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The Accuracy Potential of the Modern Bundle Block Adjustment in Aerial Photogrammetry

When the self-calibrating bundle adjustment is employed, accuracies of 2.5 μm in planimetry and 4.5 μm in height can be achieved with a minimum of effort.

INTRODUCTION

THE USE OF photogrammetric techniques for assigning coordinates to ground stations (densification of geodetic networks, cadastral surveying, engineering surveying, land problems, etc) has been a subject of continuous discussion during the last 20 years or so. Photogrammetric results have often not been accurate enough, and, unfortunately, accuracy standards for photogrammetric works have been based on such results. In retrospect, however, it must be stated that these results have for the most part been caused by inadequate treatment of analytical photogrammetry, mainly with respect to the use of approximate block adjustment methods, and partly due to improper flight planning and performance. Nowadays, the availability of self-calibrating bundle adjustment programs, the recently gained knowledge about reliability problems, and following from that the deeper insight into network structure requirements, provide the basis for exploiting completely the accuracy potential of modern aerial photographs. As a major result of this, a photogrammetric accuracy level is available and reproducible under varying circumstances, which does far beyond conventional standards and demands a revision of unjustifiable attitudes concerning practical applications of photogrammetry to assigning coordinates to survey points.

Compared with conventional surveying methods, photogrammetry has two essential advantages. Aerial photographs contain a tremendous collection of terrain information, so that they can be used for multipurpose evaluations. This feature cannot be overrated, considering the recent and future efforts in city planning and land management. Secondly, photogrammetry provides for repetitive network arrangements, which enables a simple and inexpensive prediction of the average precision of final results, e.g., by using synthetic precision models for bundle adjustment, as derived by Ebner *et al.* (1977).

Provided the specified accuracy requirements are achieved, the choice of a technical procedure should depend mainly on economic considerations. The recent increase and stabilization of photogrammetric accuracy in point determination leads to a reduction in the required effort. The

ABSTRACT: *The modern self-calibrating bundle block adjustment is a highly accurate and efficient technique for assigning coordinates to ground points. A few requirements have to be met, however, if in each project precise and reliable results are to be expected. The proper use of orthogonal bivariate polynomials as an additional parameter set and a sophisticated gross error detection strategy provide for results which are in accordance with theoretical expectations. This is shown by the processing of data of four different test field blocks, using three different camera systems. Therefore, these results can be regarded as realistic and repeatable modern photogrammetric accuracy standards, carrying analytical photogrammetry into new fields of application, such as network densification, cadastral surveying, etc.*

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necessary accuracy level in object space coordinates can now be reached by using a smaller image scale and, thus, fewer photographs, which leads to less measurement time and reduced data sets to deal with and to analyze.

It is the objective of this article to show the current accuracy potential of conventional aerial photography with the aid of empirical data processing. With previous papers (Grün, 1978, 1980a) it was proven empirically from precision and reliability considerations that sophisticated self-calibration is superior to testfield calibration (the economic advantages of self-calibration have never been doubted).

All results presented herein have been obtained under simulated economic pressure. No additional (and, as the results indicate, unnecessary) effort has been made to apply sophisticated a priori image coordinate refinement methods. Instead, only four fiducials were used, radial lens distortion was corrected according to the calibration certificate, and refraction was corrected using Bertram's modified formula.

Because a few terms with which the reader might not be familiar are used, a very brief introduction into the meaning and use of the terms *accuracy*, *precision*, and *reliability* is given first.

Aspects of Accuracy Considerations

The currently most sophisticated accuracy theory for geodetic networks was developed by Baarda (1967, 1968). He considers the accuracy of network adjustment results (coordinates; functions of coordinates such as angles, distances, and distance ratios) as consisting of two parts: precision and reliability. Much mathematics and statistics are involved in either concept, as given in the theory of S-transformations and statistical interval estimation. A brief survey of Baarda's technique and an exposition of problems, together with his definitions of accuracy, precision, and reliability, are given in Baarda (1977) and Alberda (1980). Recently, photogrammetrists have adopted portions of this theory for application to photogrammetric networks. For the investigation of the reliability of bundle systems, see Förstner (1979, 1980) and Grün (1979, 1980b, 1980c). A first attempt to introduce the concept of S-transformations into photogrammetry was made by Molenaar (1980).

It is not the intention of this paper to present or to elaborate upon Baarda's accuracy theory. However, a few basic comments regarding the terms precision, accuracy, and reliability as they are applied here need to be made in order to understand the meaning of the accuracy and precision estimators used later in the section on Empirical Accuracy Studies.

Precision

Let the linear statistical model of bundle adjustment be

$$\mathbf{l} - \mathbf{e} = \mathbf{A}\mathbf{x};\ \mathbf{P}$$
$$\mathrm{E}(\mathbf{l}) = \mathbf{A}\mathbf{x};\ \mathrm{E}(\mathbf{e}) = 0 \tag{1}$$

where

$\mathbf{x}$ is the vector of conventional bundle parameters (object point coordinates, exterior orientation elements),
$\mathbf{e}$ is the vector of true observation errors, and
$\mathbf{P}$ is the weight matrix of the observation vector $\mathbf{l}$.

Using the terminology of modern linear estimation theory for interval estimation, this model setup can be formulated as a null-hypothesis H_0 (compare Graybill (1961) and Baarda (1968)):

$$H_0\colon\ \mathbf{l} \sim \mathbf{N}(\mathbf{A}\mathbf{x}, \sigma_0^2\mathbf{P}^{-1}) \tag{2}$$

(i.e., $\mathbf{l}$ has a multidimensional normal distribution with the expectation $\mathrm{E}(\mathbf{l}) = \mathbf{A}\mathbf{x}$ and the dispersion $\mathrm{D}(\mathbf{l}) = \sigma_0^2\mathbf{P}^{-1}$; σ_0^2 is the variance factor to be estimated).

