the results obtained in Serials 14a and 15a, the tentative conclusion can be drawn that if $\bar{\sigma}_c$ is to be minimized then the mechanisms of projective compensation between the additional parameters and the exterior orientation elements must be allowed to act freely in the self-calibration adjustment. A consequence of this will undoubtedly be a fall-off in the precision of both sets of these parameters, but so long as the solution is stable there needs to be only limited concern in the context of seeking an optimum value of $\bar{\sigma}_c$.

The imposition of minimal control, Serials 1 to 14, frees the exterior orientation elements and additional parameters from implicit constraints, as does the inner constraint condition $\|\hat{\mathbf{X}}_c\|$ = minimum. Serial 16 in Table 1 lists the values of $\bar{\sigma}_c$ obtained via the inner constraint approach with $\mathbf{G}_0$ = 0 (see Equation 17) for the self-calibration and standard bundle adjustments. The two estimates for the mean positional standard error are essentially equal, with $\bar{\sigma}_c = \pm 50$ μm. This value represents a considerable improvement in precision as compared to the results obtained in Serials 1 to 15.

Inner Constraint Condition $\|\hat{\mathbf{X}}_{c'}\|$ = *Minimum.* This condition, described by Equation 20, which explicitly yields $\bar{\sigma}_{c'}$ = minimum, was imposed for $c' = 8$ object target points; the outlying points 20, 24, 38, and 41 were not included in the sample. In Serial 17 $\bar{\sigma}_{c'}$ attains its optimum value of ± 40 μm, whereas $\bar{\sigma}_c = \pm 58$ μm for the same adjustment. For the sake of comparison, the mean positional standard error $\bar{\sigma}_{c'}$ for the eight inner-most targets has been computed for Serials 2, 3, 15b, and 16, the respective magnitudes being 75, 62, 48, and 46 μm.

Relative Accuracy. Thus far the discussion has been confined to an examination of the mean positional accuracy of object points with respect to an arbitrarily imposed reference coordinate system. A number of alternative measures of precision exist, which take into account the off-diagonal terms of the variance-covariance matrix $\mathbf{\Sigma}_c$. Here, one such indicator of precision, the standard error of a computed distance between two object target points, will be considered. This relative measure of network strength is invariant with respect to translation and rotation of the object space coordinate system, but its magnitude is dependent on the adopted minimal constraint configuration. As a general rule the distance standard error $\sigma_{d_{ij}}$ will approach a minimum as $\bar{\sigma}_c$ is minimized.

Table 2 lists the computed distance standard er-

TABLE 2. COMPUTED MAGNITUDES OF DISTANCE STANDARD ERRORS $\sigma_{l_{ij}}$(UNITS ARE μM)

Distance Serial	13–17 (99.9 cm)	4–8 (73.1 cm)
2	64	79
5	66	81
7	102	112
14a	325	231
15a	260	179
16	50	77
17	50	70

rors $\sigma_{d_{ij}}$ obtained from seven minimal constraint configurations for two distances, 13-17 and 4-8 (see, for example, Fraser (1981) for appropriate formulae). The optimum estimates for $\sigma_{d_{13,17}}$ and $\sigma_{d_{4,8}}$ are obtained by imposing the inner constraint condition $\|\hat{\mathbf{X}}_{c'}\|$ = minimum. However, the respective magnitudes of 50 and 70 μm are not substantially smaller than the corresponding values for Serial 2 (seven parameters "fixed") and Serial 16 ($\|\hat{\mathbf{X}}_c\|$ = minimum). In this case, for the added computational expense of the free network adjustments, reductions of only 14 and 9 μm resulted in the distance standard errors when compared to the values obtained in the adjustment with an "optimum" minimal control point pattern. This result tends to emphasize a well-known property of inner constraint adjustments: the trace of the parameter variance-covariance matrix is minimized at the expense of higher correlation between the parameters.

When expressed in terms of the ratio $|\sigma_{d_{ij}}| / d_{ij}$ the values of $\sigma_{d_{13,17}}$ and $\sigma_{d_{4,8}}$ obtained for Serial 17 represent relative measures of precision of 1 : 20,000 and 1 : 10,400, respectively.

Concluding Remarks

The results obtained in this investigation again support the contention that close-range photogrammetry is indeed a measuring technique which can yield high precision three-dimensional coordinates of target points situated on a structure, human body, etc. However, invariably the information sought by the user of photogrammetry is obtained by considering one or more functions of the coordinates. For example, in a deformation survey coordinate difference and/or displacement vector determinations are required. Other applications may require the computation of areas, volumes, distances, or displacements of points on a structure from their design surface positions.

