M. E. O. Ali*
Gendron, Lefebvre Inc.
Laval, P. Q. H7N 1A1, Canada
A. J. Brandenberger
Department of Photogrammetry
Laval University
Quebec, P. Q. G1K 7P4, Canada

Analytical Triangulation of Space Photography

Two bundle adjustment methods were developed, and aerial triangulation tests using those methods were carried with Skylab orbital photography.

Introduction

Large portions of the world are still very poorly mapped, even at scales of 1:250,000 or smaller. The policy of making space photography available almost at no cost to the user makes mapping from space photography more and more attractive. The authors of this paper were involved in a research program to investigate the maximum possible accuracy for aerial triangulation using Skylab photography combined with very high altitude aircraft photography and utilizing Skylab orbital parameters.

Appropriate applications for space photography are for areas where there is no ground control as such, except point coordinates obtained from small scale maps. Accordingly, in such aerial triangulation the coordinates of the ground control must be treated as observed values and must be adjusted when solving the photogrammetric problem. This idea led to the developing of two new bundle adjustment methods, which could be efficiently used for the aerial triangulation of space photography and could be applied when the available ground control is of inferior quality.

In the first method the coordinates of the ground control as well as the photo measurements are used

Abstract: *A mathematical model and a computer program were developed by the authors to perform analytical aerial triangulation with space photography. In this paper, the mathematical model as well as the results of a few tests are given.*

as observations in the collinearity condition equations. Then, point by point, the contributions to the normal equations are calculated. The system of the normal equations can then be solved, yielding the corrections to both the unknown camera parameters and the unknown ground coordinates of the pass points. Finally, point by point, the residuals of the photo measurements and the ground measurements (if the points are control points) can be calculated.

In the second method, the camera parameters, the ground coordinates of the control points, and the photo measurements are used as observations in the collinearity condition equations. Although the second method can rigorously treat the case of space photography for which all orbital parameters are measured, its solution requires more computer time and memory. In this method, the contributions to the normal equations can not be calculated one point after another. Instead, first the contributions of all the control points must be calculated as one unit and then the contributions of all the pass points as another unit. The only unknowns in this case are the ground coordinates of the pass points. Therefore, only one iteration is necessary to reach the final solution. Finally, the residuals for all the measurements and their variance-covariance matrices can be calculated if needed.

* Now with the Faculty of Engineering, Al-Azhar University, Cairo, Egypt.

Photogrammetric Engineering and Remote Sensing,
Vol. 48, No. 1, January 1982, pp. 55-65.
0099-1112/82/4801-55 $02.25/0

Bundle Adjustment and Least-Squares Solution

The idea of the bundle adjustment is to use the well known collinearity equations to establish two equations for each measured photo point, and, further, to obtain a unique solution for the system of observation equations by the least-squares method. The linearized form of the collinearity equations may be given by

$$\mathbf{A}\boldsymbol{\delta} + \mathbf{B}\mathbf{V} + \mathbf{W} = 0 \tag{1}$$

where

- $\boldsymbol{\delta}$ is the correction vector to the approximate values set for the unknowns;
- $\mathbf{V}$ is the residual vector, i.e., the correction vector to the observations;
- $\mathbf{W}$ is the misclosure vector; and
- $\mathbf{A},\mathbf{B}$ are two matrices whose elements are the partial derivatives of the collinearity equations with respect to the unknowns and to the observations, respectively.

The principle of the least-squares method requires the minimizing of the quadratic form $\mathbf{V'PV}$, where $\mathbf{P}$ is the weight matrix whose elements are the weights associated with each of the observations. The least-squares solution of an equation similar to the linear form of the collinearity condition equations given by Equation 1 can be given as

$$\boldsymbol{\delta} = \mathbf{N}^{-1}\mathbf{U}, \tag{2}$$

where

$$\mathbf{N} = \mathbf{A}'\mathbf{M}^{-1}\mathbf{A}, \tag{3}$$
$$\mathbf{U} = -\mathbf{A}'\mathbf{M}^{-1}\mathbf{W}, \tag{4}$$
$$\mathbf{M} = \mathbf{B}\mathbf{P}^{-1}\mathbf{B}' \tag{5}$$

Also, the residual vector $\mathbf{V}$ can be expressed as

$$\mathbf{V} = \mathbf{P}^{-1}\mathbf{B}'\mathbf{R} \tag{6}$$

where $\mathbf{R}$ is the Lagrange multipliers vector and can be given by

$$\mathbf{R} = -\mathbf{M}^{-1}(\mathbf{A}\boldsymbol{\delta} + \mathbf{W}) \tag{7}$$

Moreover, the variance-covariance matrices for the unknowns $\boldsymbol{\Sigma}_x$ can be calculated from

$$\boldsymbol{\Sigma}_x = \sigma_0^2\mathbf{N}^{-1} \tag{8}$$

where σ_0^2 is the variance of unit weight.

By applying the method of least squares to solve the system of linearized observation equations (Equation 1), one can see that two matrices have to be inverted, $\mathbf{M}$ (Equation 5) to form the normal equations and $\mathbf{N}$ (Equation 3) to solve for the unknowns. A direct method of computing and inverting such large matrices is not practical due to both the excessive amount of computer time and memory required and also because of the rounding off of errors in machine calculations.

Several methods and computer programs have been developed using the method of adopting single bundles of rays as a unit to adjust a block of aerial photography (see, for example, Brown (1964), Keller and Tewinkel (1967), Schmid (1956-57), Schut (1978), and Wong (1971)). To overcome the difficulties of calculating and inverting such large matrices (like $\mathbf{M}$ and $\mathbf{N}$), the computer programs associated with these methods only solve special cases of the general observation equations (Equation 1). For example, Keller and Tewinkel (1967) used the ground coordinates of all points (even of the geodetic control stations) and all the camera parameters as unknowns and in this case the $\mathbf{B}$ matrix of Equation 1 is equal to a unit matrix $\mathbf{I}$. To distinguish between the unknown pass points and the observed control points, Keller and Tewinkel used what they called a position control weight to control the degree of agreement between the adjusted and the measured values of the control point coordinates.

