ATSUSHI OKAMOTO
Transportation Engineering
Kyoto University
606 Kyoto-shi, Sakyo-ku, Yoshidahonmachi, Japan

Orientation and Construction of Models

Part IV: Further Considerations in Close-Range Photogrammetry

The characteristics of lens distortion and film deformation are investigated in the orientation problem of photographs, both for the general case and for the special cases in close-range photogrammetry.

INTRODUCTION

IN THE ANALYSIS of photographs taken with non-metric cameras, the correction of lens distortion and film deformation is one of the inevitable problems. However, very little has been written that would provide a general and rigorous approach to the orientation problem of pictures containing such distortions. Hallert[1] investigated the properties of lens distortion in the orientation problem of a stereo pair of photographs and clarified that parameters of a functional form to model non-linear lens distortion can be determined from the coplanarity condition of corresponding rays. From the fact that, in two-media photogrammetry, parameters describing a boundary surface can be obtained[2,3] from the coplanarity condition of corresponding imaging rays in water, Hallert's finding is also reasonable, because the elements of non-linear lens distortion as well as those of the boundary surface belong to non-central projective parameters. On the other hand, the characteristics of linear film deformation, linear systematic errors in coordinate measurement, and other linear distortions have been studied by, e.g., Abdel-Aziz and Karara[4-6] and by others. Also, elements of these linear deformations have been revealed to be absorbed by the 11 coefficients of the general collinearity equations, which means that these parameters and the general photogrammetric orientation elements of a photograph cannot be provided separately.

ABSTRACT: *Measured photo coordinates usually include two types of systematic errors: linear deformation and non-linear distortion. Film deformation, linear systematic errors in coordinate measurement, and other linear distortions belong to the former, parameters of which are absorbed by the 11 coefficients of the general collinearity equations. On the other hand, lens distortion and deformation due to irregular film surface are involved in the latter, whose elements are non-central projective parameters. In the orientation problem of a stereo pair of photographs, these two types of systematic errors have different properties such that parameters describing linear deformation must be obtained from both the coplanarity condition of corresponding rays and object points given, while those of non-linear distortion can be calculated only from the coplanarity condition. This paper describes the orientation problem of pictures having lens distortion and film deformation for some important cases in close-range photogrammetry.*

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In this paper, the orientation theories derived in the previous paper[7] (Part I) are extended, based on the known results above, to the case where photographs have lens distortion and film deformation, because parameters of such distortions were neglected in the previous discussions. Also, the least-squares solution by means of the (determination) equations is explained.

The discussions given here are purely mathematical, and all parametes considered in this paper may not be determined accurately in practical examples on account of correlation between these elements. Such practical characteristics of the methods proposed will be clarified through future experiments.

The symbolism in this paper follows that in earlier parts, in particular, that used in Part I[7].

General Considerations of Film and Lens Distortions

Characteristics of the General Affine Transformation

The general affine transformation between original corrdinates (x,y) and the transformed coordinates (x',y') is described in the form

$$x' = a_1x + a_2y + a_3 \qquad y' = a_4x + a_5y + a_6 \tag{115}$$

in which $a_i (i = 1, \ldots, 6)$ denotes independent coefficients. This relationship is difficult to consider projectively between two planes in a three-dimensional space. From the fact that the affine transformation includes an orthogonal projective transformation as a special case, we can, however, construct the relationship (Equation 115) photogrammetrically in a three-dimensional space as follows.[4-6]

After development, a film may be deformed with different scale factors along the x and y directions, respectively. Designating a picture point at the exposure instant as $p(x,y)$ and that after film development as $p'(x',y')$, the relationship between $p(x,y)$ and $p'(x',y')$ is expressed as

$$x' = \lambda_x x \qquad y' = \lambda_y y \tag{116}$$

where λ_x and λ_y indicate scale factors along the x and y directions, respectively. Also, in the general case where the x_c-y_c plane of the comparator coordinate system (x_c,y_c,z_c) is assumed not to be parallel to the picture plane (see Figure 1 in Part I[7]), the relationship between $p'(x',y')$ and its measured image point $p_c(x_c,y_c)$ can be obtained from Equation 5 (in Part I[7]) and takes the following form

$$x_c = d_{11}x' + d_{12}y' - d_{13}c + x_{co} \qquad y_c = d_{21}x' + d_{22}y' - d_{23}c + y_{co}. \tag{117}$$

By substituting Equation 116 into Equation 117, we have

$$x_c = a_1x + a_2y + a_3 \qquad y_c = a_4x + a_5y + a_6. \tag{118}$$

Since Equations 116 and 117 have eight parameters $(\lambda_x,\lambda_y,\alpha,\beta,\gamma,x_{co},y_{co},c)$ (though only five elements among $\alpha,\beta,\gamma,x_{co},y_{co},c$ are independent), all coefficients $a_i (i = 1, \ldots, 6)$ in Equation 118 are mathematically independent. It follows that the relationship between a picture point $p(x,y)$ at the exposure instant and the measured image point $p_c(x_c,y_c)$ of $p'(x',y')$ on the deformed film is identical to the general affine transformation (Equation 115).