Suppose a minimum variance unbiased estimation of $\mathbf{x}$ and σ_0^2 is performed with

$$\hat{\mathbf{x}} = (\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l} \tag{3a}$$

$$\hat{\sigma}_0^2 = \frac{1}{r}(\mathbf{A}\hat{\mathbf{x}} - \mathbf{l})^{\mathrm{T}}\mathbf{P}(\mathbf{A}\hat{\mathbf{x}} - \mathbf{l}), \qquad r = n - u \text{ (redundancy)} \tag{3b}$$

and the residuals are denoted by

$$\mathbf{v} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{l}. \tag{4}$$

Under H_0 the distributions of $\hat{\mathbf{x}}$ and $\mathbf{v}$ are

$$\hat{\mathbf{x}} \sim \mathrm{N}(\mathbf{x}, \mathbf{K}_{xx}),\ \mathbf{K}_{xx} = \sigma_0^2\,\mathbf{Q}_{xx} \tag{5a}$$

$$\mathbf{v} \sim \mathrm{N}(0, \mathbf{K}_{vv}),\ \mathbf{K}_{vv} = \sigma_0^2\,\mathbf{Q}_{vv} \tag{5b}$$

with the weight coefficient matrices

$$\mathbf{Q}_{xx} = (\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1},\ \mathbf{Q}_{vv} = \mathbf{P}^{-1} - \mathbf{A}\mathbf{Q}_{xx}\mathbf{A}^{\mathrm{T}}. \tag{6}$$

The term *precision* is defined by the statistical features of the estimated parameters $\hat{\mathbf{x}}$, if the a priori assumptions (functional and stochastical relations) of the adjustment model (Equation 1) are considered to be true.

Hence, the covariance matrix $\mathbf{K}_{xx}$ contains all information concerning the precision of the solution $\hat{\mathbf{x}}$.

For the sake of completeness it should be mentioned that Baarda's precision measure is not the actual covariance matrix $\mathbf{K}_{xx}$, but an artificial "criterion matrix," or the deviation of $\mathbf{K}_{xx}$ therefrom, respectively.

The covariance matrices $\mathbf{K}_{xx}$, $\mathbf{K}_{vv}$ can also be formulated by using the dispersion operator D. For $\mathbf{K}_{xx}$ we get

$$\mathbf{K}_{xx} = \mathrm{D}(\hat{\mathbf{x}}) = \mathrm{E}((\hat{\mathbf{x}} - \mathrm{E}(\hat{\mathbf{x}}))(\hat{\mathbf{x}} - \mathrm{E}(\hat{\mathbf{x}}))^{\mathrm{T}}) \tag{7}$$

Under H_0 we obtain

$$\mathrm{E}(\hat{\mathbf{x}}) = \mathbf{x}, \tag{8}$$

and thus

$$\mathbf{K}_{xx} = \mathrm{D}(\hat{\mathbf{x}}) = \mathrm{E}((\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^{\mathrm{T}}) \tag{9}$$

ACCURACY

Assume the model (Equation 1) to be a false model. Then we get

$$\mathrm{E}(\mathbf{l}) \neq \mathbf{Ax}, \tag{10}$$

and the least squares estimator

$$\tilde{\mathbf{x}} = (\mathbf{A}^{\mathrm{T}}\mathbf{PA})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{Pl} = \hat{\mathbf{x}} + \nabla\mathbf{x} \tag{11}$$

is no longer unbiased, so that

$$\mathrm{E}(\tilde{\mathbf{x}}) = \mathbf{x} + \nabla\mathbf{x} \neq \mathbf{x}, \tag{12}$$

in which $\nabla\mathbf{x}$ is the bias, caused by model errors

The precision of $\tilde{\mathbf{x}}$ can again be derived by applying the dispersion operator

$$\begin{aligned} \mathrm{D}(\tilde{\mathbf{x}}) &= \mathrm{E}((\tilde{\mathbf{x}} - \mathrm{E}(\tilde{\mathbf{x}}))(\tilde{\mathbf{x}} - \mathrm{E}(\tilde{\mathbf{x}}))^{\mathrm{T}}) = \\ &= \mathrm{E}((\tilde{\mathbf{x}} - (\mathbf{x} + \nabla\mathbf{x}))(\tilde{\mathbf{x}} - (\mathbf{x} + \nabla\mathbf{x}))^{\mathrm{T}}) = \\ &= \mathrm{E}((\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^{\mathrm{T}}) = \mathbf{K}_{xx}. \end{aligned} \tag{13}$$

Hence, we obtain for the true and the false model identical *precision* measures (note: the true value σ_0^2 for the variance factor is used in both models!). In either case the precision describes the deviation of the estimated $\tilde{\mathbf{x}}$ (or $\hat{\mathbf{x}}$) from its expectation associated with the corresponding model.

For the discussion of the effects of a possibly wrong model, the term *accuracy* becomes important. In order to derive a multidimensional accuracy measure, the second moment of $\tilde{\mathbf{x}}$ about $\mathbf{x}$ (which is no longer a dispersion) is formulated as

$$\mathrm{A}(\tilde{\mathbf{x}}) = \mathrm{E}((\tilde{\mathbf{x}} - \mathbf{x})(\tilde{\mathbf{x}} - \mathbf{x})^{\mathrm{T}}). \tag{14}$$

$\mathrm{A}(\tilde{\mathbf{x}})$ defines the extent of the deviation of the estimated vector $\tilde{\mathbf{x}}$ from the true vector $\mathbf{x}$. A further development of Equation 14 leads to

$$\begin{aligned} \mathrm{A}(\tilde{\mathbf{x}}) &= \mathrm{E}(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathrm{T}}) - 2\mathrm{E}(\tilde{\mathbf{x}})\mathbf{x}^{\mathrm{T}} + \mathbf{x}\mathbf{x}^{\mathrm{T}} = \\ &= \mathrm{E}(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathrm{T}}) - 2(\mathbf{x} + \nabla\mathbf{x})\mathbf{x}^{\mathrm{T}} + \mathbf{x}\mathbf{x}^{\mathrm{T}} = \\ &= \mathrm{E}(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathrm{T}}) - 2\nabla\mathbf{x}\mathbf{x}^{\mathrm{T}} - \mathbf{x}\mathbf{x}^{\mathrm{T}}. \end{aligned} \tag{15}$$

A similar derivation of Equation 13 gives

$$\begin{aligned} \mathrm{D}(\tilde{\mathbf{x}}) &= \mathrm{E}(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathrm{T}}) - 2\mathrm{E}(\tilde{\mathbf{x}})\,(\mathbf{x} + \nabla\mathbf{x})^{\mathrm{T}} \\ &\quad + (\mathbf{x} + \nabla\mathbf{x})\,(\mathbf{x} + \nabla\mathbf{x})^{\mathrm{T}} = \\ &= \mathrm{A}(\tilde{\mathbf{x}}) - 2\mathrm{E}(\tilde{\mathbf{x}})\nabla\mathbf{x}^{\mathrm{T}} \\ &\quad + 2\mathbf{x}\nabla\mathbf{x}^{\mathrm{T}} + \nabla\mathbf{x}\nabla\mathbf{x}^{T} = \\ &= \mathrm{A}(\tilde{\mathbf{x}}) - \nabla\mathbf{x}\nabla\mathbf{x}^{\mathrm{T}}. \end{aligned} \tag{16}$$

Hence, it follows that

$$\mathrm{A}(\tilde{\mathbf{x}}) = \mathbf{K}_{xx} + \nabla\mathbf{x}\nabla\mathbf{x}^{\mathrm{T}}. \tag{17}$$

Equation 17 could be used as a measure for accuracy, comprising both the effects of precision and reliability (model errors).

