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Free Net Analysis in Close-Range Photogrammetry

The basic principles are developed and an example is given.

Abstract: *Close-range photogrammetry is analyzed following free new adjustment principles. A free net constraints elimination method, developed by the second author, combined with scale and leveling constraints is applied to the solution of object space coordinates. The results are free of distortions imposed normally by "hard point" adjustment, while their covariance matrix serves to correctly estimate the accuracy of the adjusted point coordinates and exterior orientation elements. A numerical example is provided comparing a free net solution with a "hard point" minimum constraints solution.*

Introduction

Close-range photogrammetry is similar to classical topographic photogrammetry in many respects, in particular when the solution is carried out analytically. Aerial triangulation by mono- or stereocomparator measurements and subsequent bundle adjustment calls for much the same treatment as in a large number of engineering problems treated by close-range photogrammetry.

An important and fundamental difference between the two photogrammetries is in the definition and realization of datum. In topographic photogrammetry the datum is realized through the superior, in accuracy, geodetic control points which are kept fixed or are heavily weighted in the solution. In engineering close-range photogrammetry, control is provided usually by precise measurements which are of a distinctly differential nature. Distances and elevation differences are measured by precise surveying methods between points which appear on the photographs. Occasionally, differences between linear exterior orientation elements of the camera stations are measured as partial control. There are cases where some of the angular exterior orientation elements can also be determined.

If the end result of the adjustment is the coordinates of points in object space, there is invariably a need in close-range photogrammetry to "complete" the datum by assigning, more or less arbitrarily, weights to a number of points or to exterior orientation elements. If the number of those complementary datum point coordinates or orientation elements is kept to a minimum (minimum constraints), the adjustment is acceptable and no distortions are introduced into the estimated parameters, although their covariance matrix depends on the particular choice of the datum quantities. If the constraints are not minimal, the solution is overconstrained, the sum-of-squares of the residuals is invariably larger, and there are definite distortions introduced by the adjustment process into the adjusted quantities. The measurements, which are weighted by the inverse of their variances, have to accommodate to the more or less arbitrarily chosen datum hard points.

Our conclusion is that, as far as datum is concerned, close-range photogrammetry has a built-in deficiency as compared to topographic photogrammetry. It can't rely for datum and for correction of systematic residual errors on geodetic control.

Another characteristic which has to be considered is that in close-range photogrammetry we are usually interested in relative positions of points, i.e., we can be satisfied with a partial datum which includes scale and sometimes also the direction of the vertical. However, as we may

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be interested in using adjustment procedures (and computer programs) which solve for point coordinates rather than for their differences, there is a straight-forward solution to the datum problem in close-range photogrammetry; that is, applying free net adjustment principles together with whatever differential control measurements have been made.

In this paper we present the ideas, mathematics, and adjustment procedure developed for the solution of typical close-range photogrammetry problems by free net adjustment. At present we are applying the above method for the calibration of cylindrical storage tanks by stereophotogrammetry. The same approach could, and in our opinion should, be applied to other close-range photogrammetry problems. Thus, the built-in datum problem of close-range photogrammetry can be turned to its advantage by providing a superior means for filtering out measurement errors and for obtaining a more realistic covariance matrix of the estimated quantities.

Adjustment of Free Networks

The basic property of a free net adjustment is that the trace of the covariance matrix of the estimated parameters is a minimum. There are two approaches for the solution of a free net: The first is based on generalized matrix algebra (Bjerhammar, 1973; Meissl, 1969; Mittermayer, 1972) while the second approach is based on classical adjustment methods as in (Koch, 1978; Wolf, 1972). The method presented in this section and applied by us for the solution of our close-range photogrammetry problem belongs to the second group and has been published in Perelmuter (1978, 1979, 1980) as the free net constraint elimination method.

The linearized observation equations for the solution of point coordinates in three-dimensional space are

$$\underset{n\times 1}{\mathbf{V}} = \underset{n\times m}{\mathbf{A}} \cdot \underset{m\times 1}{\mathbf{X}} - \underset{n\times 1}{\mathbf{L}} . \tag{1}$$

Denoting the rank of $\mathbf{A}$ by $\mathrm{R}(\mathbf{A})$, we write

$$\mathrm{R}(\mathbf{A}) = m - d = r$$

where d is the rank defect of $\mathbf{A}$. In our case d represents the number of datum quantities needed to define the network in three-dimensional space.

