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Calibration and Model Reconstruction in Analytical Close-Range Stereophotogrammetry

Part I: Mathematical Fundamentals*

The method has application to stereophotography, rasterstereography (using a projected grid), Moiré topography, and stereoradiography.

ABSTRACT: *The application of the photogrammetric bundle method to close-range problems is discussed with respect to stereophotography, rasterstereography (using a projected grid), Moiré topography, and stereoradiography. In Part I of this work the mathematical fundamentals are presented in a form appropriate for application in most close-range problems. A simplified procedure characterized by alternating orientation and reconstruction steps is outlined. Questions of convergence of the solution of the bundle equations are discussed and an error analysis is given.*

In the subsequent Part II, special calibration and reconstruction procedures for rasterstereography and Moiré topography are described.

INTRODUCTION

IN CLOSE-RANGE PHOTOGRAMMETRY, particularly in biostereometrics, simple uncalibrated photographic devices are very often used in various geometries which may considerably deviate from the normal case of photogrammetry. For example, oblique or convergent camera orientations are used in the case of stereophotography, or, in the case of rasterstereography (a stereophotogrammetric method using a projected grid, see Frobin and Hierholzer (1978a, 1978b)), the stereo image pair is produced by two different optical devices. In addition, in contrast to the situation in geodesy, the precision of the control point coordinates may be poor, and the cameras may be non-metric.

Consequently, a great variety of procedures for calibration and model reconstruction, often dedicated to a special project, has been developed in the past, and some of them are not very satisfactory with regard to the handling of statistical and systematic image errors. However, the analytical method of bundle adjustment which has been developed in recent years allows a generally valid determination of all unknown parameters of the photogrammetric problem, such as interior and exterior orientation, model coordinates, and control point corrections in a unified procedure which is rigorous from a statistical point of view (Brown, 1976). In the case of the self-calibrating bundle adjustment, systematic image errors can be compensated for by introducing additional parameters into the bundle equations (see, e.g., Grün, 1978), the whole method remaining a single uniform solution procedure of the photogrammetric problem.

The calculation method described in the present Part I of this paper has been worked out for the evaluation of stereoradiographs (with convergence angles of about 10°), of 90° biplane radiographs, and of rasterstereographs and conventional stereophotographs (with arbitrary conver-

* Part II of this article will be published in a subsequent issue.

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gence angles). Applying some modifications concerning the image data preparation, Moiré topograms may be evaluated too. This is due to the fact that a Moiré apparatus may be considered as a device for distance measurements by triangulation, the basic geometry being similar to that of stereophotography. In Part II of this work (Frobin and Hierholzer, 1982) special evaluation procedures for rasterstereography and Moiré topography will be presented. In contrast to aerial photogrammetry with block triangulation, we are here concerned only with image pairs.

As mentioned above, an exact determination of a priori control point coordinates may be difficult, especially in large control point distributions. Therefore, the corrections of the control point coordinates calculated in the bundle adjustment may be relatively large. In addition, the starting values for all unknown parameters in the iteration procedure of the bundle method may be quite poor. Consequently, modifications of the bundle equations as well as of the convergence criterion of the iteration are necessary as compared to the algorithms for other applications reported in the literature (Grün, 1978; Schwidefsky and Ackermann, 1976).

Formulation of the Error Equations and Normal Equations

The solution of the photogrammetric problem consists in a least-squares fit of the calculated model parameters to the observations by a variation of the unknown model parameters. The observations consist of the measured image coordinates of the control points and of the object points. Furthermore, in the case of control points which are not exactly known, their three-dimensional coordinates determined in an independent measurement are also considered as observations. The error distribution of the observations is assumed to be normal (Gaussian), possibly with different standard deviations for the different observations, and without error correlation.

The unknown model parameters consist of the elements of interior and exterior orientation of the imaging systems, possibly supplemented by additional parameters such as distortion, and of the three-dimensional coordinates of the reconstructed model points.

The relation between the observations or the corresponding calculated model parameters and the unknown model parameters is established by the imaging (or collinearity) equations. As the number of equations is generally higher than the number of unknown model parameters, the least-squares principle has to be employed to arrive at a unique solution. From a statistical viewpoint, the square sum of the observation errors has to be minimized in a rigorous analysis. However, in some applications other measures (e.g., the residuals of the collinearity or coplanarity equations) are optimized. This may lead to discrepancies in the case of strong deviations from the normal case of photogrammetry (Schwidefsky and Ackermann, 1976, chapter 3.4.1.2).