Equation 118 can be rewritten in the form

$$x_c = a_1x + a_2y - a_3'c \qquad y_c = a_4x + a_5y - a_6'c \tag{119}$$

By expressing Equation 119 in matrix notation, we obtain

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3' \\ a_4 & a_5 & a_6' \end{pmatrix} \begin{pmatrix} x \\ y \\ -c \end{pmatrix} \tag{120}$$

Also, by substituting Equation 3a (in Part I[7]) into Equation 120, the relationship relating a photographed object point $P(X,Y,Z)$ and its measured image point $p_c(x_c,y_c)$ of $p'(x',y')$ on the deformed film is given in the form of Equation 7 (in Part I[7]). Furthermore, since it contains 13 parameters $(\phi,\omega,\kappa,X_o,Y_o,Z_o,c,a_1,a_2,a_3',a_4,a_5,a_6')$, all coefficients $A_i (i = 1, \ldots, 11)$ in Equation 7 are also mathematically independent in this case. Consequently, this relationship is also identical to the general collinearity equations (Equation 1 in Part I[7]).

From the fact that the parameters (λ_x,λ_y) of linear deformation are absorbed by the 11 coefficients of the general collinearity equations, we can select three among 14 photogrammetric elements $(\phi,\omega,\kappa, X_o,Y_o,Z_o,x_{co},y_{co},c,\alpha,\beta,\gamma,\lambda_x,\lambda_y)$ arbitrarily in the orientation problem of a photograph. Thus, $\phi,\omega,\kappa,X_o,Y_o, Z_o,x_{co},y_{co},c,\alpha,\beta$ can be taken as the 11 general photogrammetric orientation parameters of a picture having linear film deformation under the assumption that

$$\lambda_x = \lambda_y = 1, \text{ and } \gamma = 0.$$

CONSIDERATION OF LENS DISTORTION

Systematic errors of picture coordinates are mainly due to film deformation and lens distortion. We have already discussed the properties of the former in the previous paragraph and have seen that the coefficients of functional form to model linear film deformation are absorbed by those of the general collinearity equations. On the other hand, non-linear lens distortion is expressed in the form[8,9]

$$\begin{aligned} \Delta x &= x(k_1r^2 + k_2r^4 + k_3r^6) + p_1(r^2 + 2x^2) + 2p_2xy \\ \Delta y &= y(k_1r^2 + k_2r^4 + k_3r^6) + 2p_1xy + p_2(r^2 + 2y^2) \end{aligned} \tag{121}$$

where x,y indicate ideal photo coordinates and

$$r^2 = x^2 + y^2.$$

The coefficients (k_1,k_2,k_3,p_1,p_2) of $\Delta x,\Delta y$ are non-central projective elements, because the functional form is non-linear with respect to the picture coordinates (x,y) and thus they cannot be absorbed by the 11 coefficients of the general collinearity equations. The general collinearity condition, including non-linear lens distortion, will be discussed as follows:

The photo coordinates (x_d,y_d) having non-linear lens distortion are described at the exposure instant as

$$\begin{aligned} x_d &= x + \Delta x \\ y_d &= y + \Delta y. \end{aligned} \tag{122}$$

After development of the film, the photo coordinates become

$$\begin{aligned} x_d' &= \lambda_x(x + \Delta x) \\ y_d' &= \lambda_y(y + \Delta y). \end{aligned} \tag{123}$$

Furthermore, in the general case (see Figure 1 in Part I[7]), the relationship between $p_d'(x_d',y_d')$ and its measured image point $p_c(x_c,y_c)$ can also be expressed by an orthogonal transformation in the form

$$\begin{aligned} x_c &= d_{11}x_d' + d_{12}y_d' - d_{13}c + x_{co} \\ y_c &= d_{21}x_d' + d_{22}y_d' - d_{23}c + y_{co}. \end{aligned} \tag{124}$$

By substituting Equation 123 into Equation 124, we get

$$\begin{aligned} x_c &= a_1(x + \Delta x) + a_2(y + \Delta y) + a_3 \\ y_c &= a_4(x + \Delta x) + a_5(y + \Delta y) + a_6. \end{aligned} \tag{125}$$

Also, considering that linear film deformation is comparatively small and the $x_c - y_c$ plane of the comparator coordinate system (x_c,y_c,z_c) is nearly parallel to the picture plane, as is in practical examples, we can assume that

$$\begin{aligned} a_1\Delta x + a_2\Delta y &\simeq \Delta x \\ a_4\Delta x + a_5\Delta y &\simeq \Delta y. \end{aligned}$$

Thus, Equation 125 can be reduced to

$$\begin{aligned} x_c - \Delta x &= a_1x + a_2y + a_3 \\ y_c - \Delta y &= a_4x + a_5y + a_6. \end{aligned} \tag{126}$$

Consequently, for the case where non-linear lens distortion is not negligibly small, the general collinearity condition can be given as

$$x_c - \Delta x = \frac{A_1 X + A_2 Y + A_3 Z + A_4}{A_9 X + A_{10} Y + A_{11} Z + 1}$$
$$y_c - \Delta y = \frac{A_5 X + A_6 Y + A_7 Z + A_8}{A_9 X + A_{10} Y + A_{11} Z + 1}. \quad (127)$$

It follows from Equation 127 that the parameters (k_1,k_2,k_3,p_1,p_2) of non-linear lens distortion must be determined from object points given in the general orientation problem of individual photographs, while they can be obtained from the coplanarity condition of corresponding rays in that of a stereo pair. Also, it will be noted that non-linear film deformation has the same behavior as non-linear lens distortion.