Brown (1964) used the two collinearity equations in the same way as Keller and Tewinkel, but, instead of using position control points, he formed extra observation equations (using them as constraints to the system) for the measured ground coordinates, the measured camera parameters, and any other available sources of information. But one can see that in Brown's method he treats the ground coordinates of the control points and the camera parameters as unknowns in the photogrammetric collinearity condition equations, and later, when adding the constraints, the same ground coordinates and the same camera parameters are treated as observations.

In this way, Brown, like Keller and Tewinkel, transfers the general form of the observation equations (Equation 1) to a special case where the $\mathbf{B}$ matrix is equal to a unit matrix $\mathbf{I}$. This special case of observation equations is known as the parametric case of least-squares adjustment, and its solution is much simpler and requires considerably less calculation when compared with the general least-squares solution case. This advantage provided Keller and Brown with a good reason to adopt their solutions, especially when considering that one of the main goals of their programs is to adjust as large a number of photographs as possible in the most economic way.

For space photography, the cost portion for adjusting the photogrammetric measurements is negligible compared with the cost for the entire mission. The main goal is to achieve the best possible accuracy for the aerial triangulation by using few photographs. This reasoning indicates the need for a new method using

- measurements of space photography;
- ground control points which are few in number and inferior in quality; and
- camera parameters for each exposure station, all employed in a simultaneous least squares adjustment.

It was hoped that such a system would produce better estimates for the measured coordinates of the ground control points and would supply accurate coordinates. Two methods and their computer programs were developed. Although these methods are also special cases of Equation 1, they are more suitable for the case of space photography.

The First Method

In the first method the two collinearity condition equations are used to calculate two equations for each measured photo point. In these equations all the camera parameters are used as unknowns, while the ground coordinates are treated in two different ways:

- the coordinates of the control points are used as observations, and
- the coordinates of the pass points are used as unknowns.

To explain this new method, assume that one starts to form the normal equations by calculating the contribution of all the observed control points, followed by the contribution of the unknown pass points. Then, the design matrices A, B, and W may be given by

$$\mathbf{A} = \left[\begin{array}{c:c} \underset{(NEQ_I,\ 6m)}{\dot{\mathbf{A}}_I} & 0 \\ \hdashline \underset{(NEQ_{II},\ 6m)}{\dot{\mathbf{A}}_{II}} & \underset{(NEQ_{II},\ 3n)}{\ddot{\mathbf{A}}} \end{array}\right] \quad \begin{array}{l} \text{(equations associated with observed control points)} \\ \text{(equations associated with unknown pass points)} \end{array} \tag{9}$$

$$\mathbf{B} = \left[\begin{array}{c:c:c} \underset{(NEQ_I,\ 3k)}{\ddot{\mathbf{B}}} & \underset{(NEQ_I,NEQ_I)}{\dddot{\mathbf{I}}_I} & 0 \\ \hdashline 0 & 0 & \underset{(NEQ_{II},NEQ_{II})}{\dddot{\mathbf{I}}_{II}} \end{array}\right] \quad \begin{array}{l} \text{(equations associated with observed control points)} \\ \text{(equations associated with unknown pass points)} \end{array} \tag{10}$$

$$\mathbf{W} = \left[\begin{array}{c} \underset{(NEQ,\ 1)}{\mathbf{W}_I} \\ \hdashline \underset{(NEQ_{II},\ 1)}{\mathbf{W}_{II}} \end{array}\right] \quad \begin{array}{l} \text{(equations associated with observed control points)} \\ \text{(equations associated with unknown pass points)} \end{array} \tag{11}$$

where

'.', '..', '...' are affixed to any elements related to the camera parameters, ground coordinates, and photo coordinates, respectively; and

'I', 'II' are affixed to the elements associated with the observed ground control points and the unknown pass points, respectively.

Accordingly, the elements of the above equations can be explained as follows:

NEQ_I, NEQ_{II} are the numbers of equations associated with the observed control points and the unknown pass points, respectively;

$\dot{\mathbf{A}}_I$, $\dot{\mathbf{A}}_{II}$ are the matrices which represent the partial derivatives of the collinearity condition equations (CCE) with respect to the unknown camera parameters and associated with the observed control points and the unknown pass points, respectively;

$\ddot{\mathbf{A}}$,$\ddot{\mathbf{B}}$ are the matrices which represent the partial derivatives of the CCE with respect to the unknown ground coordinates of the pass points and of the observed ground coordinates of the control points, respectively;

$\ddot{\mathbf{I}}_{\mathrm{I}}$, $\ddot{\mathbf{I}}_{\mathrm{II}}$ are two unit matrices which represent the partial derivatives of the CCE with respect to the photo measurements and associated with the observed control points and the unknown pass points, respectively; and

$\mathbf{W}_{\mathrm{I}}$, $\mathbf{W}_{\mathrm{II}}$ are the misclosure vectors associated with the observed control points and the unknown pass points, respectively.