To establish the imaging equations, we consider a point $\mathbf{X}$ given in the basic spatial coordinate system (X,Y,Z); this point is imaged onto the two image planes producing a "left" and a "right" image point $\mathbf{X}_{bl}$ and $\mathbf{X}_{br}$ respectively given in the image coordinate systems (x_{bl},y_{bl},z_{bl}) and (x_{br},y_{br},z_{br}). The coordinate systems together with the elements of interior and exterior orientation are illustrated in Figure 1 (the rotation angles being omitted for the sake of simplicity).

According to this choice of geometry parameters, the imaging (or collinearity) equations may be written as

$$\mathbf{X}_{bl} = \begin{pmatrix} x_{bl} \\ y_{bl} \\ 0 \end{pmatrix} = F_l \cdot (\mathbf{X}_l - \mathbf{X}_{ol}) + \mathbf{X}_{ol} \qquad (1)$$

$$\mathbf{X}_{br} = \begin{pmatrix} x_{br} \\ y_{br} \\ 0 \end{pmatrix} = F_r \cdot (\mathbf{X}_r - \mathbf{X}_{or}) + \mathbf{X}_{or} \qquad (2)$$

where

$$F_l = Z_{ol}/(Z_{ol} - Z_l),\; F_r = Z_{or}/(Z_{or} - Z_r) \qquad (3,4)$$

and $\mathbf{X}_{ol}$ and $\mathbf{X}_{or}$ are the left and right center of perspective (Figure 1). The coordinates $\mathbf{X}_l$ and $\mathbf{X}_r$ are the coordinates of point $\mathbf{X}$ transformed into the left and right image coordinate system:

$$\mathbf{X}_l = \begin{pmatrix} X_l \\ Y_l \\ Z_l \end{pmatrix} = \mathbf{D}(\phi_l, \omega_l, \kappa_l)\mathbf{X} + \mathbf{X}_{sl} \qquad (5)$$

$$\mathbf{X}_r = \begin{pmatrix} X_r \\ Y_r \\ Z_r \end{pmatrix} = \mathbf{D}(\phi_r, \omega_r, \kappa_r)\mathbf{X} + \mathbf{X}_{sr} \qquad (6)$$

Definitions for the rotation matrix $\mathbf{D}$ are given, e.g., by Finsterwalder and Hofmann (1968) or

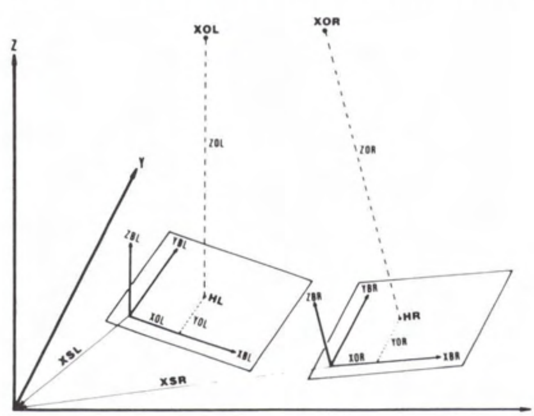

Fig. 1. Coordinate systems and elements of interior and exterior orientation of a stereoscopic image pair $\mathbf{H}_l$, $\mathbf{H}_r$ – principal points

Frobin and Hierholzer (1979). $\mathbf{X}_{sl}$ and $\mathbf{X}_{sr}$ represent the translational part of the transformation (Figure 1).

In the following calculations we combine the elements of interior and exterior orientation into nine-vectors:

$$\mathbf{O}_l = \begin{pmatrix} \mathbf{X}_{ol} \\ \mathbf{X}_{sl} \\ \Phi_l \end{pmatrix} \quad (7)$$

$$\mathbf{O}_r = \begin{pmatrix} \mathbf{X}_{or} \\ \mathbf{X}_{sr} \\ \Phi_r \end{pmatrix} \quad (8)$$

where Φ represents the three rotation angles (ϕ,ω,κ).