The Special Case

This chapter treats the orientation problem of photographs for a special case where the $x_c - y_c$ plane of the comparator coordinate system (x_c,y_c,z_c) is parallel to the picture plane and the film is deformed with different scale factors along the x and y directions, respectively.

Orientation problem of individual photographs

In this case, the relationship between $p'(x',y')$ on the deformed film and its measured image point $p_c(x_c,y_c)$ can be given in the form

$$x_c = (\cos\gamma)x' + (\sin\gamma)y' + x_H$$
$$y_c = -(\sin\gamma)x' + (\cos\gamma)y' + y_H \quad (128)$$

in which x_H,y_H indicates the principal point coordinates. By substituting Equation 116 into Equation 128, we obtain

$$x_c = (\lambda_x\cos\gamma)x + (\lambda_y\sin\gamma)y + x_H$$
$$y_c = -(\lambda_x\sin\gamma)x + (\lambda_y\cos\gamma)y + y_H. \quad (129)$$

Also, we can assume that γ is equal to zero in considering the relationship between a photographed object point and its measured image point. Thus, Equation 129 can be simplified to

$$x_c - x_H = \lambda_x x$$
$$y_c - y_H = \lambda_y y. \quad (130)$$

Further, by substitution of the conventional collinearity equations (Equation 3 in Part I[7]) into Equation 130, we have[10]

$$x_c - x_H = -c_x \frac{e_{11}(X - X_o) + e_{12}(Y - Y_o) + e_{13}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)}$$
$$y_c - y_H = -c_y \frac{e_{21}(X - X_o) + e_{22}(Y - Y_o) + e_{23}(Z - Z_o)}{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o)} \quad (131)$$

in which

$$c_x = c\lambda_x, \quad c_y = c\lambda_y.$$

Because the relationship (Equation 131) between a photographed object point $P(X,Y,Z)$ and its measured image point $p_c(x_c,y_c)$ has only ten independent parameters $(\phi,\omega,\kappa,X_o,Y_o,Z_o,x_H,y_H,c_x,c_y)$, it can not be treated as the general collinearity equations in the orientation problem of a photograph. This means that there is one constraint[11] between the 11 coefficients $A_i (i = 1, \ldots, 11)$, if we express Equation 131 in the form of Equation 7. Thus, in the orientation problem of individual photographs, these ten photogrammetric parameters will be calculated directly by linearizing Equation 131 with respect to them. Five points are mathematically required in the object space for the unique determination of the ten orientation elements. Containing non-linear lens distortion, the number of object points required increases to eight.

Orientation problem of a stereo pair of photographs

This paragraph discusses the orientation problem of a stereo pair of photographs having linear film deformation, based on the orientation theory for the usual case in close-range photogrammetry (see Part

I[7]). A stereo model is constructed in a three-dimensional space by means of the coplanarity condition of corresponding rays $\mathbf{g}_1$ and $\mathbf{g}_2$ (see Figure 29). We will first investigate which central projective one-to-one correspondence is valid between the model and object spaces. According to Part I[7], the four-dimensional central projective transformation (Equation 10) is valid between the fictitious three-dimensional film (x,y,z) and the object space (X,Y,Z) in the case where actual films for the stereo pair are not deformed. With film deformation, as in this case, the fictitious three-dimensional film is assumed to be deformed with different scale factors along the x, y, and z directions, respectively. Thus, the relationship may be described between an ideal picture point $p(x,y,z)$ and the measured image point $p_c(x_c,y_c,z_c)$ of $p'(x',y',z')$ on the deformed three-dimensional film (fictitious) in the form

$$\begin{aligned} x_c - x_H &= \lambda_x x \\ y_c - y_H &= \lambda_y y \\ z_c - z_H &= \lambda_z z \end{aligned} \tag{132}$$

under the assumption that the three-dimensional plane (x_c, y_c, z_c) of the fictitious comparator coordinate system (x_c,y_c,z_c,t_c) is parallel to the three-dimensional deformed picture plane and θ equals zero. By substituting Equation 10 (in Part I[7]) into Equation 132, we have

$$\begin{aligned} x_c - x_H &= -c_x \frac{e_{11}(X - X_o) + e_{12}(Y - Y_o) + e_{13}(Z - Z_o) + e_{14}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)} \\ y_c - y_H &= -c_y \frac{e_{21}(X - X_o) + e_{22}(Y - Y_o) + e_{23}(Z - Z_o) + e_{24}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)} \\ z_c - z_H &= -c_z \frac{e_{31}(X - X_o) + e_{32}(Y - Y_o) + e_{33}(Z - Z_o) + e_{34}(0 - T_o)}{e_{41}(X - X_o) + e_{42}(Y - Y_o) + e_{43}(Z - Z_o) + e_{44}(0 - T_o)} \end{aligned} \tag{133}$$