Moreover, the above matrices can be written in more detailed forms as follows:

$$\dot{\mathbf{A}}_{\mathrm{I}} = \begin{bmatrix} \dot{\mathbf{a}}_1 \\ \dot{\mathbf{a}}_2 \\ \vdots \\ \dot{\mathbf{a}}_k \end{bmatrix} \quad (12) \qquad \ddot{\mathbf{B}} = \begin{bmatrix} \ddot{\mathbf{b}}_1 & 0 & 0 & \cdot & 0 \\ 0 & \ddot{\mathbf{b}}_2 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdots & 0 & \ddot{\mathbf{b}}_k \end{bmatrix} \quad (13)$$

$$\dot{\mathbf{A}}_{\mathrm{II}} = \begin{bmatrix} \dot{\mathbf{a}}_{k+1} \\ \dot{\mathbf{a}}_{k+2} \\ \vdots \\ \dot{\mathbf{a}}_{k+n} \end{bmatrix} \quad (14) \qquad \ddot{\mathbf{A}} = \begin{bmatrix} \ddot{\mathbf{a}}_{k+1} & 0 & 0 & \cdot & 0 \\ 0 & \ddot{\mathbf{a}}_{k+2} & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 & \ddot{\mathbf{a}}_{k+n} \end{bmatrix} \quad (15)$$

$$\mathbf{W}_{\mathrm{I}} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_3 \end{bmatrix} \quad (16) \qquad \mathbf{W}_{\mathrm{II}} = \begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{w}_{k-2} \\ \vdots \\ \mathbf{w}_{k+n} \end{bmatrix} \quad (17)$$

where

$\underset{(\mathrm{neq}_i,6m)}{\dot{\mathbf{a}}_i}$ are the partial derivatives of the CCE with respect to the unknown camera parameters and associated with the ground point i;

$\underset{(\mathrm{neq}_i,3)}{\ddot{\mathbf{a}}_i}$ are the partial derivatives of the CCE with respect to the unknown ground coordinates X_G, Y_G, Z_G of the pass point i ($\ddot{\mathbf{a}}_i = 0$ when processing the observed control point, i.e., when $i \leq k$);

$\underset{(\mathrm{neq}_i,3)}{\ddot{\mathbf{b}}_i}$ are the partial derivatives of the CCE with respect to the observed ground coordinates (X_G, Y_G, Z_G) of the control point i ($\ddot{\mathbf{b}}_i = 0$ when processing the unknown pass points, i.e., when $i > k$);

$\underset{(\mathrm{neq}_i,1)}{\mathbf{w}_i}$ is the misclosure vector associated with the ground point i;

m is the number of camera stations;

k,n are the numbers of points with observed and unknown ground coordinates, respectively; and

neq_i is the number of equations associated with the ground point i, and this number is equal to twice the number of photographs with the image of this ground point.

Further, using the definition of the **B** matrix given by Equation 10, the **M** matrix of Equation 5 may be given as follows:

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{M}_{\mathrm{I}} & 0 \\ \hline 0 & \mathbf{M}_{\mathrm{II}} \end{array}\right] \begin{array}{l} \text{(associated with observed ground control)} \\ \text{(associated with unknown pass points)} \end{array} \quad (18)$$

where

$$\mathbf{M}_{\mathrm{I}} = \begin{bmatrix} \mathbf{m}_1 & 0 & 0 & \cdot & 0 \\ 0 & \mathbf{m}_2 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 & \mathbf{m}_k \end{bmatrix} \quad (19) \qquad \mathbf{M}_{\mathrm{II}} = \begin{bmatrix} \mathbf{m}_{k+1} & 0 & 0 & \cdot & 0 \\ 0 & \mathbf{m}_{k+2} & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 & \mathbf{m}_{k+n} \end{bmatrix} \quad (20)$$

and

$\underset{(\mathrm{neq}_i,\mathrm{neq}_i)}{\mathbf{m}_i}$ is the nonzero submatrix of the **M** matrix associated with the ground point i, and

$\mathbf{m}_i = \dot{\mathbf{b}}_i\, \ddot{\mathbf{p}}_i^{-1}\, \dot{\mathbf{b}}_i' + \bar{\mathbf{p}}_i^{-1}$ for $i = 1, 2, \ldots, k$ (associated with control points (21)

or

$\mathbf{m}_i = \bar{\mathbf{p}}_i^{-1}$ for $i = k+1, k+2, \ldots, k+n$ (associated with pass points) (22)

where

$\underset{(3,3)}{\ddot{\mathbf{p}}_i}$ is the weight matrix for a ground control point i, and

$\underset{(\mathrm{neq}_i,\mathrm{neq}_i)}{\bar{\mathbf{p}}_i}$ is the weight matrix for the measured photo coordinates associated with the ground point i.

If the image coordinates are measured using a comparator, then $\bar{\mathbf{p}}_i$'S are diagonal matrices and the diagonal elements are proportional to the reciprocals of the variances of the measured photo coordinates.

It can be seen from Equations 19 and 20 that the inversion of the **M** matrix can be easily and directly obtained by inverting its block diagonal submatrices.

Using the definitions of the **A** and **M** matrices (Equations 9 and 18) by utilizing the detailed forms of $\dot{\mathbf{A}}_{\mathrm{I}}$, $\dot{\mathbf{A}}_{\mathrm{II}}$, $\ddot{\mathbf{A}}$, $\mathbf{M}_{\mathrm{I}}$, and $\mathbf{M}_{\mathrm{II}}$ matrices given by Equations 12, 14, 15, 19, and 20, respectively; and using the definitions for the **N** matrix and the **U** vector given by Equations 3 and 4, respectively; the system of normal equations can be written in detail as follows:

$$\left[\begin{array}{c|cccc} \dot{\mathbf{N}} & \bar{\mathbf{n}}_{k+1} & \bar{\mathbf{n}}_{k+2} & \cdot & \bar{\mathbf{n}}_{k+n} \\ \hline \bar{\mathbf{n}}_{k+1}' & \ddot{\mathbf{n}}_{k+1} & 0 & \cdot & 0 \\ \bar{\mathbf{n}}_{k+2}' & 0 & \ddot{\mathbf{n}}_{k+2} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{\mathbf{n}}_{k+n}' & 0 & 0 & \cdot & \ddot{\mathbf{n}}_{k+n} \end{array}\right] \left[\begin{array}{c} \dot{\boldsymbol{\delta}} \\ \hline \ddot{\boldsymbol{\delta}}_{k+1} \\ \ddot{\boldsymbol{\delta}}_{k+2} \\ \cdot \\ \ddot{\boldsymbol{\delta}}_{k+n} \end{array}\right] = \left[\begin{array}{c} \dot{\mathbf{U}} \\ \hline \ddot{\mathbf{u}}_{k+1} \\ \ddot{\mathbf{u}}_{k+2} \\ \cdot \\ \ddot{\mathbf{u}}_{k+n} \end{array}\right] \tag{23}$$