Under H_0 we get

$$\nabla\mathbf{x} = 0,$$

and the accuracy $\mathrm{A}(\tilde{\mathbf{x}})$ equals the precision $\mathrm{D}(\tilde{\mathbf{x}})$.

The bias $\nabla\mathbf{x}$ is usually unknown, except that external information is available which is not being used in the estimation model (Equation 1), as for instance check point coordinates. Baarda has overcome this problem by replacing Equation 17 through a formula, which contains the combined effect of both random errors and nondetectable model errors on $\mathbf{x}$ (leading to his so-called "boundary values" $\nabla_0\mathbf{x}$, which are scalar measures).

RELIABILITY AND DATA-SNOOPING

The term *reliability* defines the quality of the adjustment model with respect to the detection of model errors. Those errors can be blunders, systematic errors (errors in the functional assumptions), and weight errors (errors in the stochastical assumptions). It was Baarda (1967, 1968) who developed a rather complete reliability theory which was in use for geodetic network adjustments in the Netherlands for a couple of years and which recently has also been adopted by photogrammetrists to investigate the reliability of photogrammetric network structures.

Currently the term reliability is mainly used with respect to blunder detection and location. This is correct, if a sophisticated self-calibrating program provides for the compensation of systematic errors, and because the problem of weight improvement should be treated separately by weight estimation.

Definitions are available for

- *Internal reliability:* It gives the amount of a gross error in an observation, which is just non-detectable at a certain probability level.
- *External reliability:* It indicates the effect of this non-detectable gross error on the estimated quantities (of special interest: object space coordinates of ground points). For details concerning the internal and external reliability of bundle systems see Förstner (1979, 1980) and Grün (1979, 1980b, 1980c).
- *Data-snooping technique:* A test procedure for blunder detection.

As the true errors ($\mathbf{e}$) of the adjustment system (Equation 1) are not available, a blunder detection test procedure must be restricted to $\mathbf{x}$ and/or $\mathbf{v}$.

So the null-hypotheses

$$\mathrm{H}_{0i}^{x}\colon\ \mathrm{E}(\hat{x}_i) = x_i, \tag{18a}$$

$$\mathrm{H}_{0i}^{v}\colon\ \mathrm{E}(v_i) = 0 \tag{18b}$$

could be used for testing.

The expectations x_i for the solution vector are usually not known; exceptions are control and check point coordinates.

In the case of check point coordinates the tests would lead to the test criteria

$$\mathrm{t}_i = \frac{\hat{\mathrm{x}}_i - \mathrm{x}_i}{\hat{\sigma}_{\mathrm{x}_i}}, \text{ with } \hat{\sigma}_{\mathrm{x}_i} = \hat{\sigma}_0\sqrt{q_{\mathrm{x}_i\mathrm{x}_i}} \tag{19}$$

in which $q_{\mathrm{x}_i\mathrm{x}_i}$ is the i^{th} diagonal element of Q_{xx} and if $\hat{\sigma}_0$ has to be used instead of the expectation σ_0.

Under $\mathrm{H}_{0_i}^{x}$ the test criterion t_i is distributed as Student's t with r degrees of freedom

$$\mathrm{t}_i \sim t(r). \tag{20}$$

In practical projects check point coordinates are

usually not available. If the objective of a photogrammetric test project is to prove the power of the applied technique and if recommendations for the profession are derived from that, the test criteria (Equation 19) should not be applied in the test projects, because then the so-gained results would not be in accordance with practical requirements. So the results presented in this paper were not subject to the test of Equation 19, i.e., the photogrammetric check point coordinates have not been cleaned by a direct a posteriori rejection procedure.

If control point coordinates are introduced as stochastic quantities (compare Grün, 1978) test criteria similar to Equation 19 can be constructed to test the deviation of the photogrammetrically determined coordinates from their geodetic values, as

$$t^c{}_i = \frac{v_i{}^c}{\hat{\sigma}_{x_i}{}^c}, \tag{21}$$

where c stands for "control point."

$t_i{}^c$ has the character of a standardized residual and is no longer Student distributed, since $v_i{}^c$ is dependent on $\hat{\sigma}_0$ and as such on $\hat{\sigma}_{x_i}{}^c$. For the proof of this statement examine theorem 4.21 in Graybill (1961, p. 88).

The correct distribution of $t^c{}_i$ is the τ-distribution (compare the application of τ in Pope (1975) for blunder detection)

$$t^c{}_i \sim \tau(r), \tag{22a}$$

with

$$\tau^2(r) = \frac{r \cdot t^2(r-1)}{r - 1 + t^2(r-1)}. \tag{22b}$$

The testing of the image coordinate observations, based on Equation 18b, could be performed by using Baarda's data-snooping technique. Again for photogrammetric blocks the estimated $\hat{\sigma}_0$ has mostly to be used instead of the expectation σ_0. Then the test criteria are resulting in

$$w_i = \frac{-v_i}{\hat{\sigma}_{v_i}}, \text{ with } \hat{\sigma}_{v_i} = \hat{\sigma}_0 \sqrt{q_{v_i v_i}} \tag{23}$$

in which $q_{v_i v_i}$ is the i^{th} diagonal element of $\mathbf{Q}_{vv}$. If $H_{0_i}^v$ is true, w_i is τ-distributed for the same reason mentioned above: i.e.,

$$w_i \sim \tau(r). \tag{24}$$

The τ-distribution is not much investigated so far; there are still specific problems with respect to the distribution of w_i under alternative hypotheses to be solved. Fortunately, the redundancy in photogrammetric blocks is rather large, so that τ can be replaced by the Student distribution

$$\tau(r) \approx t(r), \tag{25}$$

which is far easier to handle.