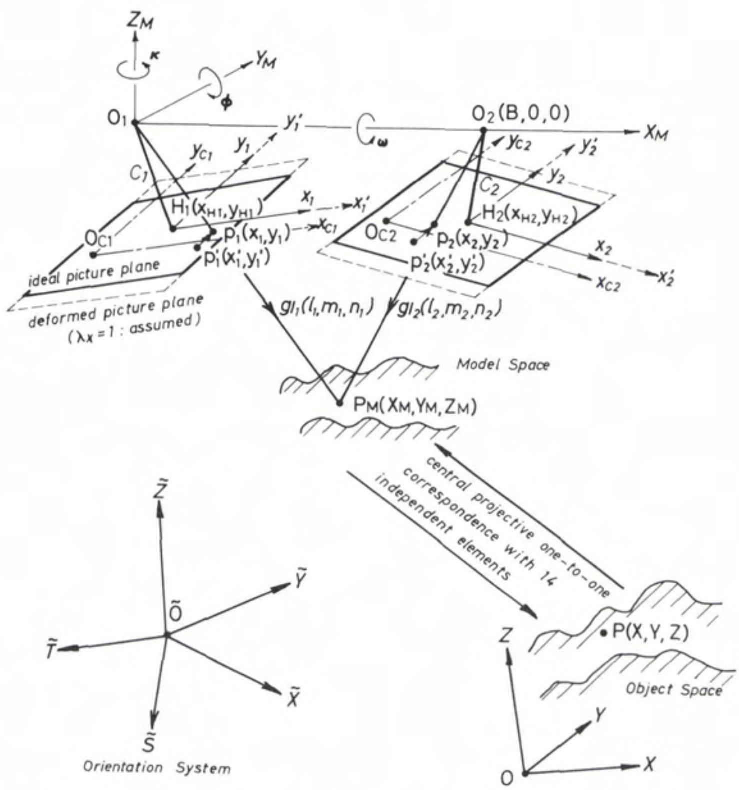

FIG. 29. Orientation problem of a stereo pair of photographs having linear film deformation.

in which

$$c_x = \lambda_x c, \quad c_y = \lambda_y c, \quad c_z = \lambda_z c.$$

Also, this relationship (Equation 133) has 14 independent parameters ($\phi,\omega,\kappa,\mu,X_o,Y_o,Z_o,T_o,x_H,y_H,z_H,c_x,c_y,c_z$). Thus, four points with the space coordinates given and one point with the planimetric coordinates known are mathematically necessary for the unique determination of the one-to-one correspondence (Equation 133).

From the above discussions, it can be easily understood that the one-to-one correspondence with 14 independent parameters is valid between the model space (X_M,Y_M,Z_M) and the object space (X,Y,Z), if a stereo pair of photographs (actual) are deformed with different scale factors along the x and y directions, respectively, and the $x_c - y_c$ plane (actual) of the comparator coordinate system (x_c,y_c,z_c) is parallel to the picture plane. Accordingly, the coplanarity condition of corresponding rays provides mathematically six independent orientation parameters, because the stereo pair of photographs have 20 independent elements to be determined.

The 20 orientation parameters will be calculated as follows. It can be seen from Equation 131 that one element of c,λ_x,λ_y can be taken arbitrarily. Thus, we may select $\phi,\omega,\kappa,X_o,Y_o,Z_o,x_H,y_H,c,\lambda_y$ as the 10 independent orientation parameters of a photograph to be determined under the assumption that λ_x is equal to one. The relationship (Equation 130) between an ideal picture point $p(x,y)$ and the measured image point $p_c(x_c,y_c)$ of $p'(x',y')$ on the deformed film is then rewritten in the form

$$\begin{aligned} x_c - x_H &= x \\ y_c - y_H &= \lambda_y y. \end{aligned} \tag{134}$$

Also, the (reduced) transformed picture coordinates ($_M\bar{X}_p,{}_M\bar{Y}_p,{}_M\bar{Z}_p$) of the ideal picture point can be expressed as

$$\begin{pmatrix} _M\bar{X}_p \\ _M\bar{Y}_p \\ _M\bar{Z}_p \end{pmatrix} = \mathbf{D}_\phi \mathbf{D}_\omega \mathbf{D}_\kappa \begin{pmatrix} x_c - x_H \\ \dfrac{y_c - y_H}{\lambda_y} \\ -c \end{pmatrix} \tag{135}$$

By means of Equation 135 we can construct the equations of corresponding rays $\mathbf{g}_1$ and $\mathbf{g}_2$ (Equations 25 and 26 in Part I[7]) in the model coordinate system (X_M,Y_M,Z_M), and derive the coplanarity condition (Equation 35 in Part I[7]) which contains, however in this case, 13 orientation parameters ($\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2,x_{H1},y_{H1},c_1,\lambda_{y1},x_{H2},y_{H2},c_2,\lambda_{y2}$). It must be noted that ($\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2$) are selected as the conventional relative orientation elements.