where

$$\underset{(6m,6m)}{\dot{\mathbf{N}}} = \sum_{i=1}^{k+n} \dot{\mathbf{a}}_i'\, \mathbf{m}_i^{-1} \dot{\mathbf{a}}_i \tag{24}$$

$$\underset{(3,3)}{\ddot{\mathbf{n}}_i} = \ddot{\mathbf{a}}_i'\, \mathbf{m}_i^{-1} \ddot{\mathbf{a}}_i,\ i = k+1, k+2, \ldots, k+n \tag{25}$$

$$\underset{(6m,3)}{\bar{\mathbf{n}}_i} = \dot{\mathbf{a}}_i'\, \mathbf{m}_i^{-1} \ddot{\mathbf{a}}_i \text{ and } i = k+1, k+2, \ldots, k+n \tag{26}$$

$$\underset{(6m,1)}{\dot{\mathbf{U}}} = \sum_{i=1}^{k+n} -\dot{\mathbf{a}}_i'\, \mathbf{m}_i^{-1}\, \mathbf{w}_i \tag{27}$$

$$\underset{(3,1)}{\ddot{\mathbf{u}}_i} = -\ddot{\mathbf{a}}_i'\, \mathbf{m}_i^{-1}\, \mathbf{w}_i \text{ and } i = k+1, k+2, \ldots, k+n \tag{28}$$

$\underset{(3,1)}{\ddot{\boldsymbol{\delta}}_i}$ represent the unknown corrections to the ground coordinates X_G, Y_G, and Z_G of the pass point i, and $i = k+1, k+2, \ldots, k+n$,

$\underset{(6m,1)}{\dot{\boldsymbol{\delta}}}$ represent the unknown corrections to the camera parameters.

One space photograph covers a relatively large area, and it may be practically sufficient to adjust few of the photographs at a time. Then, if the unknown ground coordinate vectors $\ddot{\boldsymbol{\delta}}_i$ are eliminated, the reduced system of normal equations can be written as:

$$\left[\dot{\mathbf{N}} - \sum_{i=k+1}^{k+n} \bar{\mathbf{n}}_i\, \ddot{\mathbf{n}}_i^{-1}\, \bar{\mathbf{n}}_i'\right] \left[\dot{\boldsymbol{\delta}}\right] = \left[\dot{\mathbf{U}} - \sum_{i=k+1}^{k+n} \bar{\mathbf{n}}_i\, \ddot{\mathbf{n}}_i^{-1}\, \ddot{\mathbf{u}}_i\right] \tag{29}$$

or in abbreviated form as

$$\underset{(6m,6m)}{\mathbf{N}_R}\ \underset{(6m,1)}{\dot{\boldsymbol{\delta}}} = \underset{(6m,1)}{\mathbf{U}_R} \tag{30}$$

Since the size of the $\mathbf{N}_R$ matrix is relatively small, the unknown corrections to the camera parameters $\dot{\boldsymbol{\delta}}$ can be easily calculated as follows:

$$\underset{(6m,1)}{\dot{\boldsymbol{\delta}}} = \mathbf{N}_R^{-1}\,\mathbf{U}_R \tag{31}$$

From Equation 23 one can find out that the unknown corrections, $\ddot{\boldsymbol{\delta}}_i$, to the ground coordinates for any point i can be calculated from equations similar to the following equations

$$\underset{(3,1)}{\ddot{\boldsymbol{\delta}}_i} = \ddot{\mathbf{n}}_i^{-1}\,\ddot{\mathbf{u}}_i - \ddot{\mathbf{n}}_i^{-1}\,\bar{\mathbf{n}}_i'\,\dot{\boldsymbol{\delta}} \tag{32}$$

The last step is to present the method by which the variance-covariance matrices of the unknowns are calculated.

The weight coefficient matrix for the unknown camera parameters is provided by the inversion of the $\mathbf{N}_R$ matrix when the $\dot{\boldsymbol{\delta}}$ are calculated (Equation 31).

The weight coefficient matrix, $\ddot{\mathbf{Q}}_{\ddot{\delta}_i}$, for the unknown ground coordinates of point i can be calculated by applying the error propagation law to equations similar to Equation 32 and the final result may be given as follows:

$$\underset{(3,3)}{\ddot{\mathbf{Q}}_{\ddot{\delta}_i}} = \ddot{\mathbf{n}}_i^{-1} + \ddot{\mathbf{n}}_i^{-1}\,\bar{\mathbf{n}}_i'\,\mathbf{N}_R^{-1}\,\bar{\mathbf{n}}_i\,\ddot{\mathbf{n}}_i^{-1} \tag{33}$$

Although Equation 33 only provides the submatrix associated with point i of the full weight coefficient matrix for all the unknown ground coordinates $\ddot{\boldsymbol{\delta}}$, this information is sufficient for the photogrammetric applications considered.