In the preceding treatment the validity of the collinearity equations is assumed. If necessary, the imaging equations may be modified to account for image distortion and other systematic image errors by introducing additional parameters which may be treated in much the same way as the orientation elements. Such a procedure is discussed, for example, by Grün (1978), but we shall not go into details of this.

To establish the error equations of the photogrammetric problem, we consider a spatial distribution of n points. The first n_p of them are control points with approximately known coordinates $\mathbf{X}_p$, the rest being object points with unknown coordinates $\mathbf{X}$. From the observed image pairs $\mathbf{X}_{ml}$ and $\mathbf{X}_{mr}$ of all points and from the imaging Equations 1 and 2, we obtain the error equations

$$\mathbf{X}_{bl_i} = \mathbf{X}_{bl}(\mathbf{X}_i, \mathbf{O}_l) = \mathbf{X}_{ml_i} + \mathbf{V}_{l_i} \quad (9)$$

$$\mathbf{X}_{br_i} = \mathbf{X}_{br}(\mathbf{X}_i, \mathbf{O}_r) = \mathbf{X}_{mr_i} + \mathbf{V}_{r_i} \quad (10)$$

$$(i = 1, \ldots, n)$$

with the residuals $\mathbf{V}_{l_i}$ and $\mathbf{V}_{r_i}$ due to the measuring errors and the overdeterminacy of the system. These are equations for the unknown point coordinates $\mathbf{X}_i$ (object points *and* control points) and for the unknown orientation elements $\mathbf{O}_l$ and $\mathbf{O}_r$. As the approximately measured control point coordinates $\mathbf{X}_{p_i}$ are considered as observations, too, additional error equations have to be established for the n_p control points only:

$$\mathbf{X}_i = \mathbf{X}_{p_i} + \mathbf{W}_i \quad (11)$$
$$(i = 1, \ldots, n_p)$$

The system of Equations 9, 10, and 11 has to be solved with the constraint that the (generally weighted) square sum of the residuals is minimized:

$$\sum_1^n \mathbf{V}_{l_i}{}^{\mathrm{T}} \mathbf{G}_{l_i} \mathbf{V}_{l_i} + \sum_1^n \mathbf{V}_{r_i}{}^{\mathrm{T}} \mathbf{G}_{r_i} \mathbf{V}_{r_i} + \sum_1^{n_p} \mathbf{W}_i{}^{\mathrm{T}} \mathbf{H}_i \mathbf{W}_i = \text{Min.} \quad (12)$$

$\mathbf{G}_{l_i}$, $\mathbf{G}_{r_i}$, and $\mathbf{H}_i$ are weight factors or, more generally, weight matrices of the left and right image measurements and of the control point measurements. The weight factors may be determined by a priori estimations, proportional to the inverse of the observation errors. The observation errors are generally assumed to be uncorrelated. The weight matrices are then diagonal.

To simplify the following calculations, we note that the third sum in Equation 12 may be extended to n, if weights of zero are introduced for the object points. Thus, the control points and the object points may be treated in exactly the same way.

We now collect the vectors and matrices occurring into "hypervectors" and "hypermatrices" to obtain the error equations and the minimum condition in compressed form

$$\mathbf{X}_b = \mathbf{X}_m + \mathbf{V} \quad (13)$$
$$\mathbf{X} = \mathbf{X}_p + \mathbf{W} \quad (14)$$
$$\mathbf{V}^{\mathrm{T}}\mathbf{G}\mathbf{V} + \mathbf{W}^{\mathrm{T}}\mathbf{H}\mathbf{W} = \text{Minimum} \quad (15)$$

with the abbreviations

$$\mathbf{X}_b = \begin{Bmatrix} \mathbf{X}_{bl} \\ \mathbf{X}_{br} \end{Bmatrix} = \begin{Bmatrix} \mathbf{X}_{bl_1} \\ \cdot \\ \cdot \\ \mathbf{X}_{bl_n} \\ \mathbf{X}_{br_1} \\ \cdot \\ \cdot \\ \mathbf{X}_{br_n} \end{Bmatrix}; \quad \mathbf{X}_m = \begin{Bmatrix} \mathbf{X}_{ml} \\ \mathbf{X}_{mr} \end{Bmatrix} = \begin{Bmatrix} \mathbf{X}_{ml_1} \\ \cdot \\ \cdot \\ \mathbf{X}_{ml_n} \\ \mathbf{X}_{mr_1} \\ \cdot \\ \cdot \\ \mathbf{X}_{mr_n} \end{Bmatrix};$$