From the coplanarity condition of corresponding rays, only six independent orientation parameters can be provided and a stereo model is constructed in a three-dimensional space of a multi-dimensional space having more than four dimensions (see Figure 29). The constructed stereo model is, however, not similar to the object, because the three-dimensional similarity transformation is not satisfied between both spaces. In order to calculate, directly, the traditional photogrammetric orientation elements, it is necessary to make the model and object similar by introducing the similarity condition (see Part I[7]). In other words, we must construct the stereo model in the same three-dimensional space as that of the object by using the coplanarity condition of corresponding rays and the similarity condition between the model and object spaces simultaneously.

The similarity condition in this case will be discussed as follows. For the unique determination of the central projective one-to-one correspondence between the model and object spaces, four points with the space coordinates (X,Y,Z) given and one point with the planimetric coordinates (X,Y) known are mathematically required. These five points are equivalent to five points, four of which lie on a plane, as is demonstrated in Figure 30. Since the degrees of freedom of such five points is eight, the similarity condition can be constructed by setting up Equation 37 (in Part I[7]) for eight corresponding line segments in the model and object spaces. Thus, seven independent equations are obtained. Accordingly, we have 13 independent equations (six coplanarity equations plus seven equations based on the similarity condition) for the unique determination of the 13 orientation parameters included in the coplanarity condition of corresponding rays. Then, a stereo model similar to the object is constructed, and it can be transformed into the object space by means of the three-dimensional similarity transformation (Equation 38 in Part I[7]).

In the case where used camera lens is distorted, the parameters describing non-linear lens distortion can be obtained from the coplanarity condition of corresponding rays.

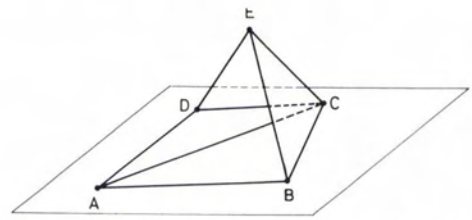

FIG. 30. Degrees of freedom of five points, four of which lie on a plane.

PARTICULAR CONSIDERATIONS IN CLOSE-RANGE PHOTOGRAMMETRY

As for cameras used in close-range photogrammetry, the particular case may sometimes occur in which only the planimetric coordinates of the principal point are known, or where only the principal distance is given. We will discuss the orientation problem of a stereo pair of photographs taken with such cameras.

PREPARATIONS

First, the central projective one-to-one correspondence between a flat terrain and the picture taken with a non-metric camera (lens distortion and film deformation are not considered) will be discussed. The transformation between these two planes (X,Y) and (x_c,y_c) is given by means of Equation 96 (in Part III[12]). Thus, this relationship has eight independent photogrammetric parameters $(\phi,\omega,\kappa,X_o,Y_o,Z_o,x_H,y_H)$, because Z_o and c cannot be provided separately for the case of a flat object space. Furthermore, this one-to-one correspondence can be divided into two transformations: a two-dimensional similarity transformation with four independent parameters and a (central projective) transformation with four independent elements into the same two-dimensional space as that of the object. The four parametes of the latter are classified into two groups: elements (x_H,y_H) determining the principal point coordinates and parameters (ϕ,ω) related to the scale of individual picture points.

The four-dimensional central projective one-to-one correspondence (Equation 55 in Part I[7]) includes 12 independent parameters $(\phi,\omega,\kappa,\mu,X_o,Y_o,Z_o,T_o,x_H,y_H,z_H,c)$ and is also divided into two transformations: a three-dimensional similarity transformation with seven independent elements and a (central projective) transformation with five independent parameters into the same three-dimensional space as that of the object. This can be explained more precisely as follows. The exterior orientation elements of the fictitious three-dimensional picture (the model) include eight parameters $(\phi,\omega,\kappa,\mu,X_o,Y_o,Z_o,T_o)$, and its interior orientation elements include four parameters (x_H,y_H,z_H,c). The case will first be discussed where the interior orientation parameters are known. By regarding $\phi,\omega,\kappa,X_o,Y_o,Z_o,T_o$ $(=m)$ as the conventional absolute orientation parameters, the remaining one element, μ, is considered to pertain to the scale of individual picture points on the fictitious three-dimensional photograph (the model). Also, the fictitious three-dimensional picture becomes similar to the object by determining only this element, μ. For the case where the interior orientation parameters are not given, we must determine five elements (x_H,y_H,z_H,c,μ) so as to make both spaces similar.