From the above formulas the following interesting conclusions can be drawn:

- The submatrix $\mathbf{m}_i$ and consequently its inverse $\mathbf{m}_i^{-1}$, associated with the ground point i, is dependent only on terms associated with this point ($\ddot{\mathbf{b}}_i$, $\ddot{\mathbf{p}}_i$, $\bar{\mathbf{p}}_i$); accordingly, $\mathbf{m}_i^{-1}$ can be calculated for any ground point independently of the other points;
- For the unknown ground pass points when $\ddot{\mathbf{b}}_i = 0$, $\mathbf{m}_i^{-1}$ is a diagonal matrix and the diagonal elements are proportional to the variance of the photo measurements. In this case $\mathbf{m}_i$ need not be inverted for such points;
- From the definition for $\dot{\mathbf{N}}$, $\ddot{\mathbf{n}}_i$, $\bar{\mathbf{n}}_i$, $\dot{\mathbf{U}}$ and $\ddot{\mathbf{u}}_i$ given in Equations 24, 25, 26, 27, and 28, respectively, it can be concluded that the contribution of any ground point i to the reduced system of normal equations (Equation 29) can be calculated independently one point after another. The same idea is applied in the process of back substitution to calculate the unknown ground coordinates $\ddot{\boldsymbol{\delta}}_i$ (Equation 32). Also, because $\mathbf{m}_i$, $\mathbf{m}_i^{-1}$, $\ddot{\mathbf{n}}_i$, $\ddot{\mathbf{n}}^{-1}$, $\dot{\mathbf{N}}$, $\mathbf{N}_R$, and $\mathbf{N}_R^{-1}$ are symmetric positive definite matrices, only their upper or lower triangle need to be calculated, and a computer subroutine using the Choleski method for the matrix inversion may be used wherever inversions of such matrices are needed.

Finally, by using the refined forms for the design matrices (**A, B,** and **W**), the residual vectors can be calculated from

$$\ddot{\mathbf{v}}_i = \ddot{\mathbf{p}}_i^{-1}\,\ddot{\mathbf{b}}'\,\mathbf{r}_i \tag{34}$$

$$\bar{\mathbf{v}}_i = \bar{\mathbf{p}}_i^{-1}\,\mathbf{r}_i \tag{35}$$

where

$\underset{(3,1)}{\ddot{\mathbf{v}}_i}$ is the residual subvector of the measured ground coordinates, X_G, Y_G, and Z_G of the point i;

$\underset{(neq_i,1)}{\bar{\mathbf{v}}_i}$ is the residual subvector of the measured photo coordinates associated with the ground point i; and

$\underset{(neq_i,1)}{\mathbf{r}_i}$ is the Lagrange multipliers subvector associated with the ground point i, i.e.,

$$\mathbf{r}_i = -\,\mathbf{m}_i^{-1}\,(\dot{\mathbf{a}}_i\,\dot{\boldsymbol{\delta}} + \mathbf{w}_i) \text{ when } i = 1, 2, \ldots, k \tag{36}$$

or

$$\mathbf{r}_{k+i} = -\,\mathbf{m}_{k+i}^{-1}\,(\dot{\mathbf{a}}_{k+i}\,\dot{\boldsymbol{\delta}} + \mathbf{w}_{k+i}) - \mathbf{m}_{k+i}^{-1}\,\ddot{\mathbf{a}}_{k+i}\,\ddot{\boldsymbol{\delta}}_i \text{ when } i = 1, 2, \ldots, n. \tag{37}$$

Also, here, it is worth mentioning that all the submatrices (except $\dot{\boldsymbol{\delta}}$) which are needed to calculate $\mathbf{r}_i$, $\ddot{\mathbf{v}}_i$, and $\bar{\mathbf{v}}_i$ (Equations 36, 37, 34, 35) for any ground point i are the submatrices associated with this ground point.

The Second Method

In the second method, the two collinearity condition equations are also used to calculate two equations for each measured photo point. The application of these equations is as follows:

- All the camera parameters are used as observations, and
- The ground coordinates are used in two different ways:
  - The coordinates of the control points are used as observations, and
  - The coordinates of the pass points are used as unknowns.

To form the normal equations for the least-squares adjustment, one has to calculate the **M** matrix given by Equation 5. Here, in the second method, the **B** matrix has partial derivatives of the collinearity condition equations corresponding both to the coordinates of the ground control points and to the camera parameters. Hence, it is impossible to calculate the contributions to the normal equations for either a point or a photo independently of the others. Accordingly, the whole **A**, **B**, **W**, and $\mathbf{M}^{-1}$ matrices have to be calculated before it is possible to calculate any contributions to the normal equations. However, to reduce the computing time and memory, all the ground control points are used as one group followed by all the pass points as another group. In this way, the **M** matrix is partitioned into four smaller submatrices which can be calculated one after another.

Assuming that one starts to calculate the observation equations associated with all the control points and then calculates the observation equations of all the pass points, then the design matrices **A**, **B**, and **W** can be written as follows:

$$\mathbf{A} = \begin{array}{l} \left[\begin{array}{c} 0 \\ \hline \underset{(\mathrm{NEQ_{II}},3n)}{\ddot{\mathbf{A}}} \end{array}\right] \end{array} \begin{array}{l} \text{equations associated with the control points} \\ \text{equations associated with the pass points} \end{array}$$

$$\mathbf{B} = \left[\begin{array}{c:c:c:c} \underset{(\mathrm{NEQ_I},6j)}{\dot{\mathbf{B}}_{\mathrm{I}}} & \underset{(\mathrm{NEQ_I},3k)}{\ddot{\mathbf{B}}} & \underset{(\mathrm{NEQ_I},\mathrm{NEQ_I})}{\bar{\mathbf{I}}_{\mathrm{I}}} & 0 \\ \hdashline \underset{(\mathrm{NEQ_{II}},6j)}{\dot{\mathbf{B}}_{\mathrm{II}}} & 0 & 0 & \underset{(\mathrm{NEQ_{II}},\mathrm{NEQ_{II}})}{\bar{\mathbf{I}}_{\mathrm{II}}} \end{array}\right] \begin{array}{l} \text{equations associated with the control points} \\ \text{equations associates with the pass points} \end{array}$$

$$\mathbf{W} = \left[\begin{array}{c} \underset{(\mathrm{NEQ_I},1)}{\mathbf{W}_{\mathrm{I}}} \\ \hdashline \underset{(\mathrm{NEQ_{II}},1)}{\mathbf{W}_{\mathrm{II}}} \end{array}\right] \begin{array}{l} \text{equations associated with the control points} \\ \text{equations associated with the pass points} \end{array}$$

where

$\ddot{\mathbf{A}}, \ddot{\mathbf{B}}, \mathbf{W}_{\mathrm{I}}, \mathbf{W}_{\mathrm{II}}$ have their definitions and their detailed expressions as in the first method (Equations 15, 13, 16, and 17, respectively);

j is the number of stations with observed camera parameters; and

$\dot{\mathbf{B}}_{\mathrm{I}}, \dot{\mathbf{B}}_{\mathrm{II}}$ are two matrices which represent the partial derivatives of the collinearity condition equations with respect to the observed camera parameters and associated with the observed ground control points and ground pass points respectively.