$$\mathbf{V} = \begin{Bmatrix} \mathbf{V}_l \\ \mathbf{V}_r \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}_{l_1} \\ \cdot \\ \cdot \\ \mathbf{V}_{l_n} \\ \mathbf{V}_{r_1} \\ \cdot \\ \cdot \\ \mathbf{V}_{r_n} \end{Bmatrix} \quad \begin{matrix} (16) \\ (17) \\ (18) \end{matrix}$$

and similar expressions for $\mathbf{X}$, $\mathbf{X}_p$, and $\mathbf{W}$. The matrices are given by

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_l & \mathrm{O} \\ \mathrm{O} & \mathbf{G}_r \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{l_1} & & & & \\ & \ddots & & \mathrm{O} & \\ & & \mathbf{G}_{l_n} & & \\ & & & \mathbf{G}_{r_1} & \\ & \mathrm{O} & & & \ddots \\ & & & & \mathbf{G}_{r_n} \end{bmatrix} \quad (19)$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \ddots \\ & & \mathbf{H}_n \end{bmatrix} \quad (20)$$

assuming no observation error correlation. The dimensions of the vectors are $4n$ for $\mathbf{X}_b$, $\mathbf{X}_m$, and $\mathbf{V}$; $3n$ for $\mathbf{X}$, $\mathbf{X}_p$, and $\mathbf{W}$ (the last $3n - 3n_p$ components being irrelevant); and $4n \times 4n$ and $3n \times 3n$ respectively for the matrices.

From the error equation we obtain by linearization

$$\mathbf{A}\,\mathbf{dO} + \mathbf{B}\,\mathbf{dX} = \mathbf{a} + \mathbf{V} \tag{21}$$

$$\mathbf{dX} = \mathbf{b} + \mathbf{W} \tag{22}$$

where

- $\mathbf{A}$ = functional matrix containing all derivatives of the calculated image coordinates (Equations 1 and 2) with respect to the orientation elements ($4n \times 18$ elements);
- $\mathbf{B}$ = functional matrix containing all derivatives of the calculated image coordinates (Equations 1 and 2) with respect to the control point and object point coordinates ($4n \times 3n$ elements);
- $\mathbf{dO}$ = correction vector of the orientation (18 elements);
- $\mathbf{dX}$ = correction vector of the control point and object point coordinates ($3n$ elements);
- $\mathbf{a} = \mathbf{X}_m - \mathbf{X}_{b_0}$ = image error at the point of consideration; and
- $\mathbf{b} = \mathbf{X}_p - \mathbf{X}_0$ = error of the control points (the last $3n - 3n_p$ components are irrelevant).

From Equations 21 and 22 we obtain in the usual way, utilizing the weight matrices, the normal equations

$$\mathbf{A}^T\mathbf{GA}\,\mathbf{dO} + \mathbf{A}^T\mathbf{GB}\,\mathbf{dX} = \mathbf{A}^T\mathbf{Ga} \tag{23}$$

$$\mathbf{B}^T\mathbf{GA}\,\mathbf{dO} + (\mathbf{B}^T\mathbf{GB} + \mathbf{H})\,\mathbf{dX} = \mathbf{B}^T\mathbf{Ga} + \mathbf{Hb} \tag{24}$$

Starting from some pre-estimated values for the unknown orientation, $\mathbf{O}_0$, and for the control point and object point coordinates $\mathbf{X}_0$, the solution of these normal equations yields corrections $\mathbf{dO}$ and $\mathbf{dX}$ of the initial values. A reasonable choice for the initial control point coordinates (first $3n_p$ components of $\mathbf{X}_0$) would of course be their observed values $\mathbf{X}_p$. By iteration, we arrive at final values fulfilling the minimum condition (Equation 15).