Because the seven parameters of the three-dimensional similarity transformation are given only by exterior orientation elements of a stereo pair of photographs (actual), the five elements of the (central projective) transformation of the model into the same three-dimensional space as that of the object are considered to be functions of only their interior orientation elements $(x_{H1},y_{H1},c_1,x_{H2},y_{H2},c_2)$. Also, the five parameters (x_H,y_H,z_H,c,μ) of the latter are classified into two groups: elements (x_H,y_H,z_H) defining the principal point coordinates of the fictitious three-dimensional picture and parameters (c,μ) pertaining to the scale of the individual picture points. Furthermore, among the six interior orientation parameters $(x_{H1},y_{H1},c_1,x_{H2},y_{H2},c_2)$ of the stereo pair of photographs (actual), $(x_{H1},y_{H1},x_{H2},y_{H2})$ indicate the principal point coordinates and c_1,c_2 are related to the scale. Consequently, the five elements (x_H,y_H,z_H,c,μ) can be expressed as

$$
\begin{aligned}
x_H &= x_H(x_{H1},y_{H1},x_{H2},y_{H2}) \\
y_H &= y_H(x_{H1},y_{H1},x_{H2},y_{H2}) \\
z_H &= z_H(x_{H1},y_{H1},x_{H2},y_{H2}) \\
c &= c(c_1,c_2) \\
\mu &= \mu(c_1,c_2).
\end{aligned}
\tag{136}
$$

THE FIRST CASE

This paragraph treats the case where the principal point coordinates are given to a stereo pair of photographs. First, we will discuss which central projective one-to-one correspondence is valid between the model and object spaces, based on the properties of the four-dimensional central projective transformation (Equation 55 in Part I[7]). It can be easily understood from Equation 136 that independent parameters included in Equation 55 become nine elements $(\phi,\omega,\kappa,\mu,X_o,Y_o,Z_o,T_o,c)$ in this case. Accordingly, the coplanarity condition of corresponding rays provides mathematically five independent elements (the relative exterior orientation parameters in conventional photogrammetry), because we must determine 14 independent parameters in the orientation problem of the stereo pair.

The traditional orientation elements of the stereo pair are calculated by introducing the similarity condition between the model and object spaces into the model construction process. Also, the similarity condition is described as follows. For the unique determination of the one-to-one correspondence between the model and object spaces, three points with the space coordinates (X,Y,Z) known must be mathematically given in the object space. As the degrees of freedom of three points is three, we must apply Equation 37 (in Part I[7]) to three corresponding line segments in both spaces so as to make them similar. Thus, two independent equations can be obtained as the similarity condition. Consequently, we have seven independent equations (five coplanarity equations plus two equations based on the similarity condition) during the phase of model construction in the same three-dimensional space as that of the object. By solving these seven equations with respect to the seven orientation unknowns $(\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2,c_1,c_2)$, a stereo model can be constructed which is similar to the object.

THE SECOND CASE

When the principal distance is given for a stereo pair of photographs, 16 parameters must be determined in the orientation problem. Ten elements among them can be obtained from the one-to-one correspondence between the model and object spaces, because two elements (c,μ) of the fictitious three-dimensional photograph are considered to be given in this case (see Equation 136). Thus, the coplanarity condition of corresponding rays determines six independent orientation elements.

The similarity condition in this case will be explained as follows. We must have three points with the space coordinates (X,Y,Z) given and one point with a coordinate known for the unique determination of the one-to-one correspondence between the model and object spaces. Also, because these four points are equivalent to four points, three of which lie on a straight line (see Figures 31), the degrees of freedom of those four points is four. Thus, we can construct the similarity condition by setting up Equation 37 (in Part I[7]) for four corresponding line segments in the model and object spaces. By solving the coplanarity condition (six independent equations) and the similarity condition (three independent equations) with respect to the nine orientation unknowns $(\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2,x_{H1},y_{H1},x_{H2},y_{H2})$ included in the coplanarity condition, a stereo model can be constructed in the same three-dimensional space as that of the object.

From the discussions given in this chapter, it follows that one of the principal point coordinates $(x_{H1},y_{H1},x_{H2},y_{H2})$ of a stereo pair of photographs can be mathematically obtained from the coplanarity condition of corresponding rays. (Koelbl[13] noted that the position of the principal point in one direction can be determined from the coplanarity condition of corresponding rays by means of convergent pictures.)

Determination of Unknown Orientation Parameters

In addition to the systematic errors due to lens distortion and film deformation discussed in the preceding chapters, measured image coordinates (x_c,y_c) may include linear lens distortion and linear systematic measurement errors caused by the non-orthogonality of the x_c and y_c axes of the comparator coordinate system (x_c,y_c,z_c). In such cases, the relationship between an ideal picture point $p(x,y)$ and the measured image point $p_c(x_c,y_c)$ of $p'(x',y')$ on the deformed film can also be given[4-6] by the general

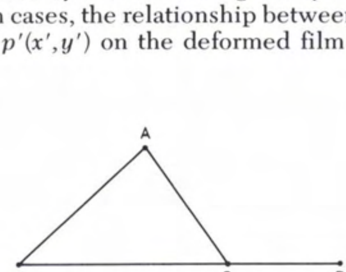

FIG. 31. Degrees of freedom of four points, three of which lie on a straight line.

affine transformation (reduced) (Equation 126), even when the x_c-y_c plane of the comparator coordinate system (x_c, y_c, z_c) is parallel to the deformed picture plane. This chapter describes some techniques to calculate the coefficients $A_i(i = 1, \ldots, 11)$ of the general collinearity equations and the parameters $(k_1, k_2, k_3, p_1, p_2)$ describing non-linear lens distortion simultaneously, and also to obtain, directly, the 11 general photogrammetric orientation parameters $(\phi, \omega, \kappa, X_o, Y_o, Z_o, x_{co}, y_{co}, c, \alpha, \beta)$ of a photograph with the elements of non-linear lens distortion together.