The test criteria w_i are of considerable practical importance for the cleaning of data sets. They provide for a rather sensitive blunder detection procedure, since they include via $\hat{\sigma}_{v_i}$ ($q_{v_i v_i}$) the network structure. For large systems, as they usually appear in aerial triangulation, the strict computation of the complete $\mathbf{Q}_{vv}$-matrix or even of its diagonal elements $q_{v_i v_i}$ only becomes very costly. Considering in addition the uncertainty in the assumption of the weight matrix $\mathbf{P}$, the idea of applying a modified data-snooping technique has been brought up by Grün (1979, 1980c). Based on the use of approximate q_{vv}- values derived from internal reliability models, the image coordinates of the testfields Jämijärvi and Willunga, which are referred to later in the section on Empirical Accuracy Studies have been checked by this method.

Test Criteria for the Assessment of Block Adjustment Results

For the assessment of block adjustment results, it is crucial to use test criteria which can provide for an answer to the question "is the applied estimation model correct or not?" $\mathbf{K}_{xx}$ has been defined as a measure of precision for the solution $\hat{\mathbf{x}}$ (or $\tilde{\mathbf{x}}$). The complete matrix $\mathbf{K}_{xx}$ however, although of utmost information density, is usually a very complicated precision measure, and so scalar precision measures are mostly preferred. In photogrammetric testblock investigations, the means of the variances of the adjusted ground point coordinates are used as theoretical precision measures (here the symbols **X, Y, Z** refer to the vectors of ground point coordinates, i.e., to subsets of the vector **x** in Equation 1:

$$\sigma_x{}^2 = \frac{\text{tr}(\mathbf{K}_{xx}^x)}{n_{x,y}}, \quad \sigma_y{}^2 = \frac{\text{tr}(\mathbf{K}_{xx}^y)}{n_{x,y}} \tag{26a}$$

$$\sigma_{x,y}^2 = \frac{\sigma_x{}^2 + \sigma_y{}^2}{2}, \quad \sigma_z{}^2 = \frac{\text{tr}(\mathbf{K}_{xx}^z)}{n_z} \tag{26b}$$

in which $\mathbf{K}_{xx}^x$, $\mathbf{K}_{xx}^y$, $\mathbf{K}_{xx}^z$ are corresponding parts of $\mathbf{K}_{xx}$ for **X, Y, Z;** and $n_{x,y}$, n_z are the numbers of **X, Y,** and **Z** coordinates. The values (Equations 26a and 26b) are of special practical interest, as they are closely related to quadratic forms of the vectors of estimated coordinate differences at check points, as it is shown in the following.

The difference between photogrammetrically estimated coordinates and independently determined coordinates (assumed to be free of errors compared with the expected photogrammetric accuracy) are

$$\begin{aligned} \Delta\hat{\mathbf{X}} &= \hat{\mathbf{X}} - \mathbf{X}, \\ \Delta\hat{\mathbf{Y}} &= \hat{\mathbf{Y}} - \mathbf{Y}, \\ \Delta\hat{\mathbf{Z}} &= \hat{\mathbf{Z}} - \mathbf{Z}. \end{aligned} \tag{27}$$

in which $(\mathbf{X}^T, \mathbf{Y}^T, \mathbf{Z}^T)^T$ is the vector of "true" coordinates and $(\hat{\mathbf{X}}^T, \hat{\mathbf{Y}}^T, \hat{\mathbf{Z}}^T)^T$ is the vector of photogrammetric coordinates. Since the true values **X, Y, Z** are included in $\Delta\hat{\mathbf{X}}$, $\Delta\hat{\mathbf{Y}}$, $\Delta\hat{\mathbf{Z}}$ estimators for the accuracy can be defined as

$$\tilde{\mu}_x{}^2 = \frac{\Delta\hat{\mathbf{X}}^T\Delta\hat{\mathbf{X}}}{n_{x,y}}, \quad \tilde{\mu}_y{}^2 = \frac{\Delta\hat{\mathbf{Y}}^T\Delta\hat{\mathbf{Y}}}{n_{x,y}}, \tag{28a}$$

$$\tilde{\mu}_{x,y}^2 = \frac{\tilde{\mu}_x{}^2 + \tilde{\mu}_y{}^2}{2}, \quad \tilde{\mu}_z{}^2 = \frac{\Delta\hat{\mathbf{Z}}^T\Delta\hat{\mathbf{Z}}}{n_z}. \tag{28b}$$

Under the null-hypotheses

$$H_0^{x,y}: \begin{pmatrix} \Delta\hat{\mathbf{X}} \\ \Delta\hat{\mathbf{Y}} \end{pmatrix} \sim N\ (0, \mathbf{K}_{xx}^{x,y}) \tag{29a}$$

$$H_0^z: \Delta\hat{\mathbf{Z}} \sim N\ (0, \mathbf{K}_{xx}^z) \tag{29b}$$

the estimators $\tilde{\mu}_{x,y}^2$ and $\tilde{\mu}_z^2$ are unbiased estimators for the theoretical precision measures $\sigma_{x,y}^2$, σ_z^2

$$E(\tilde{\mu}_{x,y}^2) = \sigma_{x,y}^2, \tag{30a}$$

$$E(\tilde{\mu}_z{}^2) = \sigma_z{}^2. \tag{30b}$$

Hence the "traced statistics" $\tilde{\mu}_{x,y}^2$ and $\tilde{\mu}_z{}^2$ (terminology after Bjerhammar (1960), as $E(\tilde{\mu}_{x,y}^2) = \mathrm{tr}(\mathbf{K}_{xx}^{x,y})/2n_{x,y}$, $E(\tilde{\mu}_z{}^2) = \mathrm{tr}(\mathbf{K}_{xx}^z)/n_z$) are well-suited as accuracy predictors in practical project planning and are also useful in empirical accuracy investigations, as presented in this paper.