As mentioned earlier, the last $n - n_p$ submatrices of $\mathbf{H}$ belonging to the unknown object points are zero. Thus, the corresponding components of $\mathbf{dX}$ and $\mathbf{b}$ have no influence in Equation 24. However, for the control points the contribution of $\mathbf{Hb}$ in this equation is quite important, at least if the observed values of the control point coordinates are not very accurate and if the initial value $\mathbf{O}_0$ of the orientation is only a rough estimation. In such situations, which often occur in close-range photogrammetry, the iteration of the normal equations (Equations 23 and 24) must be continued testing the least squares sum (Equation 15) for reaching the minimum. Testing of the corrections $\mathbf{dO}$ and $\mathbf{dX}$ against a threshold, as sometimes reported in literature, is only a necessary, not a sufficient, condition for the least squares solution.

If the initial control point coordinates are chosen as their observation values, then $\mathbf{b} = 0$ in the first iteration step. However, omitting the term $\mathbf{Hb}$ in Equation 24 in every iteration step (see, e.g., Grün, 1978; Schwidefsky and Ackermann, 1976, chapter 3.4.4.2) appears to be equivalent to utilizing a zero weight matrix $\mathbf{H}$ for the control points, if the iteration is continued to the least squares minimum (Equation 15). That is, the final solution is then not bounded to (weighted) minimal control point corrections, and it is, in general, independent of the actual choice of $\mathbf{H}$. The only effect of introducing $\mathbf{H}$ on the left, but not on the right side of Equation 24 is to change the convergence rate of the iteration.

This can be understood from the following considerations. First, omitting $\mathbf{Hb}$ means that only the image errors $\mathbf{a}$ but not the control point errors $\mathbf{b}$ are taken into account in the normal equations; consequently, the observed control point coordinates are of no influence on the final solution ($\mathbf{O}$, $\mathbf{X}$) of the least squares problem.

Second, if we assume that a least-squares solution ($\mathbf{O}$, $\mathbf{X}$) (with respect to the image errors) has already been obtained, then

$$\mathbf{A}^T\mathbf{Ga} = 0; \qquad \mathbf{B}^T\mathbf{Ga} = 0$$

From Equations 23 and 24 it then follows (with $\mathbf{Hb} = 0$) that

$$\mathbf{dO} = 0; \qquad \mathbf{dX} = 0$$

and the solution is stationary, independent of $\mathbf{H}$. If the final solution ($\mathbf{O}$, $\mathbf{X}$) is not yet achieved, the correction $\mathbf{dX}$ calculated from the normal equations is apparently the less the larger the elements of $\mathbf{H}$ are chosen. Thus, the rate of convergence, but not the final stationary solution, is affected by the choice of $\mathbf{H}$, if $\mathbf{b} = 0$ throughout the iteration.

In the above analysis we have tacitly assumed that initially only a rough estimation of the orientation elements is known. This is the case in most close-range applications. However, the orientation may also be treated as an observation and can thus be introduced into the normal equation (Equation 23) using an appropriate weight matrix $\mathbf{F}$ and an error vector $\mathbf{c}$ in analogy to $\mathbf{H}$ and $\mathbf{b}$ in Equation 24 (see, e.g., Brown, 1976).

The system of normal equations (Equations 23 and 24) is solved by the well known procedure of partial reduction yielding

$$\mathbf{dO} = (\mathbf{A}^T\mathbf{QA})^{-1}\,\mathbf{A}^T\,(\mathbf{Qa} - \mathbf{GB}(\mathbf{B}^T\mathbf{GB} + \mathbf{H})^{-1}\,\mathbf{Hb}) \tag{25}$$

using the pseudo weight matrix

$$\mathbf{Q} = \mathbf{G} - \mathbf{GB}(\mathbf{B}^T\mathbf{GB} + \mathbf{H})^{-1}\,\mathbf{B}^T\mathbf{G}. \tag{26}$$

The correction **dX** is then calculated by back substitution of Equation 25 into

$$\mathbf{dX} = (\mathbf{B}^T\mathbf{G}\mathbf{B} + \mathbf{H})^{-1}\,(\mathbf{B}^T\mathbf{G}\mathbf{a} + \mathbf{H}\mathbf{b} - \mathbf{B}^T\mathbf{G}\mathbf{A}\;\mathbf{dO}) \tag{27}$$

derived from Equation 24.

Of course, the above solution is possible only for a proper control point distribution with nonvanishing weights. Otherwise, the necessary matrix inversions are not possible due to singularity.