DLT METHOD FOR INDIVIDUAL PHOTOGRAPHS

The Direct Linear Transformation (DLT) method of solving the general collinearity condition of photogrammetry was developed by Abdel-Aziz and Karara in 1971. However, the solution by means of Equation 127 was not fully explained in their paper.[4] Hence, a direct linear solution approach will be discussed in this paragraph (cf. Wong[14]).

The functional form to model non-linear lens distortion can be rewritten as

$$\begin{aligned} \Delta x &= f_1 k_1 + f_2 k_2 + f_3 k_3 + f_4 p_1 + f_5 p_2 \\ \Delta y &= g_1 k_1 + g_2 k_2 + g_3 k_3 + g_4 p_1 + g_5 p_2 \end{aligned} \tag{137}$$

where

$$f_i = f_i(x,y), \qquad g_i = g_i(x,y)$$

Also, ideal picture coordinates (x,y) are given approximately as

$$x = x_c - \bar{x}_H, \qquad y = y_c - \bar{y}_H.$$

in which $(\bar{x}_H, \bar{y}_H)$ denote the first approximations of the coordinates of the principal point referred to the comparator coordinate system. Thus, the following expressions can be derived directly from Equation 127:

$$\begin{aligned} &XA_1 + YA_2 + ZA_3 + A_4 - (Xx_c)A_9 - (Yx_c)A_{10} - (Zx_c)A_{11} \\ &+ (f_1D)k_1 + (f_2D)k_2 + (f_3D)k_3 + (f_4D)p_1 + (f_5D)p_2 - x_c = 0 \\ &XA_5 + YA_6 + ZA_7 + A_8 - (Xy_c)A_9 - (Yy_c)A_{10} - (Zy_c)A_{11} \\ &+ (g_1D)k_1 + (g_2D)k_2 + (g_3D)k_3 + (g_4D)p_1 + (g_5D)p_2 - y_c = 0 \end{aligned} \tag{138}$$

in which

$$D = A_9X + A_{10}Y + A_{11}Z + 1.$$

The unknown parameters (A_9, A_{10}, A_{11}) also appear in the coefficients of the non-linear lens distortion elements $(k_1, k_2, k_3, p_1, p_2)$ in Equation 138. Therefore, rigorously, we cannot treat Equation 138 as linear equations with respect to $A_i(i = 1, \ldots, 11)$ and k_1, k_2, k_3, p_1, p_2. However, by introducing approximations of A_9, A_{10}, A_{11} into the coefficients of the non-linear lens distortion elements, Equation 138 can be regarded as a pair of linear equations. The solution is performed iteratively in the following way: In the first step, we find the first approximations of A_9, A_{10}, A_{11}, calculate $D_i(i = 1, \ldots, n$ (the number of object points given)) with those parameters, and then set up Equation 138 for the object points. Solving these equations with respect to the 16 unknowns by a least-squares adjustment of indirect observations, we get the second approximations of A_9, A_{10}, A_{11}. In the second step, the same is repeated. The iteration is carried out until discrepancies between the approximations and obtained solutions of A_9, A_{10}, A_{11} become negligibly small.

ALGEBRAIC METHOD FOR A STEREO PAIR OF PHOTOGRAPHS

The fundamental equations for the general orientation problem of a stereo pair of pictures having lens distortion and film deformation are the (reduced) general collinearity equations (Equation 127) between a photographed object point $P(X,Y,Z)$ and its measured image point $p_c(x_c, y_c)$. In the algebraic approach (see Part I[7]), the coefficients ${}_iA_j(i = 1,2; j = 1, \ldots, 11)$ of the general collinearity equations and the parameters $(k_{i1}, k_{i2}, k_{i3}, p_{i1}, p_{i2})$ $(i = 1,2)$ describing non-linear lens distortion are calculated directly and simultaneously for the stereo pair of pictures. The (determination) equations are given by means of

Equations 19 and 20, if we replace $x_{c1}, y_{c1}, x_{c2}, y_{c2}$ in Equations 19 and 20 by $x_{r1}, y_{r1}, x_{r2}, y_{r2}$, which are given as

$$x_{r1} = x_{c1} - \Delta x_1, \qquad y_{r1} = y_{c1} - \Delta y_1$$
$$x_{r2} = x_{c2} - \Delta x_2, \qquad y_{r2} = y_{c2} - \Delta y_2.$$

Equation 20 (modified) provides mathematically 15 among the 22 coefficients of the general collinearity equations for the stereo pair, if five points are given in the object space. On the other hand, the remaining seven coefficients and the ten parameters describing non-linear lens distortion can be obtained from Equation 19 (modified) by setting up Equation 19 for 17 corresponding measured image coordinates.

Since the (determination) equations are non-linear with respect to the unknowns, we must first linearize both equations and find the solution by means of an iterative approach. In the case of more than five points given in the object space, the least-squares adjustment with conditions (the general case)[15] (or the least-squares adjustment by conditions having unknowns[16]) may be applied, because Equations 19 and 20 have different qualities.