However, if $\Delta\hat{\mathbf{X}}$, $\Delta\hat{\mathbf{Y}}$, $\Delta\hat{\mathbf{Z}}$ have to be tested on normal distribution (or on their deviation therefrom), the Maximum Likelihood estimators

$$\hat{\mu}_{x,y}^2 = \frac{(\Delta\hat{\mathbf{X}}^T,\Delta\hat{\mathbf{Y}}^T)\ \mathbf{Q}_{xx}^{x,y\,-1}\ (\Delta\hat{\mathbf{X}}^T,\Delta\hat{\mathbf{Y}}^T)^T}{2\,n_{x,y}} \tag{31a}$$

$$\hat{\mu}_z{}^2 = \frac{\Delta\hat{\mathbf{Z}}^T\,\mathbf{Q}_{xx}^{z\,-1}\,\Delta\hat{\mathbf{Z}}}{n_z} \tag{31b}$$

are the better estimators (see Grün, 1980a; Molenaar, 1978).

It is an interesting statistical characteristic that under $H_0^{x,y}$ and H_0^z, the estimators $\hat{\mu}_{x,y}^2$ and $\hat{\mu}_z{}^2$ are unbiased estimators for the variance factor:

$$E(\hat{\mu}_{x,y}^2) = \sigma_0{}^2, \quad E(\hat{\mu}_z{}^2) = \sigma_0{}^2. \tag{32}$$

Both $\hat{\mu}_{x,y}^2/\hat{\sigma}_0{}^2$ and $\hat{\mu}_z{}^2/\hat{\sigma}_0{}^2$ follow (under $H_0^{x,y}$, H_0^z) Fisher distributions:

$$T_{x,y} = \frac{\hat{\mu}_{x,y}^2}{\hat{\sigma}_0{}^2} \sim F(2n_{x,y}, r), \tag{33a}$$

$$T_z = \frac{\hat{\mu}_z{}^2}{\hat{\sigma}_0{}^2} \sim F\ (n_z, r). \tag{33b}$$

If the matrices $\mathbf{Q}_{xx}^{x,y\,-1}$ and $\mathbf{Q}_{xx}^{z\,-1}$ are available, which are the "partly reduced normal equations" for **X, Y,** and **Z,** interval estimation for **X, Y,** and **Z** using the estimators (Equations 31a and 31b) can easily be performed.

Assume a type I error size $1 - \alpha$ (α is the probability to reject a true null-hypothesis), then the equations

$$T_{x,y} = F(1 - \alpha, 2n_{x,y}, r), \tag{34a}$$

$$T_z = F(1 - \alpha, n_z, r) \tag{34b}$$

are defining two confidence-hyperellipsoids with the centers in $(\hat{\mathbf{X}}^T,\hat{\mathbf{Y}}^T)^T$ and $\hat{\mathbf{Z}}$.

Thus, $1 - \alpha$ is the probability that under $H_0^{x,y}$ and H_0^z, respectively, the true values $(\mathbf{X}^T,\mathbf{Y}^T)^T$, $\mathbf{Z}$ fall into their corresponding hyperellipsoid.

The estimators $\hat{\mu}_{x,y}^2$, $\hat{\mu}_z^2$ can be checked by using the critical values $c_{x,y}$, c_z from the Fisher tables as

$$P(\hat{\mu}_{x,y}^2 \leq c_{x,y} = \hat{\sigma}_0{}^2 F(1 - \alpha, 2n_{x,y}, r)/H_0^{x,y}) = 1 - \alpha, \tag{35a}$$

$$P(\hat{\mu}_z{}^2 \leq c_z = \hat{\sigma}_0{}^2 F(1 - \alpha, n_z, r)/H_0^z) = 1 - \alpha. \tag{35b}$$

In contrast to this procedure the interval estimation for $(\mathbf{X}^T,\mathbf{Y}^T)^T$, $\mathbf{Z}$ by means of Equation 28b is much more expensive.

Approximations for the distributions of $\tilde{\mu}_{x,y}^2$, $\tilde{\mu}_z^2$ under $H_0^{x,y}$, H_0^z are given by Molenaar (1978) with

$$\tilde{\mu}_{x,y}^2 \sim \sigma_0{}^2 \frac{\mathrm{tr}(\mathbf{Q}_{xx}^{x,y})}{2n_{x,y}} F(K_{x,y}, \infty), \tag{36a}$$

$$\tilde{\mu}_z{}^2 \sim \sigma_0{}^2 \frac{\mathrm{tr}(\mathbf{Q}_{xx}^z)}{n_z} F(K_z, \infty). \tag{36b}$$

From this it can be deduced that

$$\tilde{\mu}_{x,y}^2 \sim \hat{\sigma}_0{}^2 \frac{\mathrm{tr}(\mathbf{Q}_{xx}^{x,y})}{2n_{x,y}} F(K_{x,y}, r), \tag{37a}$$

$$\tilde{\mu}_z{}^2 \sim \hat{\sigma}_0{}^2 \frac{\mathrm{tr}(\mathbf{Q}_{xx}^z)}{n_z} F(K_z, r). \tag{37b}$$

If the eigenvalues of $\mathbf{Q}_{xx}^{x,y}$, $\mathbf{Q}_{xx}^z$ are denoted by $\lambda_i^{x,y}$ $(i = 1, \ldots, 2n_{x,y})$, $\lambda_j^z (j = 1, \ldots, n_z)$, then the degrees of freedom $K_{x,y}$, K_z are to be computed as

$$K_{x,y} = \mathrm{int}\left(\frac{\left(\sum_{\nu=1}^{2n_{x,y}} \lambda_\nu^{x,y}\right)^2}{\sum_{\nu=1}^{2n_{x,y}} (\lambda_\nu^{x,y})^2}\right), \tag{38a}$$

$$K_z = \mathrm{int}\left(\frac{\left(\sum_{\nu=1}^{n_z} \lambda_\nu^z\right)^2}{\sum_{\nu=1}^{n_z} (\lambda_\nu^z)^2}\right) \tag{38b}$$

in which int is the integer conversion operator.