The vector $\mathbf{X}$ can be partitioned into $\underset{r\times 1}{\mathbf{X}_1}$ and $\underset{d\times 1}{\mathbf{X}_2}$ in a way such that $\mathbf{X}_2$ is a set of parameters of size d which complements the datum definition of the net. It should be pointed out that the partitioning $\mathbf{X}_1\,\mathbf{X}_2$ is not unique, i.e., there are many $\mathbf{X}_2$ sets in $\mathbf{X}$ which could fulfill the datum definition. Equation 1 can be written now as follows:

$$\mathbf{V} = [\mathbf{A}_1\ \mathbf{A}_2]\cdot\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix} - \mathbf{L} \tag{1$'$}$$

As is well known (see Grafarend and Schaffrin (1976) and Welsch (1979)), an unbiased estimate of parameters in an adjustment process is possible only if the $\mathbf{A}$ matrix of the observation equations is of full rank, i.e., $d = 0$. If $d \neq 0$, an unbiased solution can be obtained only for $r = m - d$ parameters. The above can be achieved by transforming the original observation equations through the introduction of a linear relationship between $\mathbf{X}_1$ and $\mathbf{X}_2$ as, for example,

$$\mathbf{X}_2 = \mathbf{G}_1^{\mathrm{T}}\cdot\mathbf{X}_1 \tag{2}$$

which is equivalent to

$$\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix} = \begin{bmatrix}\mathbf{I}\\ \mathbf{G}_1^{\mathrm{T}}\end{bmatrix}\cdot\mathbf{X}_1 \tag{3}$$

We substitute Equation 3 into Equation 1′ and obtain

$$\mathbf{V} = [\mathbf{A}_1\ \mathbf{A}_2]\cdot\begin{bmatrix}\mathbf{I}\\ \mathbf{G}_1^{\mathrm{T}}\end{bmatrix}\cdot\mathbf{X}_1 - \mathbf{L} \tag{4}$$

written also as

$$\mathbf{V} = \overset{*}{\mathbf{A}}\cdot\mathbf{X}_1 - \mathbf{L} \tag{5}$$

where

$$\overset{*}{\mathbf{A}} = \mathbf{A}_1 + \mathbf{A}_2\cdot\mathbf{G}_1^{\mathrm{T}}$$

and

$$\mathrm{R}(\overset{*}{\mathbf{A}}) = r \qquad \text{– a full rank}$$

It should be pointed out that Equation 1′ together with Equation 2 form in effect a case of observation equations with conditions between the unknowns. As shown by Mittermayer (1972), the condition of minimum trace of the covariance matrix of $\mathbf{X}$ is equivalent to the condition of $\mathbf{X}^{\mathrm{T}}\cdot\mathbf{X} = \min$. One way of obtaining a minimum for $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ is by defining the matrix $\mathbf{G}_1^{\mathrm{T}}$ as shown in Perelmuter (1978); i.e.,

$$\mathbf{G}_1^{\mathrm{T}} = (\mathbf{A}_2^{\mathrm{T}}\cdot\mathbf{A}_1)\cdot(\mathbf{A}_1^{\mathrm{T}}\cdot\mathbf{A}_1)^{-1} \tag{6}$$

from which we have also

$$\mathbf{A}_2 = \mathbf{A}_1\cdot\mathbf{G}_1 \tag{6$'$}$$

Substituting Equation 6′ into Equation 5, we obtain finally

$$\mathbf{V} = \bar{\mathbf{A}}\cdot\mathbf{X}_1 - \mathbf{L} \tag{7}$$

where

$$\bar{\mathbf{A}} = \mathbf{A}_1\cdot(\mathbf{I} + \mathbf{G}_1\cdot\mathbf{G}_1^{\mathrm{T}}) = \mathbf{A}_1\cdot\mathbf{S}$$

The unbiased estimate of $\mathbf{X}_1$ and its weight coefficients matrix are obtained from

$$\begin{aligned}\mathbf{X}_1 &= (\bar{\mathbf{A}}^{\mathrm{T}}\cdot\mathbf{P}\cdot\bar{\mathbf{A}})^{-1}\cdot(\bar{\mathbf{A}}^{\mathrm{T}}\cdot\mathbf{P}\cdot\mathbf{L})\\ \mathbf{Q}_{11} &= (\bar{\mathbf{A}}^{\mathrm{T}}\cdot\mathbf{P}\cdot\bar{\mathbf{A}})^{-1} = \mathbf{S}^{-1}\cdot(\mathbf{A}_1^{\mathrm{T}}\cdot\mathbf{P}\cdot\mathbf{A}_1)^{-1}\cdot\mathbf{S}^{-1}\end{aligned} \tag{8}$$