Rather than seeking to optimize the mean positional accuracy of object target points with respect to the arbitrarily assigned datum, it may be more appropriate to consider the precision of the mathematical function of the X,Y,Z coordinates which is of interest; e.g., the standard error of a distance, as detailed in the previous section. For a particular linear function $\hat{\mathbf{Y}} = \mathbf{F}(\hat{\mathbf{X}})$, the precision of $\hat{\mathbf{Y}}$ is given according to the law of propagation of variances $\mathbf{\Sigma}_{\hat{Y}} = \mathbf{F}\, \mathbf{\Sigma}_{\hat{X}}\, \mathbf{F}^T$. From this investigation it may be concluded that, for the majority of functions $\mathbf{F}$ which are commonly utilized by photogrammetrists, an optimum $\mathbf{\Sigma}_{\hat{Y}}$ will only be obtained when the trace of a selected submatrix of $\mathbf{\Sigma}_{\hat{X}}$ is a minimum, i.e., when either $\bar{\sigma}_c$ or $\bar{\sigma}_{c'}$ is minimized.

In order to obtain this optimum precision, close attention must be paid to the selection of the type and configuration of the minimal constraint. The choice between "fixing seven parameters" and turning to free network methods is perhaps most dependent on the nature of the desired function $\mathbf{F}$ of the X,Y,Z coordinates. For those who consider that the free network methds produce a less than fully meaningful variance-covariance matrix of the parameters (note that $\mathbf{N}^+$ is meaningful for many linear combinations of $\hat{\mathbf{X}}$), the choice of a minimal constraint configuration boils down to which seven object space parameters are to be held "fixed." Here it has been demonstrated that this choice can greatly influence the overall precision of the minimally constrained close-range photogrammetric network adjustment.

References

Albert, A., 1972. *Regression and the Moore-Penrose Pseudoinverse.* Academic Press, New York, 180 p.

Baarda, W., 1967. *Statistical Concepts in Geodesy.* Netherlands Geodetic Commission, Publications on Geodesy, New Series, Vol. 2, No. 4, 54 p.

———, 1968. A Testing Procedure for Use in Geodetic Networks. *Netherlands Geodetic Commission,* New Series, Vol. 2, No. 5, 97 p.

Blaha, G., 1971. *Inner Adjustment Constraints with Emphasis on Range Observations.* Dept. of Geodetic Science Report No. 148, Ohio State University, Columbus, 85 p.

———, 1980. Discussion of Five Methods for Adjusting Free Networks. *Manuscripta Geodaetica* (to appear).

Bossler, J. D., 1972. *Bayesian Inference in Geodesy.* Ph.D. Dissertation, Ohio State University, Columbus, 79 p.

———, 1973. A Note on the Meaning of Generalized Inverse Solutions in Geodesy. *J. Geophys. Res.,* Vol. 78, pp. 2616-2617.

Brown, D. C., 1980. *Application of Close-Range Photogrammetry to Measurements of Structures in Orbit.* GSI Technical Report No. 80-012, Geodetic Services Inc., Melbourne, Florida, Vol. 1, 131 p.

Draper, N. R., and R. C. van Nostrand, 1979. Ridge Regression and James-Stein Estimation: Review and Comments. *Technometrics,* Vol. 21, pp. 451-466.

Ebner, H., and A. Grün, 1979. Accuracy Aspects of Bundle Adjustment Application to Cadastre and Network Densification. *Presented Paper, Aerial Triangulation Symposium,* University of Queensland, Brisbane, Oct. 15-17.

Förstner, W., 1979. On Internal and External Reliability of Photogrammetric Coordinates. *Proceedings of ASP Annual Meeting,* Washington, D.C., pp. 294-310.

———, 1980. The Theoretical Reliability of Photogrammetric Coordinates. *Presented Paper, XIV Congress of ISP, Comm. III,* Hamburg.

Fraser, C. S., 1980. On Variance Analysis of Minimally Constrained Photogrammetric Adjustments. *Aust. J. Geod. Phot. Surv.,* No. 33, pp. 39-56.

———, 1981. Accuracy Aspects of Multiple Focal Setting Self-Calibration Applied to Non-Metric Cameras. *Photogrammetria,* Vol. 36, pp. 121-132.

Granshaw, S. I., 1980. Bundle Adjustment Methods in

Engineering Photogrammetry. *Photogrammetric Record*, Vol. 10, pp. 181-207.

Grün, A., 1978. Accuracy, Reliability and Statistics in Close-Range Photogrammetry. *Presented paper, Inter-Congress Symposium of ISP, Comm. V*, Stockholm.