After matrix multiplication, the **M** matrix (Equation 5) can be written as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}'_{12} & \mathbf{M}_{22} \end{bmatrix}$$

where

$$\underset{(\mathrm{NEQ_I},\mathrm{NEQ_I})}{\mathbf{M}_{11}} = \dot{\mathbf{B}}_{\mathrm{I}}\,\dot{\mathbf{P}}^{-1}\,\mathbf{B}'_{\mathrm{I}} + \ddot{\mathbf{B}}\,\ddot{\mathbf{P}}^{-1}\,\ddot{\mathbf{B}}' + \bar{\mathbf{P}}_{\mathrm{I}}^{-1},$$

$$\underset{(\mathrm{NEQ_I},\mathrm{NEQ_I})}{\mathbf{M}_{12}} = \dot{\mathbf{B}}_{\mathrm{I}}\,\dot{\mathbf{P}}^{-1}\,\dot{\mathbf{B}}'_{\mathrm{II}}$$

$$\underset{(\mathrm{NEQ_{II}},\mathrm{NEQ_{II}})}{\mathbf{M}_{22}} = \dot{\mathbf{B}}_{\mathrm{II}}\,\dot{\mathbf{P}}^{-1}\,\dot{\mathbf{B}}'_{\mathrm{II}} + \dddot{\mathbf{P}}_{\mathrm{II}}^{-1},$$

and

$\dot{\mathbf{P}}$ is the weight matrix for the measured camera parameters.

The inversion of the $\mathbf{M}$ matrix can be obtained by the method of partitioning and may be expressed as:

$$\mathbf{M}^{-1} = \begin{bmatrix} \hat{\mathbf{M}}_{11} & \hat{\mathbf{M}}_{12} \\ \hat{\mathbf{M}}'_{12} & \hat{\mathbf{M}}_{22} \end{bmatrix}$$

where

$$\underset{(\mathrm{NEQ_{II}},\mathrm{NEQ_{II}})}{\hat{\mathbf{M}}_{22}} = (\mathbf{M}_{22} - \mathbf{M}'_{12}\,\mathbf{M}_{11}^{-1}\,\mathbf{M}_{12})^{-1}$$

$$\underset{(\mathrm{NEQ_{I}},\mathrm{NEQ_{II}})}{\hat{\mathbf{M}}_{12}} = -\,\mathbf{M}_{11}^{-1}\,\mathbf{M}_{12}\,\hat{\mathbf{M}}_{22}$$

$$\underset{(\mathrm{NEQ_{I}},\mathrm{NEQ_{I}})}{\hat{\mathbf{M}}_{11}} = \mathbf{M}_{11}^{-1} - \hat{\mathbf{M}}_{12}\,\mathbf{M}'_{12}\,\mathbf{M}_{11}^{-1}$$

The only unknowns in the case of the second method are the ground coordinates of the pass points $\ddot{\boldsymbol{\delta}}$ which can be calculated directly from

$$\underset{(3n,1)}{\ddot{\boldsymbol{\delta}}} = \mathbf{N}^{-1}\,\mathbf{U}$$

where

$$\mathbf{N} = \ddot{\mathbf{A}}'\,\hat{\mathbf{M}}_{22}\,\ddot{\mathbf{A}}$$

$$\mathbf{U} = -(\ddot{\mathbf{A}}\,\hat{\mathbf{M}}'_{12}\,\mathbf{W}_{\mathrm{I}} + \ddot{\mathbf{A}}'\,\hat{\mathbf{M}}_{22}\,\mathbf{W}_{\mathrm{II}})$$

Since the collinearity condition equations can be transformed to a linear form with regard to the unknowns (see Manual of Photogrammetry (1966), p. 51), only one iteration is needed to reach the final solution. The residual vectors (Equation 6) may be calculated from

$$\underset{(6j,1)}{\dot{\mathbf{V}}} = \dot{\mathbf{P}}^{-1}\,\dot{\mathbf{B}}'_{\mathrm{I}}\,\mathbf{R}_{\mathrm{I}} + \dot{\mathbf{P}}^{-1}\,\dot{\mathbf{B}}'_{\mathrm{II}}\,\mathbf{R}_{\mathrm{II}}$$

$$\underset{(3k,1)}{\ddot{\mathbf{V}}} = \ddot{\mathbf{P}}^{-1}\,\ddot{\mathbf{B}}'\,\mathbf{R}_{\mathrm{I}}$$

$$\underset{(\mathrm{NEQ_{I}},1)}{\dddot{\mathbf{V}}_{\mathrm{I}}} = \dddot{\mathbf{P}}_{\mathrm{I}}^{-1}\,\mathbf{R}_{\mathrm{I}}$$

$$\underset{(\mathrm{NEQ_{II}},1)}{\dddot{\mathbf{V}}_{\mathrm{II}}} = \dddot{\mathbf{P}}_{\mathrm{II}}^{-1}\,\mathbf{R}_{\mathrm{II}}$$

where

- $\dot{\mathbf{V}}$, $\ddot{\mathbf{V}}$ are the residuals of the measured camera parameters and of the measured ground coordinates of the control points, respectively;
- $\dddot{\mathbf{V}}_{\mathrm{I}}$, $\dddot{\mathbf{V}}_{\mathrm{II}}$ are the residuals of the measured photo coordinates associated with the control points and the pass points, respectively; and
- $\mathbf{R}_{\mathrm{I}}$, $\mathbf{R}_{\mathrm{II}}$ are two Lagrange multipliers sub-vectors, and they may be given by

$$\mathbf{R}_{\mathrm{I}} = -(\hat{\mathbf{M}}_{11}\,\mathbf{W}_{\mathrm{I}} + \hat{\mathbf{M}}_{12}\,(\ddot{\mathbf{A}}\,\ddot{\boldsymbol{\delta}} + \mathbf{W}_{\mathrm{II}}))$$

$$\mathbf{R}_{\mathrm{II}} = -(\hat{\mathbf{M}}'_{12}\,\mathbf{W}_{\mathrm{I}} + \hat{\mathbf{M}}_{22}\,(\ddot{\mathbf{A}}\,\ddot{\boldsymbol{\delta}} + \mathbf{W}_{\mathrm{II}}))$$