According to Equation 2, $\mathbf{X}_2$ and its weight matrix are obtained following the solution of $\mathbf{X}_1$ and $\mathbf{Q}_{11}$ from

$$\begin{aligned}\mathbf{X}_2 &= \mathbf{G}_1^{\mathrm{T}}\cdot\mathbf{X}_1\\ \mathbf{Q}_{22} &= \mathbf{G}_1^{\mathrm{T}}\cdot\mathbf{Q}_{11}\cdot\mathbf{G}_1\end{aligned} \tag{9}$$

The problem treated above is a particular case of a more general situation where we seek a minimum for only a part of the $\mathbf{X}^{\mathrm{T}}\cdot\mathbf{X}$ sum. That would mean partitioning $\mathbf{X}_1$ into $\mathbf{X}_{11}$ and $\mathbf{X}_{12}$ so that the minimum condition applies to the following sum:

$$\mathbf{X}_{12}\cdot\mathbf{X}_{12}^{\mathrm{T}} + \mathbf{X}_2^{\mathrm{T}}\cdot\mathbf{X}_2 = \min$$

The new form of the observation and condition equations would be (see also Papo and Perelmuter (1981)):

$$\begin{bmatrix}\mathbf{V}\\ \\ \mathbf{O}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_2\\ & & \\ \mathbf{O} & \overset{*}{\mathbf{G}}{}_1^{\mathrm{T}} & -\mathbf{I}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{X}_{11}\\ \\ \mathbf{X}_{12}\\ \mathbf{X}_2\end{bmatrix} - \begin{bmatrix}\mathbf{L}\\ \\ \mathbf{O}\end{bmatrix} \tag{10}$$

The $\overset{*}{\mathbf{G}}{}_1^{\mathrm{T}}$ matrix is treated in the following subsection. Considerations which could guide us in the partitioning of $\mathbf{X}_1$ are discussed in the section on Applications to Close-Range Photogrammetry as well as in Wolf (1977) and in Papo and Perelmuter (1981).

Equation 6 can be regarded as a general method for computing $\mathbf{G}_1^{\mathrm{T}}$.

In cases where the parameters $\mathbf{X}$ are corrections to point coordinates in three-dimensional space, there is another approach for the evaluation of $\mathbf{G}_1^{\mathrm{T}}$ or rather $\overset{*}{\mathbf{G}}{}_1^{\mathrm{T}}$ which is geometrically meaningful (see also Koch (1978), Mittermayer (1972), and Perelmuter (1978)).

According to Meissl (1969) a free network can be obtained from a given arbitrary network by a Helmert transformation, consisting of three translations, three rotations, and a scale change which are all differentially small. The Helmert transformation matrix for a network of u points and all seven degrees of freedom is as follows:

$$\mathbf{C}^{\mathrm{T}} = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 1 & 0 & 0 & 1\\
0 & -z_1 & y_1 & 0 & -z_2 & y_2 & 0 & \cdots & y_{u-1} & 0 & -z_u & y_u\\
z_1 & 0 & -x_1 & z_2 & 0 & -x_2 & z_3 & \cdots & -x_{u-1} & z_u & 0 & -x_u\\
-y_1 & x_1 & 0 & -y_2 & x_2 & 0 & -y_3 & \cdots & 0 & -y_u & x_u & 0\\
x_1 & y_1 & z_1 & x_2 & y_2 & z_2 & x_3 & \cdots & z_{u-1} & x_u & y_u & z_u
\end{bmatrix} \tag{11}$$

The first three rows are associated with translations along the x, y, z axes, respectively; the next three to rotations around the x, y, z axes, respectively; and the last row to scale change. According to Meissl (1969) a free net adjustment in all seven degrees of freedom satisfies the condition $\mathbf{C}^{\mathrm{T}}\cdot\mathbf{X} = 0$ which, when added to the original observation equations (Equation 1), results in the following adjustment system:

$$\begin{bmatrix}\mathbf{V}\\ \\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_2\\ & & \\ \mathbf{C}_{11}^{\mathrm{T}} & \mathbf{C}_{12}^{\mathrm{T}} & \mathbf{C}_2^{\mathrm{T}}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{X}_{11}\\ \mathbf{X}_{12}\\ \mathbf{X}_2\end{bmatrix} - \begin{bmatrix}\mathbf{L}\\ \\ \mathbf{O}\end{bmatrix} \tag{12}$$