———, 1980. Precision and Reliability Aspects in Close-Range Photogrammetry. *Invited Paper, XIV Congress of ISP, Comm. V*, Hamburg.

Lindley, D. V., and A. F. M. Smith, 1972. Bayes Estimates for the Linear Model (with discussion). *J. Roy. Statist. Soc. Ser. B*, Vol. 34, pp. 1-41.

Luenberger, D. G., 1969. *Optimization by Vector Space Methods*. John Wiley, New York, 326 p.

Marquardt, D. W., 1970. Generalized Inverses, Ridge Regression, Biased Linear Estimation, and Non-Linear Estimation. *Technometrics*, Vol. 12, pp. 591-612.

Meissl, P., 1969. Zusammenfassung und Ausbau der inneren Fehlertheorie eines Punkthaufens. *Deutsche Geodätische Kommission*, Series A, 61, pp. 8-21.

Papo, H., and A. Perelmuter, 1980. Free Net Analysis of Storage Tank Calibration. *Presented paper, XIV Congress of ISP, Comm. V*, Hamburg.

Pelzer, H., 1979. Some Criteria for the Accuracy and Reliability of Networks. *Presented paper, XVII General Assembly of IUGG*, Canberra.

Pope, A. J., 1971. Transformation of Covariance Matrices due to Changes in Minimal Control. *Presented paper, AGU Fall Meeting*, San Francisco.

Pope, A. J., and R. H. Hanson, 1972. An Algorithm for the Pseudo-Inverse of Sparse Matrices. *Presented paper, AGU Spring Meeting*, Washington, D.C.

Sprinsky, W. H., 1976. *The Design of Special-Purpose Horizontal Geodetic Control Networks*. Defense Mapping Agency, Technical Report, DMA 76-003, Washington, D.C.

Welsch, W., 1979. A Review of the Adjustment of Free Networks. *Survey Review*, Vol. XXV, pp. 167-180.

(Received 29 April 1981; revised and accepted 13 November 1981)

Forthcoming Articles

William Befort and *Joseph J. Ulliman*, Dot-Grid Area Measurement on Panoramic Photographs.
P. D. Carman, Sensitometry in Canadian Aerial Survey.
Jule Caylor, *John Pierce*, and *Walter Salazar*, Optical Bar Panoramic Photography for Planning Timber Salvage in Drought-Stressed Forests.
William M. Ciesla, *Ray A. Allison*, and *Frederick P. Weber*, Panoramic Aerial Photography in Forest Pest Management.
R. D. Dillman and *W. B. White*, Estimating Mountain Pine Beetle-Killed Ponderosa Pine Over the Front Range of Colorado with High Altitude Panoramic Photography.
H. H. Elfick and *M. J. Fletcher*, The Qasco SD-4.
Lawrence W. Fritz, Testing Procedures for Analytical Plotters.
Clifford W. Greve, APPS-IV, Improving the Basic Instrument.
William R. Hafker and *Warren R. Philipson*, Landsat Detection of Hardwood Forest Clearcuts.
Alex Hasler, The Analytical Stereoplotter Wild Aviolyt BC 1.
Michael Jackson, *Clifford Greve*, *George Hoffman*, and *David Ackerman*, A Parameterization of the Itek KA-80-A Panoramic Camera.
J. A. Kirchner, *S. Youkhana*, and *J. A. Smith*, Influence of Sky Radiance Distribution on the Ratio Technique for Estimating Bidirectional Reflectance.
William H. Klein, Estimating Bark Beetle-Killed Lodgepole Pine with High Altitude Panoramic Photography.
Donald G. Leckie, An Error Analysis of Thermal Infrared Line-Scan Data for Quantitative Studies.
Richard L. Liston, Photogrammetric Methods for Mapping Resource Data from High Altitude Panoramic Photography.
Jack McCarthy, *Charles E. Olson, Jr.*, and *John A. Witter*, Evaluation of Spruce-Fir Forests Using Small-Format Photographs.
Timothy P. Roberts and *Hans J. Wehrli*, A Unique Concept in Analytical Plotters.
Charles J. Robinove, Computation with Physical Values from Landsat Digital Data.
George H. Rosenfield, Sample Design for Estimating Change in Land Use and Land Cover.
Richard H. Seymour, US-2 Analytical Stereoplotter.
Howard Turner, The Canadian Marconi ANAPLOT II System.
J. E. Unruh and *E. M. Mikhail*, Mensuration Tests Using Digital Images.
Kenneth C. Winterberger and *Karl M. Hegg*, A Low-Cost Photo-Scale Protractor for Small Scale Photography.
Sherman S. C. Wu and *Francis J. Schafer*, Photogrammetry of the Viking Lander Imagery.