GEOMETRIC METHOD FOR A STEREO PAIR OF PHOTOGRAPHS

The general orientation problem of a photograph having non-linear lens distortion and linear film deformation can also be solved by using the 11 general photogrammetric orientation parameters $(\phi,\omega,\kappa,X_o,Y_o,Z_o,x_{co},y_{co},c,\alpha,\beta)$ and the elements of non-linear lens distortion instead of $A_i (i = 1, \ldots, 11)$ of the general collinearity equations and those of non-linear lens distortion in the algebraic method. In this geometric approach (see Part I[7]) a stereo model is first constructed by means of the coplanarity condition of corresponding rays and the similarity condition between the model and object spaces. The parameters determined in this process are the relative (exterior) orientation elements $(\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2)$, the interior orientation parametrs $(x_{coi},y_{coi},c_i,\alpha_i,\beta_i)$ $(i = 1,2)$, and those $(k_{i1},k_{i2},k_{i3},p_{i1},p_{i2})$ $(i = 1,2)$ of nonlinear lens distortion. The coplanarity condition is given by means of Equation 35 (in Part I[7]). Also, the (reduced) transformed picture coordinates $({}_M\bar{X}_p, {}_M\bar{Y}_p, {}_M\bar{Z}_p)$ are described as

$$\begin{pmatrix} {}_M\bar{X}_p \\ {}_M\bar{Y}_p \\ {}_M\bar{Z}_p \end{pmatrix} = \mathbf{D}_\phi \mathbf{D}_\omega \mathbf{D}_\kappa \begin{pmatrix} (x_r - x_{co} + c \sin\alpha \cos\beta) \sec\alpha - (y_r - y_{co} - c \sin\beta) \tan\alpha \tan\beta \\ (y_r - y_{co} - c \sin\beta) \sec\beta \\ -c \end{pmatrix} \tag{139}$$

in which

$$x_r = x_c - \Delta x, \qquad y_r = y_c - \Delta y.$$

The coplanarity condition of corresponding rays determines mathematically the seven (general) relative orientation parameters $(\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2$, one of $(x_{co1},y_{co1},x_{co2},y_{co2})$, and one of $(\alpha_1,\beta_1,\alpha_2,\beta_2))$ and the ten elements $(k_{i1},k_{i2},k_{i3},p_{i1},p_{i2})$ $(i = 1,2)$ of non-linear lens distortion. On the other hand, the similarity condition (Equation 37 in Part I[7]) provides the eight remaining interior orientation elements of the stereo pair, if nine distances (lengths) are given in the object space.

If we treat m (model scale) as an unknown in the model construction process, a stereo model congruent to the object can be constructed, which is sometimes the final product in close-range photogrammetry. In the case of more than nine distances given, the least-squares solution by means of the coplanarity condition of corresponding rays and the similarity condition between the model and object spaces is performed iteratively with the adjustment by conditions having unknowns[16], because these two conditions are non-linear with respect to the 25 parameters to be determined. Furthermore, the constructed stereo model can be transformed into the object space by means of the three-dimensional similarity tranformation (Equation 38 in Part I[7])

Concluding Discussion

The properties of lens distortion and film deformation have been investigated in the orientation problem of pictures, and some new interesting facts have been revealed:

- All parameters of linear systematic deformations of a photograph are absorbed by the 11 coefficients of the general collinearity equations.[4–6] Thus, the general orientation problem of pictures having linear systematic errors can also be solved by using the 11 general photogrammetric orientation parameters $(\phi,\omega,\kappa,X_o,Y_o,Z_o,x_{co},y_{co},c,\alpha,\beta)$ for a photograph.
- In the case where a picture is deformed with different scale factors along the x and y directions, respectivley, and the x_c-y_c plane of the comparator coordinate system (x_c,y_c,z_c) is parallel to the picture plane, the photograph has only ten independent orientation elements. In the orientation problem of the stereo pair, the central projective one-to-one correspondence with 14 independent parameters is satisfied between the model and

object spaces. Hence, the coplanarity condition of corresponding rays provides six independent orientation elements.

Next, the characteristics of the conventional interior orientation parameters (three elements of the principal point) have been discussed in the orientation problem of a stereo pair of photographs in the following two cases: (1) where the planimetric coordinates of the principal point are given and (2) where the principal distance is known. Also, the following results have been obtained:

- For the first case, the coplanarity condition of corresponding rays provides mathematically only five independent orientation elements, because nine independent parameters determine the one-to-one correspondence between the model and object spaces uniquely.
- For the second case, the central projective one-to-one correspondence with ten independent parameters is valid between the model and object spaces. Thus, the coplanarity condition determines mathematically six independent orientation elements.
- Consequently, one of the planimetric coordinates of the principal points for the stereo pair can be obtained[13] from the coplanarity condition.

The situation often occurs that the solution does not converge in the orientation calculation of pictures with various errors. This is due to the following reasons: correlations between selected unknowns in the case where the used pictures have random errors, or dependent relationships between the selected unknowns. However, because of the many parametes involved in close-range photogrammetry, the cause of ill convergence is sometimes confusing. The discussions given here are very helpful to avoid such misunderstandings.

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