Aerial Triangulation Test Results

Several aerial triangulation tests were performed using the two computer programs to adjust some of the Skylab space photography (scale 1:2,900,000), and a combination of Skylab and very high altitude aerial photography (scale 1:120,000) with and without utilization of Skylab orbital parameters. Although some test results were given in previous publications (Ali, 1976; Ali and Brandenberger, 1978), the full details of all the tests and their analysis are given in the author's thesis (Ali, 1980). Here, only a few tests will be described in order to show the efficiency and the capability of the developed programs.

One model of Skylab photography (S-190 A camera), covering the areas of Windsor in Canada and parts of the State of Michigan in the U.S.A., was adjusted using the developed bundle adjustment pro-

TABLE 1. RMSE AND AVMR FOR THE ROUNDED CONTROL POINTS

		Ontario area					
		RMSE			AVMR		
Coord. Rounded to	Number of points	X (m)	Y (m)	Z (m)	X (m)	Y (m)	Z (m)
100 m	28	27	33	28	48	49	46
200	28	62	61	64	99	100	97
300	28	93	85	86	148	140	144
400	28	122	119	141	190	197	199
500	28	152	127	182	248	222	246
600	28	162	174	201	296	300	297
700	28	210	193	216	346	310	303
800	28	233	248	240	373	387	372
900	28	267	254	266	432	449	444
1000	28	282	267	278	446	488	461

grams. Seventy-six points were identified in the model, and their ground coordinates were measured from either 1:25,000 or 1:24,000 scale maps of Canada and the U.S.A., respectively. It was assumed that space photography would not be used for aerial triangulation in areas for which large scale maps exist. Consequently, and to simulate real conditions, map coordinates were rounded off to the nearest 100, 200, . . . to 1,000 m. The differences (or residuals) between the map coordinates and their rounded values, as well as the Root Mean Square Errors (RMSE) of the differences for each case were calculated. Table 1 shows the RMSE's as well as the Absolute Values of the Maximum Residuals (AVMR) for each case. Figure 1 shows the relation between the RMSE's and the values to which the map coordinates were rounded.

The map coordinates and their rounded values for 28 well distributed points were used as control in a series of eleven aerial triangulation tests. The differences between the adjusted ground coordinates of the control and pass points and the map coordinates and the RMSE's of these differences were calculated for all tests. Table 2 shows the RMSE's and the AVMR's for the eleven tests nos. 1 to 11 when the coordinates of the 28 control points were either the map coordinates or their values rounded to the nearest 100, 200, . . . or 1,000 m, respectively. Figure 2 shows the relation between the RMSE's for the adjusted coordinates of both the control and the pass points and the accuracy of the control points used. Table 2 also shows the RMSE's and AVMR's for one test (Test no. 12) using the second bundle adjustment method and when the coordinates of the control points were rounded to the nearest 500 m.

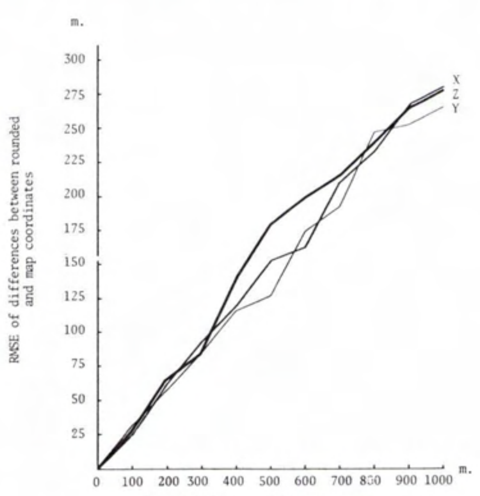

FIG. 1. Graphs showing the RMSE's for the differences between the rounded coordinates and the map coordinates as a function of the rounded coordinates values (Ontario model).

TABLE 2. RMSE AND AVMR (ONTARIO MODEL USING THE FIRST AND SECOND BUNDLE ADJUSTMENT METHOD)

	RMSE of control points			AVMR of control points			RMSE of pass points			AVMR of pass points		
Serial no. of test	X (m)	Y (m)	Z (m)	X (m)	Y (m)	Z (m)	X (m)	Y (m)	Z (m)	X (m)	Y (m)	Z (m)
First bundle adjustment method												
1	01	01	00	02	01	00	35	40	148	104	127	394
2	22	30	28	46	80	45	40	39	162	92	127	394
3	37	45	62	91	96	99	37	46	128	116	122	374
4	52	44	84	112	104	151	52	34	148	96	136	397
5	53	66	138	105	104	220	46	58	194	141	183	532
6	76	84	175	138	166	243	69	44	130	149	168	325
7	34	76	196	113	165	292	39	53	191	98	126	385
8	68	65	209	116	161	313	75	62	205	168	155	386
9	114	107	228	247	221	337	100	88	210	203	248	406
10	86	111	241	155	273	396	88	88	193	181	239	385
11	40	63	280	91	133	474	36	58	286	122	108	588
Second bundle adjustment method												
12	34	53	170	76	159	242	36	41	157	100	129	357

CONCLUSIONS

The aerial triangulation results of the Skylab model, using the first bundle adjustment method when the coordinates of the ground control points and the check points were measured from 1:25,000 scale maps (Table 2), show RMSE's of 53 m in planimetry (17 μm in photo plane) and 148 m in height (0.035 percent of orbital height). Although the planimetric accuracy seems of reasonable order, it is lower than the accuracy claimed by some other investigators (Kubik and Kure, 1971; Brown, 1976) for the adjustment of normal aircraft photography using the bundle adjustment technique. The reason for the relatively poor accuracy could be due to the fact that the ground coordinates used to check the results were measured from maps. The height accuracy is poor as expected; this is mainly due to the small base/height ratio of the Skylab photographs.