If the initial control point coordinates are chosen as their observed values, then initially $\mathbf{b} = 0$. However, this is not true in the subsequent iterations, as discussed above.

The functional matrices **A** and **B** may be built up from the contributions of the individual control and object points, as reported for example by Frobin and Hierholzer (1979) in a special representation.

Alternating Orientation and Reconstruction

Loosely speaking, the calculation of the orientation, e.g., according to Equation 25, may be denoted as "orientation," and the calculation of the point coordinates, Equation 27, as "reconstruction." Rewriting the normal equations leads to an alternative for the simultaneous solution described above. From Equations 23 and 24 we obtain

$$\mathbf{A}^T\mathbf{G}\mathbf{A}\;\mathbf{dO} = \mathbf{A}^T\mathbf{G}\mathbf{a}'' \tag{28}$$

$$(\mathbf{B}^T\mathbf{G}\mathbf{B} + \mathbf{H})\mathbf{dX} = \mathbf{B}^T\mathbf{G}\mathbf{a}' + \mathbf{H}\mathbf{b} \tag{29}$$

with the abbreviations

$$\mathbf{a}'' = \mathbf{a} - \mathbf{B}\;\mathbf{dX} \tag{30}$$

$$\mathbf{a}' = \mathbf{a} - \mathbf{A}\;\mathbf{dO} \tag{31}$$

Assuming $\mathbf{a}''$ to be known, we may calculate the orientation from Equation 28. This is possible in the case of exactly known control points (leaving the object points out of consideration for the moment). On the other hand, if the orientation elements are exactly known, the point coordinates may be reconstructed from Equation 29. Now if the orientation elements and the control point coordinates are already known with some accuracy, it should be possible to replace the simultaneous solution of the system (Equations 23 and 24) by an alternating solution of Equations 28 and 29 at least for small corrections **dO** and **dX**, i.e., in linear approximation. The procedure is as follows:

> In the first step the control point coordinates are considered to be accurate. The orientation elements are determined by iteration of Equation 28 alone with $\mathbf{dX} = 0$.
>
> In the second step the new orientation is used to calculate corrections of the control point coordinates (and possibly of the object points) by iteration of Equation 29 with $\mathbf{dO} = 0$.
>
> The corrected control points coordinates are inserted into Equation 28 again to obtain better orientation values, and so forth.

It should be noted that in the first step only the relatively accurate control point coordinates, but not the roughly estimated object points, are used to calculate the initial orientation. However, after the second step the object points are likewise known with good accuracy and may thus be used in Equation 28 from the third step on. In other words, the whole procedure may be regarded as a higher order correction of the basic photogrammetric procedure utilizing exactly known control points for calibration, and the then exactly known orientation for reconstruction.

The advantage of this procedure consists in the simpler structure of the equations as compared to the bundle equations. However, it appears that the computing time savings are not very important, as more iterations are necessary than for the simultaneous solution. A symbolic comparison of the two methods is shown in Figure 2. The minimum on the hyper-surface of the square sum of errors (Equation 15) is located either by proceeding in the direction of the gradient (simultaneous correction **dO** and **dX** in the case of the bundle method) or by alternating orthogonal steps in the **dO** and **dX** direction. Evidently more steps are necessary in the case of the alternating method to arrive at the minimum.

Error Analysis

For a complete error analysis of the solution (**O, X**) of the normal equations (Equations 23 and 24) in terms of variances and covariances the matrix of the system

$$\mathbf{N} = \begin{pmatrix} \mathbf{A}^T\mathbf{G}\mathbf{A} & \mathbf{A}^T\mathbf{G}\mathbf{B} \\ \mathbf{B}^T\mathbf{G}\mathbf{A} & \mathbf{B}^T\mathbf{G}\mathbf{B} + \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{T} \end{pmatrix} \tag{32}$$

has to be inverted. In most cases this is impractical because of its large number $(18 + 3n)$ of rows and

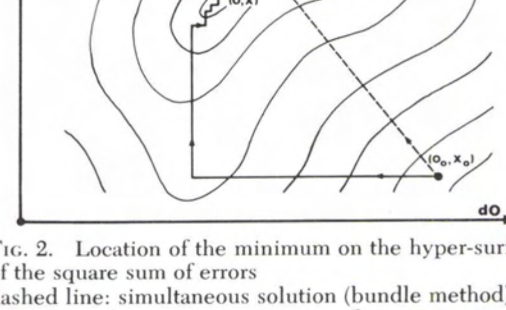

Fig. 2. Location of the minimum on the hyper-surface of the square sum of errors
dashed line: simultaneous solution (bundle method)
solid line: alternating orientation and reconstruction

columns. However, the covariances defined by the diagonal elements of the inverse matrix may be calculated much less elaborately by dispensing with determination of all error correlations.