The number of rows taken from the full $\mathbf{C}^{\mathrm{T}}$ matrix (Equation 11) is equal to the defect d of Equation 1 and their identity is determined by the particular degrees of freedom of the network. We note also that $\mathbf{C}^{\mathrm{T}}$ is partitioned into $\mathbf{C}_{11}^{\mathrm{T}}$ which is set to zero, $\mathbf{C}_{12}^{\mathrm{T}}$ and $\mathbf{C}_2^{\mathrm{T}}$ related to $\mathbf{X}_{12}$ and $\mathbf{X}_2$, respectively. $\mathbf{C}_2^{\mathrm{T}}$ is square and nonsingular. Its columns are chosen so that they can remove the rank defect of the original observation equations. The second row of equations in Equation 12 can be multiplied from the left by the negative inverse of $\mathbf{C}_2^{\mathrm{T}}$, the result being identical to Equation 10, where

$$\overset{*}{\mathbf{G}}{}_1^{\mathrm{T}} = -(\mathbf{C}_2^{\mathrm{T}})^{-1}\cdot\mathbf{C}_1^{\mathrm{T}} \tag{13}$$

As shown in the next section, we have chosen this approach for forming our $\overset{*}{\mathbf{G}}{}_1^{\mathrm{T}}$ matrix. The free net constraints elimination method can be characterized by its flexibility in allowing us the choice of leaving out part of the $\mathbf{X}^{\mathrm{T}}\cdot\mathbf{X} = \min$ condition, by the ease of forming the $\overset{*}{\mathbf{G}}{}_1^{\mathrm{T}}$ matrix and finally by the reduction in the overall size of the normal matrix to be inverted: i.e., $(m - d)\times(m - d)$ instead of $(m + d)\times(m + d)$.

Applications to Close-Range Photogrammetry

Let us consider a typical close-range photogrammetry situation composed of p camera stations, u points in object space, and a number of control measurements (distances and leveling) involving k out of the u points.

As indicated in the previous section and also in Meissl (1969), a network in three-dimensional space has seven degrees of freedom. If the above control measurements are incorporated in the adjustment system as observations with certain weights corresponding to their variances, we can easily see that part of the quantities needed to define the datum are provided by the control measurements, namely:

- Scale is defined by the distances measured;
- orientation in space of the z (vertical axis) is defined by the leveling measurements.

Thus, the balance of datum quantities, d, still needed to define completely the network in space is as follows:

- three quantities for defining the datum origin;
- one quantity for defining the orientation of the x,y axes (a rotation around z).

According to the above we can easily select the relevant four rows from the Helmert transformation matrix $\mathbf{C}^T$, i.e., rows 1, 2, 3, and 6. We apply the free net condition ($\mathbf{X}^T \cdot \mathbf{X} = \min$) to the k points only, mainly from practical considerations associated with computer programming.

Due to the control measurements between the k points, the portion of the normal matrix pertaining to their coordinates is a full matrix and so the folding-in technique employed for the rest of the $(u-k)$ points (see Papo and Perelmuter (1980)) cannot be applied. By limiting the application of free net constraints to the same k points, the size of the matrix to be inverted ($3k \times 3k$) does not increase but rather is reduced in size down to $(3k - d) \times (3k - d)$.

The observation and condition equations of our problem are written following partitioning of the unknowns as suggested in the previous section.

$$\begin{matrix} n_1 \\ n_2 \\ d \end{matrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \mathbf{O} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ \mathbf{O} & \mathbf{O} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ \mathbf{O} & \mathbf{O} & \overset{*}{\mathbf{G}}{}_3^T & -\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \mathbf{X}_4 \end{bmatrix} - \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \\ \mathbf{O} \end{bmatrix} \tag{14}$$

where

- row 1 represents the observation equations of the comparator measurements to which a weight matrix $\mathbf{P}_1$ is assigned;
- row 2 represents the control measurement observation equations with $\mathbf{P}_2$ weight matrix;
- row 3 represents the free net conditions as applied to $\mathbf{X}_3$ and $\mathbf{X}_4$ only;
- $\mathbf{X}_1$ are corrections to the $6p$ exterior orientation elements of the p camera stations;
- $\mathbf{X}_2$ are the $3(u - k)$ corrections to point coordinates;
- $\mathbf{X}_3$, $\mathbf{X}_4$ are the $3k$ corrections to the coordinates of the points on which the $\mathbf{X}^T \cdot \mathbf{X}$ = minimum condition is applied;
- $\mathbf{X}_4$ are the d preselected datum definition quantities; and
- n_1, n_2 are the respective numbers of comparator and control measurements.