Tests no. 1 to 11 using the first bundle adjustment program show that using ground control of different accuracy is not the main factor which influences the accuracy of the aerial triangulation; the differences of the RMSE's are statistically insignificant (at the 95 percent confidence level). In fact, in Test no. 11 for the Ontario model, better results were obtained when the coordinates of the ground control were rounded to 1000 m compared with the results for coordinates rounded to 300 m. This, to some extent,

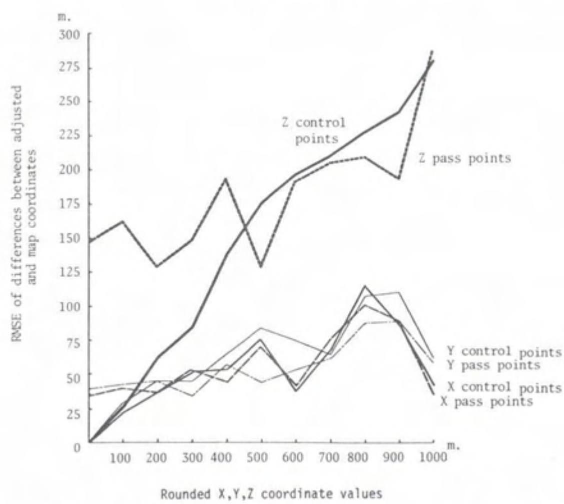

FIG. 2. Graphs showing the RMSE's for the differences between the adjusted coordinates using the first bundle adjustment algorithm and map coordinates as a function of the rounded coordinate values for the control points (Ontario model).

coincides with theoretical studies done by Ebner (1972) concerning the effect of the accuracy of the control on the results of aerial triangulation.

Comparing the similar tests using the second and the first bundle adjustment methods (Tests no. 12 and 6, respectively), it can be seen that a significant improvement in planimetry was obtained by using the second method. But during the investigation, the camera parameters supplied to the authors by NASA were not well defined. Moreover, due to some inherent difficulties during the mission, the parameters were also not reliable. Accordingly, the camera parameters and their weights resulting from the solution using the first method were used as observations in the second method. Hence, the significant improvement of the accuracy of the results obtained from the second method compared with those of the first method should not be used to draw any final conclusion.

Acknowledgment

Sponsoring of the project by the Dept. of Energy, Mines & Resources and the National Research Council (Now Natural Sciences & Engineering Research Council), Canada, is gratefully acknowledged.

References

Ali, M. E. O., 1976. Aerial Triangulation with SKYLAB Photography, Presented paper, Congress of the International Society of Photogrammetry, Helsinki.

Ali, M. E. O., and A. J. Brandenberger, 1978. Combined SKYLAB and High Altitude Aircraft Photography Space Triangulation, *Proceedings, International Symposium of New Technology for Mapping,* Ottawa, October.

———, 1980. *Aerial Triangulation with SKYLAB and High Altitude Aircraft Photography, Consideration of SKYLAB Orbital Parameters,* Ph.D. Thesis, Department of Photogrammetry, Laval University, Canada.

Brown, D. C., 1964. *Research in Mathematical Targeting. The Practical and Rigorous Adjustment of Large Photogrammetric Nets,* Technical Documentary Report No. RADC-TDR-64-353, Rome Air Development Center, Air Force Systems Command, Griffiss Air Force Base, New York, October.

———, 1976. The Bundle Adjustment—Progress and Report, XIII Congress of the International Society for Photogrammetry, Helsinki.

Ebner, H., 1972. Theoretical Accuracy Models for Block Triangulation, *Bildmessung und Luftbildwesen* (214-221).

Keller, M., G. Tewinkel, 1967. *Block Analytic Aerotriangulation,* U.S. Department of Commerce, Coast and Geodetic Survey, Technical Bulletin (35), November.

Kubik, K., and J. Kure, 1971. ISP Investigation into the Accuracy of Photogrammetric Triangulation (Intermediate Report), Aerial Triangulation Proceedings of a Symposium Held at University College London 9-13 August.

Schmid, H., 1956-57. An Analytical Treatment of the Problem of Triangulation by Stereophotogrammetry, *Photogrammetria* 12 (2 and 3).

Schut, G. H., 1978. *Adjustment of Bundles,* National Research Council Canada, Ottawa P-PR-47, NRC-16832, June.

Wong, K. W., 1971. *Simultaneous Adjustment of Photogrammetric Geodetic Observations (SAPGO),* Civil Engineering Studies, Photogrammetry Series No. 30, U. of Illinois, Urbana.

American Society of Photogrammetry, 1966. *Manual of Photogrammetry,* Third Edition.

(Received 21 June 1980; revised and accepted 23 June 1981)

Symposium
Visibility: An Update of the Field

La Jolla Village Inn, La Jolla, California
1-5 March 1982

Presented by the University of California, San Diego, and led by Dr. Seibert Quimby Duntley, the Symposium will include the following sessions: Introduction and Overview, Elementary Visibility Calculations, Daytime Visibility, Model for a Non-Absorbing Atmosphere, The Information Content of Images, Visual Performance, Appearance of the Earth and the Oceans, Effects of Atmospheric Absorption, and Visibility through Water.

For further information please contact

Deborah Eck
University Extension, X-001
University of California, San Diego
La Jolla, CA 92093
Tele. (714) 452-3450