The estimators $\tilde{\mu}_{x,y}^2$, $\tilde{\mu}_z^2$ can be checked by using the critical values $\bar{c}_{x,y}$, $\bar{c}_z$ as

$$P(\tilde{\mu}_{x,y}^2 \leq \bar{c}_{x,y} = \hat{\sigma}_0{}^2 \frac{\mathrm{tr}(\mathbf{Q}_{xx}^{x,y})}{2n_{x,y}} F(1 - \alpha, K_{x,y}, r)/H_0^{x,y}) = 1 - \alpha, \tag{39a}$$

$$P(\tilde{\mu}_z{}^2 \leq \bar{c}_z = \hat{\sigma}_0{}^2 \frac{\mathrm{tr}(\mathbf{Q}_{xx}^z)}{n_z} F(1 - \alpha, K_z, r)/H_0^z) = 1 - \alpha. \tag{39b}$$

The more $\mathbf{Q}_{xx}^{x,y}$, $\mathbf{Q}_{xx}^{z}$ tend towards multiples of the unity matrix, which is certainly more likely with dense rather than with sparse control, the more the estimators $\tilde{\mu}_{x,y}^2$, $\tilde{\mu}_z^2$ correspond with $\hat{\mu}_{x,y}^2$, $\hat{\mu}_z^2$.

Suppose

$$\mathbf{Q}_{xx}^{x,y} = f\mathbf{I}. \tag{40}$$

in which f is a scalar factor and $\mathbf{I}$ is the unity matrix.

Then

$$\mathrm{tr}(\mathbf{Q}_{xx}^{x,y}) = 2n_{x,y}f, \tag{41}$$

$$K_{x,y} = 2n_{x,y}, \tag{42}$$

and

$$\tilde{\mu}_{x,y}^2 \sim \hat{\sigma}_0^2 f\mathrm{F}(2n_{x,y}, r). \tag{43}$$

For $\hat{\mu}_{x,y}^2$ we obtain

$$\hat{\mu}_{x,y}^2 = \frac{(\Delta\hat{\mathbf{X}}^{\mathrm{T}}, \Delta\hat{\mathbf{Y}}^{\mathrm{T}})\begin{pmatrix}\Delta\hat{\mathbf{X}}\\ \Delta\hat{\mathbf{Y}}\end{pmatrix}}{2n_{x,y}f}, \tag{44}$$

and finally

$$\hat{\mu}_{x,y}^2 = \frac{\tilde{\mu}_{x,y}^2}{f} \sim \hat{\sigma}_0^2\mathrm{F}(2n_{x,y}, r). \tag{45}$$

For $f = 1$, the distributions for $\hat{\mu}_{x,y}^2$ and $\tilde{\mu}_{x,y}^2$ are identical. A similar derivation can readily be performed for the **Z**-coordinate.

For more details, see Koch (1975), Molenaar (1978), and Grün (1980a). The computation of the estimators $\hat{\mu}_{x,y}$ and $\hat{\mu}_z$ was not possible for these investigations, since, for the blocks used, the $\mathbf{Q}_{xx}^{x,y\,-1}$, $\mathbf{Q}_{xx}^{z-1}$-matrices were not available.

Since the estimators $\tilde{\mu}_{x,y}^2$, $\tilde{\mu}_z^2$ comprise both precision and reliability features, they can be regarded as accuracy estimators and as such be used to review test block adjustment results as it is done in the section on Empirical Accuracy Studies.

Empirical Accuracy Studies

Empirical photogrammetric accuracy studies using testblocks have been executed for the past several years. These studies were mostly based on geodetic control of insufficient accuracy, and, to a minor extent on approximation methods for data processing. So the actual photogrammetric accuracy level could never be proved. Recently, however, startling results have been reported by Roberts (1976) and Slama (1978), both using reseau camera systems and investing considerable effort in a priori calibration and systematic error compensation works. Although the data processing procedures suggested by both authors cannot be recommended as efficient for any project, their investigations showed up the accuracy potential inherent in photogrammetry.

High accuracy applications need some basic requirements:

- use of targeted points;
- flights with at least fourfold photo coverage, either with 60 percent sidelap or with 20 percent sidelap and cross flights;
- relatively dense perimeter control in planimetry and a relatively dense grid of height control points;
- measurement of image coordinates and application of rigorous bundle block adjustment; including a method for systematic error compensation (self-calibration is recommended); and
- sophisticated gross error detection strategy, sufficiently strict in a statistical sense, thereby not wasting computing time.

The application of the self-calibration technique and related results achieved are described in several publications (see Brown (1976), Ebner (1976), Grün (1978, 1980a), and other authors). A strategy for gross error detection in bundle adjustment is suggested by Grün (1979).

In the following sections, the results of several testblock investigations carried out at different testfields, with different cameras and image scales, but based on the aforementioned requirements, will be presented.

Table 1 shows the project parameters of the testfields Jämijärvi (Finland), and Willunga and Kapunda (both Australia). This article presents only the results obtained with simultaneous self-

Table 1. Testblock Specifications

Camera	Area	Size[1] (km²)	Overlap f/s	No. of photos	Image Scale	Control Precision (mm) Plan	Height
RMK A2 WA	Jämijärvi	2 × 2	60/60	51	1: 4,000	5	0.6/km
MRB WA	Jämijärvi	2 × 2	60/60	51	1: 4,000	5	0.6/km
RMK AR WA	Willunga	5.8 × 5.8	60/60	48	1:12,700	25	30
RMK AR WA	Kapunda	24 × 24	60/60	81	1:50,000	?	?

[1] Area of coordinated points.

calibration; comparisons of testfield calibration and self-calibration are to be found in Grün (1978, 1980a). The additional parameter sets used were the orthogonal ones developed by Ebner (1976) and Grün (1978). The image coordinates were measured with Zeiss PSK stereocomparators of the Department of Lands (Adelaide, Australia), the Institute of Photogrammetry of the Helsinki University of Technology (Helsinki, Finland), and the Chair for Photogrammetry of the Technical University Munich (Munich, West Germany). Results obtained with analytical plotter measurements (Zeiss Planicomp C 100) are reported in Ebner (1979) and Eder and Grün (1980).

For economic reasons, only the absolutely necessary measurements and computations have been carried out. So the refinements of the image coordinates were restricted to

- affine transformation (six parameters) using four fiducials (although Willunga and Kapunda are reseau materials, and the MRB-Jämijärvi photos have an "edge reseau", only the four midpoints of the sides were selected as fiducials while the others were ignored);
- correction of radial distortion according to the calibration report; and
- correction of refraction using Bertram's modified formula.

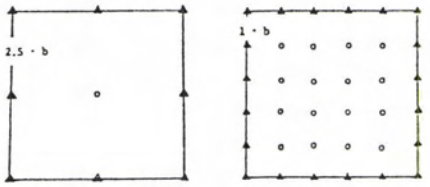
For the bundle adjustment the refined image coordinates have been regarded as uncorrelated and of equal weights (=1), while the geodetic control coordinates were introduced with infinite weights (free of errors).