The first step in the solution is the transformation of the system from biased into an unbiased (full rank) system as follows:

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \bar{\mathbf{A}}_{13} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{23} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix} - \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix} \tag{15}$$

where

$$\bar{\mathbf{A}}_{13} = \mathbf{A}_{13} + \mathbf{A}_{14} \cdot \overset{*}{\mathbf{G}}{}_3^T \tag{15'}$$

and

$$\bar{\mathbf{A}}_{23} = \mathbf{A}_{23} + \mathbf{A}_{24} \cdot \overset{*}{\mathbf{G}}{}_3^T$$

Next, the normal equations are formed

$$\begin{bmatrix} \mathbf{A}_{11}^T \cdot \mathbf{P}_1 \cdot \mathbf{A}_{11} & \mathbf{A}_{11}^T \cdot \mathbf{P}_1 \cdot \mathbf{A}_{12} & \mathbf{A}_{11}^T \cdot \mathbf{P}_1 \cdot \bar{\mathbf{A}}_{13} \\ \mathbf{A}_{12}^T \cdot \mathbf{P}_1 \cdot \mathbf{A}_{11} & \mathbf{A}_{12}^T \cdot \mathbf{P}_1 \cdot \mathbf{A}_{12} & \mathbf{O} \\ \bar{\mathbf{A}}_{13}^T \cdot \mathbf{P}_1 \cdot \mathbf{A}_{11} & \mathbf{0} & \bar{\mathbf{A}}_{13}^T \cdot \mathbf{P}_1 \cdot \bar{\mathbf{A}}_{13} + \bar{\mathbf{A}}_{23}^T \cdot \mathbf{P}_2 \cdot \bar{\mathbf{A}}_{23} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}^T \cdot \mathbf{P}_1 \cdot \mathbf{L}_1 \\ \mathbf{A}_{12}^T \cdot \mathbf{P}_1 \cdot \mathbf{L}_1 \\ \bar{\mathbf{A}}_{13}^T \cdot \mathbf{P}_1 \cdot \mathbf{L}_1 + \bar{\mathbf{A}}_{23}^T \cdot \mathbf{P}_2 \cdot \mathbf{L}_2 \end{bmatrix} \tag{16}$$

The normal equations can be written now as follows:

$$\begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{21}^T & \mathbf{N}_{31}^T \\ \mathbf{N}_{21} & \mathbf{N}_{22} & \mathbf{O} \\ \mathbf{N}_{31} & \mathbf{O} & \mathbf{N}_{33} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{bmatrix} \tag{16'}$$

where $\mathbf{N}_{11}$, $\mathbf{N}_{33}$, $\mathbf{N}_{21}$, $\mathbf{N}_{31}$ are in general full matrices and $\mathbf{N}_{22}$ is a block diagonal matrix with 3×3 blocks and quite large overall dimensions.

The system of normal equations is solved by a double fold-in operation as shown in Papo and Shmutter (1978). The results are the $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4$ adjusted unknowns and their respective variance-covariance matrices. We note that the particular pattern of the normal matrices, where $\mathbf{N}_{22}$ is block diagonal (as is usual in a bundle adjustment), has not been disturbed by the introduction of the free net constraints.

As a demonstration of the above procedure, we simulated the following simple situation:

A horizontal level area of 2.5 by 2.5 m was photographed by a terestrial metric camera, which resulted in two normal photographs with 60 percent overlap. Principal distance of the camera was taken to be 100 mm. Six points marked on the ground were numbered according to Figure 1. Points 1 and 2 were chosen so as to lie on the verticals passing through the two exposure centers. The vertical distance from the exposure centers to the ground was 1.1 m.

Ground control measurements were performed as follows:

- horizontal distances were measured between the points 1-2, 2-3, and 2-5; and
- leveling was performed between the points 1-2, 1-3, and 2-5.