Following the procedure of Schmid (1959), we may write

$$\mathbf{N}\cdot\mathbf{N}^{-1} = \begin{pmatrix} \mathbf{R} & \mathbf{S} \\ \mathbf{S}^{T} & \mathbf{T} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{L}^{T} & \mathbf{M} \end{pmatrix} = \mathbf{I} \qquad (33)$$

The submatrices **K** and **M** can be calculated explicitly. We merely state the results. For a full derivation see Frobin and Hierholzer (1979). The submatrix **K** relates to the orientation elements; we obtain

$$\mathbf{K} = (\mathbf{A}^{T}\mathbf{Q}\mathbf{A})^{-1} \qquad (34)$$

This expression had already to be calculated for Equation 25. Thus, no additional calculations are necessary for the error analysis of the orientation. Furthermore, correlations between the orientation elements may also be obtained from Equation 34.

The variances of the coordinates can be determined from the diagonal elements of **M. M** consists of 3 × 3 submatrices of the form

$$\mathbf{M}_{ij} = \boldsymbol{\delta}_{ij}\,\mathbf{T}_j^{-1} + \mathbf{U}_i^{T}\,\mathbf{K}\mathbf{U}_j \qquad (35)$$

each submatrix containing contributions of only two (control or object) points numbered i and j. Here we are concerned only with the diagonal submatrices

$$\mathbf{M}_{ii} = \mathbf{T}_i^{-1} + \mathbf{U}_i^{T}\,\mathbf{K}\mathbf{U}_i \qquad (36)$$

The submatrices **U** and **T** are given by

$$\mathbf{U}_i = \begin{pmatrix} \mathbf{A}_{l_i}^{T}\,\mathbf{G}_{l_i}\,\mathbf{B}_{l_i} \\ \mathbf{A}_{r_i}^{T}\,\mathbf{G}_{r_i}\,\mathbf{B}_{r_i} \end{pmatrix} \mathbf{T}_i^{-1} \qquad (37)$$

$$\mathbf{T}_i = \mathbf{B}_{l_i}^{T}\,\mathbf{G}_{l_i}\,\mathbf{B}_{l_i} + \mathbf{B}_{r_i}^{T}\,\mathbf{G}_{r_i}\,\mathbf{B}_{r_i} + \mathbf{H}_i \qquad (38)$$

All matrices occurring have already been calculated during the partial reduction of the system of normal equations (see Equations 23 through 27). Thus, only a few additional matrix operations have to be performed to obtain the desired variances.

From the number of degrees of freedom f of the system of error equations 13 and 14, we calculate the standard deviation of unit weight, or, more accurately, an estimation of it:

$$s^2 = (\mathbf{V}^{T}\mathbf{G}\mathbf{V} + \mathbf{W}^{T}\mathbf{H}\mathbf{W})/f \qquad (39)$$

where
$$f = n + 3n_p - 18 \qquad (40)$$

From the diagonal elements of Equations 34 and 36, we calculate the confidence limits

$$\delta\mathrm{O}_j = t_S(f)\,\sqrt{s^2\,K_{jj}} \quad (j = 1 \ldots 18) \qquad (41)$$

$$\delta X_{i_k} = t_S(f)\,\sqrt{s^2\,M_{ii_{kk}}} \quad \begin{pmatrix} i = 1 \ldots n \\ k = 1 \ldots 3 \end{pmatrix} \qquad (42)$$

for the 18 orientation elements O_j and the $3n$ control point and object point coordinates X_{i_k}, respectively, the index i denoting the point number; k relates to the x, y, and z coordinate. $t_S(f)$ is the t distribution of confidence S for f degrees of freedom (see, e.g., Spiegel, 1972).

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