In each case secant plane control coordinates were used, thus making image coordinate correction for Earth curvature unnecessary.

The additional parameters have been treated as free unknowns (nonstochastic values). After the first iteration step of the bundle adjustment, the significant additional parameters were determined with Student's t-test (α-level : 0.05). For the second iteration step the non-significant additional parameters were excluded from further computations.

Both Jämijärvi and Willunga image coordinates have been subject to an approximate data-snooping procedure, providing for blunder detection and elimination (compare Grün (1979, 1980c)). Two different control versions as shown in Figure 1 were used in each project. The adjustment results are shown in Table 2. To be independent of the photo scale, the accuracy estimators $\tilde{\mu}_{x,y}$, $\tilde{\mu}_z$ of this table are given in μm.

As the results show, self-calibration was very successful. Accuracy improvements up to a factor of 2.7 were achieved. According to theoretical expectation and practical experience, the improvement is larger with sparse control (see also the previous experiments in Ebner (1976) and Grün (1978), a very nice feature of self-calibration. The



▲ combined planimetric and height control point
o height control point

FIG. 1. Control distributions for the Jämijärvi, Willunga, and Kapunda testblock computations.

application of the 44 additional-parameter set was only feasible with Jämijärvi data (5 by 5 image point distribution); here the results in height turned out to be better (improvement factors 1.2 to 1.6) than the corresponding results obtained with the 12 additional parameter set (improvement factors 1.1). The results in planimetry, however, are of equal accuracy.

It is surprising that even within the extended set of 44 additional parameters the correlation coefficients remained less than $|r| = 0.5$ (mutual correlations of additional parameters and between additional parameters and exterior orientation elements). Due to those small correlations, the statistical testing on significance of additional parameters, based on the one-dimensional Student test, was admissible.

It may seem somewhat peculiar at a first glance that in two cases out of 20 the results obtained with sparse control are better than the corresponding results with dense control (planimetry of Jämijärvi MRB and height of Kapunda RMK). This is explained by the fact that the estimators $\tilde{\mu}^2_{x,y}$, $\tilde{\mu}_z{}^2$ are themselves not true values, but stochastic quantities with a certain expectation and a specified range of variation (interval, based on probability assumptions).

It is well known that the precision measures (Equation 26b) reach almost their minimum with the control bridging distance $i = 2b$. A further decrease of i will not cause a significant improvement in **X**, **Y**, and **Z** (improvement < 15 percent).

Hence, if the estimator $\tilde{\mu}^2$ for sparse control is denoted by $\tilde{\mu}^2_{(S)}$ and the one for dense control by $\tilde{\mu}^2_{(D)}$, we get for our control distributions ($i = b$, $i = 2.5b$) the approximation

$$E(\tilde{\mu}^2_{(S)}) \approx E(\tilde{\mu}^2_{(D)}) \tag{46}$$

for both planimetry and height.

In addition, it is to be expected that the interval for $\tilde{\mu}^2$ increases as the amount of control decreases. So the fact that in our examples the empirical accuracy estimators obtained with sparse control *can* be smaller than those obtained with dense control is not surprising any more. Unfortunately, the correct interval estimation for $\tilde{\mu}^2_{x,y}$, $\tilde{\mu}_z{}^2$ becomes fairly costly as it has been shown in the previous chap-

TABLE 2. RESULTS OF JÄMIJÄRVI, WILLUNGA, AND KAPUNDA COMPUTATIONS

Block version	Calibration version	Accuracy estimators-block adjustment $\hat{\sigma}_0$ [μm]	$\tilde{\mu}_{x,y}$ [μm]	$\tilde{\mu}_{x,y}$ factor[1]	$\tilde{\mu}_z$ [μm]	$\tilde{\mu}_z$ factor[1]	number of checkpoints plan.	number of checkpoints height
JA1 RMK	0	4.3	2.8	—	5.0	—		
(dense)	12APs	3.7	2.3	1.2	4.5	1.1	100	84
	44APs	3.6	2.2	1.3	4.2	1.2		
JA2 RMK	0	4.3	3.3	—	7.2	—		
(sparse)	12APs	3.7	2.5	1.3	6.5	1.1	112	111
	44APs	3.6	2.5	1.3	5.0	1.4		
JA1 MRB	0	4.3	3.5	—	6.5	—		
(dense)	12APs	4.1	2.6	1.4	5.8	1.1	100	84
	44APs	3.8	2.8	1.2	5.0	1.3		
JA2 MRB	0	4.1	4.2	—	12.0	—		
(sparse)	12APs	4.0	2.8	1.5	10.5	1.1	112	111
	44APs	3.8	2.6	1.6	7.5	1.6		
WI1	0	5.0	3.7	—	5.6	—	73	59
(dense)	12APs	3.5	2.6	1.4	4.4	1.3		
WI 2	0	4.7	4.9	—	14.3	—	83	82
(sparse)	12APs	3.3	2.8	1.8	6.6	2.2		
KA1	0	5.4	4.8	—	6.1	—	21	42
(dense)	12APs	3.7	3.0	1.6	4.6	1.3		
KA2	0	4.9	8.0	—	7.9	—	37	68
(sparse)	12APs	3.7	3.0	2.7	4.4	1.8		

JA RMK . . . Jämijärvi (RMK A2)
JA MRB . . . Jämijärvi (MRB)
WI . . . Willunga
KA . . . Kapunda
[1] Factor of accuracy improvement
0 . . . zero version (conventional bundle adjustment without additional parameters)
12APs . . . 12 orthogonal additional parameters (Ebner, 1976)
44APs . . . 44 orthogonal additional parameters (Grün, 1978)

ter, so that it could not be performed for this investigation.