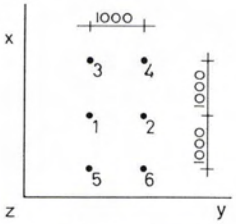

FIG. 1. Test area

The standard deviations of the simulated measurements were as follows:

- comparator $\sigma_x = \sigma_y = 10\ \mu m$
- distances $\sigma_l = 0.1$ mm.
- leveling $\sigma_h = 0.01$ mm.

The variance of unit weight, taken as 100, was used to calculate the respective weights.

The observation equations, formed according to Equation 14, have dimensions evaluated from the following:

$$p = 2;\ u = 6;\ k = 4;\ n_1 = 12;\ n_2 = 6;\ d = 4.$$

The free net constraints are applied to points 1, 2, 3, and 5. The datum definition quantities are x_1, y_1, z_1, and y_3. The $\overset{*}{G}{}_3^T$ matrix is evaluated according to Equation 13 resulting in

Sum of X_3, X_4 variances:

Free Net Solution: 335.16
Hard Point Solution: 838.49

TABLE 1. VARIANCES FROM FREE NET AND HARD POINT SOLUTIONS

		Photo 1 FN	Photo 1 HP	Photo 2 FN	Photo 2 HP
X_1	x_0	403.0	306.2	287.3	358.2
	y_0	461.5	489.6	519.5	557.0
	z_0	122.5	123.1	277.0	277.7
	κ	0.338	0.945	0.525	1.229
	ϕ	1.472	1.472	0.734	0.734
	ω	3.297	3.297	3.474	3.474

	i	x_i FN	x_i HP	y_i FN	y_i HP	z_i FN	z_i HP
X_2	4	520.3	611.0	218.0	237.7	573.1	573.1
	6	520.3	595.9	218.0	406.4	574.0	575.0
X_3	2	19.18	158.9	24.36	55.2	0.375	0.999
	5	89.79	195.1	15.61	230.2	0.874	1.995
	3	89.79	195.1			0.875	0.999
X_4	3			15.61	0.0		
	1	53.45	0.0	24.87	0.0	0.374	0.0

$$\overset{*}{G}{}_3^T = -\begin{bmatrix} x_3 & -y_1 & x_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -y_2 & x_2 & 0 & -y_5 & x_5 & 0 & -y_3 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\overset{*}{G}{}_3^T = \frac{1}{x_1 - x_3} \cdot \begin{bmatrix} y_1 - y_2 & x_2 - x_1 & 0 & y_1 - y_5 & x_5 - x_1 & 0 & y_1 - y_3 & 0 \\ x_3 - x_1 & 0 & 0 & x_3 - x_1 & 0 & 0 & x_3 - x_1 & 0 \\ y_2 - y_1 & x_3 - x_2 & 0 & y_5 - y_1 & x_3 - x_5 & 0 & y_3 - y_1 & 0 \\ 0 & 0 & x_3 - x_1 & 0 & 0 & x_3 - x_1 & 0 & x_3 - x_1 \end{bmatrix}$$

The normal matrices were formed and solved by APL on an IBM 370/168 computer using the double fold-in technique.

A second minimum constraints solution was performed, based on the same measurements. Instead of free net constraints, the datum was defined by fixing (hard points) the x_1, y_1, z_1, and y_3 coordinates.

Table 1 shows the covariance matrix diagonal elements (variances) of the X_1, X_2, X_3, and X_4 unknowns. As expected, according to Meissl (1969), the free net solution has a covariance matrix of X_3 and X_4 with a significantly smaller trace.

Variances in the table are in
$mm^2 \times 10^{-4}$ for $x_0, y_0, z_0, x_i, y_i, z_i$ and
radians squared $\times 10^{-8}$ for κ, ϕ, ω.

CONCLUSIONS

We have demonstrated theoretically and also by the above numerical example that close-range photogrammetry combined with precise surveying measurements and processed by free net adjustment techniques is an extremely powerful tool which can and should be applied to a wide range of engineering problems.

The residuals of a free net adjustment can best disclose the existence of certain unmodeled systematic effects. Additional studies should be conducted on the application of free net principles for the solution of photogrammetric systems which include self-calibration parameters. It appears that high correlations between parameters or their combinations could be effectively treated by free net adjustment principles.

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- Utilization of digitally processed data related to Earth resources.

Crop inventory techniques including spectral characterization, species identification, area estimation, condition assessment, and yield prediction will be emphasized.

Plenary session themes are (I) The Role of Remote Sensing in Meeting Crop Production Information Needs, and, (II) Issues and Perspectives in Earth Observation and Resource Information Systems.

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