The total accuracy level is exceptionally high and very consistent. Averaging the results of all projects, we get as empirical accuracy measures:

$$\begin{matrix} \tilde{\mu}_{x,y} & = 2.6\ \mu\text{m} \\ \tilde{\mu}_z & = 4.5\ \mu\text{m} \end{matrix} \text{ with dense control}$$

$$\begin{matrix} \tilde{\mu}_{x,y} & = 2.7\ \mu\text{m} \\ \tilde{\mu}_z & = 5.9\ \mu\text{m} \end{matrix} \text{ with sparse control}$$

Using an average $\hat{\sigma}_0 = 3.6\ \mu$m (see Table 2), we get for the dense control version

$$\frac{\tilde{\mu}_{x,y}}{\hat{\sigma}_0} = 0.7, \quad \frac{\tilde{\mu}_z}{\hat{\sigma}_0} = 1.2,$$

TABLE 3. AVERAGE EMPIRICAL ACCURACY VALUES, TRANSFERRED TO THE GROUND SYSTEM

	at image scale [μm]	Jämijärvi 1:4,000 [cm]	Willunga 1:12,700 [cm]	Kapunda 1:50,000 [cm]
$\tilde{\mu}_{x,y}$	2.6	1.0	3.3	13
$\tilde{\mu}_z$	4.5	1.8	5.7	22.5

which is in sufficient agreement with the theoretical precision expectations presented in Ebner *et al.* (1977).

Of course the terms "dense" ($i = 1\,b$) and "sparse" ($i = 2.5\,b$) are of relative character and related to control distributions, usually found in cadastre, land consolidation, and network densification projects in Europe. If in extended areas even sparser distributions have to be used, then one has to consider a somewhat reduced accuracy level. Suitable precision predictors for practical projects can be derived from synthetic precision models (Ebner *et al.*, 1977).

Transferring the average accuracy values for planimetry from the image system into the ground system of each individual project, the heavy demands to be made on an adequate geodetic planimetric control accuracy can be demonstrated (see Table 3). The average height accuracy of $\tilde{\mu}_z = 4.5$ μm corresponds to 0.03‰ of the flying height, a very remarkable level too. The residuals at check points, shown in Figures 2 and 3 for Jämijärvi (RMK) and Willunga testfields, look sufficiently random, which indicates the proper work of block-invariant additional parameters.

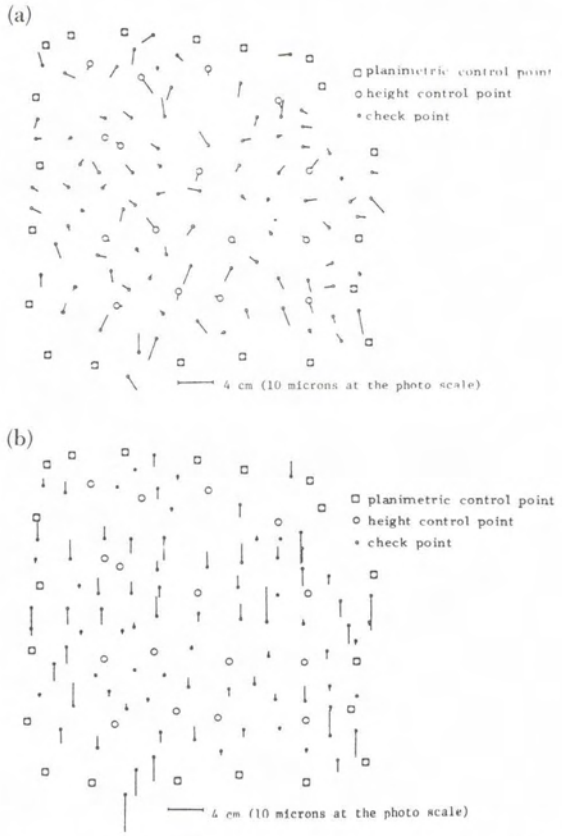


FIG. 2. (a) Residuals at planimetric check points of the Jämijärvi testfield (control version JA1 (RMK), additional parameters: 44 orthogonal); (b) Residuals at height check points of the Jämijärvi testfield (control version JA1 (RMK), additional parameters: 44 orthogonal).

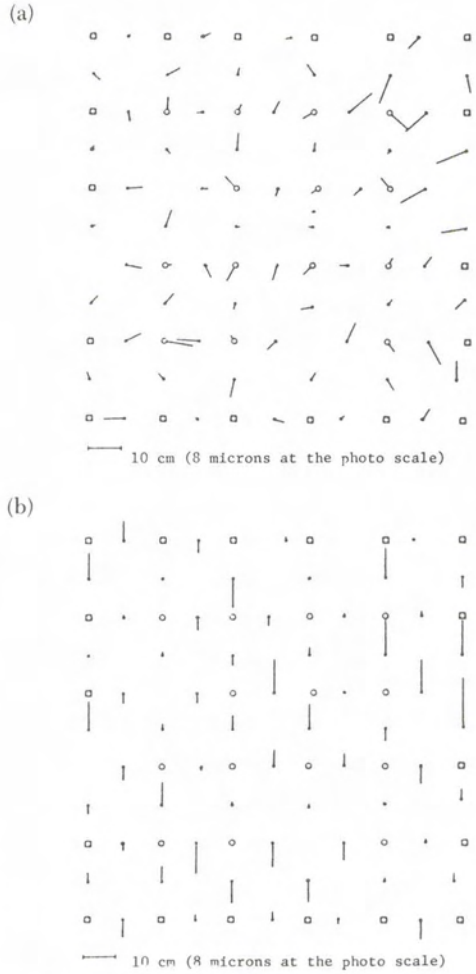


FIG. 3. (a) Residuals at planimetric check points of the Willunga testfield (control version WI 1, additional parameters: 12 orthogonal); (b) Residuals at height check points of the Willunga testfield (control version WI 1, additional parameters: 12 orthogonal).

CONCLUDING REMARKS

It has been proved empirically that the accuracies of photogrammetric coordinates of $\tilde{\mu}_{x,y} = 2.5\ \mu m$ for planimetry and $\tilde{\mu}_z = 4.5\ \mu m$ for height ($= 0.03‰$ of flying height) can be achieved. This was obtained with a minimum of effort and costs by self-calibrating bundle adjustment. The use of relatively dense control ($i \approx 2b$) and 60/60 percent overlap, together with a sophisticated additional parameter set is indispensable to arrive continuously at such a high accuracy level.

These accuracy levels are in agreement with the theoretical precision expectations and, therefore, are not surprisingly high. They carry analytical photogrammetry into new fields of application, such as network densification, cadastral surveying, and many other subjects dealing with the provision of coordinates to ground points in rural and urban areas.

The present values are to be regarded as realistic and repeatable photogrammetric accuracy standards, and it is up to responsible institutions, authorities and private companies to utilize this accuracy potential to obtain more economic tools for point determinations purposes.

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