S. Kondo
R. Ohba
K. Murata
Department of Applied Physics
Faculty of Engineering
Hokkaido University
Sapporo 060, Japan

The Location of Three-Dimensional Linear Objects by Using Multiple Projections

A least-squares method is developed for locating linear objects, such as straight and curved lines, which contain no distinct points or marks.

Introduction

Several approaches have been utilized for locating objects in a three-dimensional space. Typical methods among them use stereo cameras,[1-5] a three-degrees-of-freedom instrumented arm, or an ultrasonic transducer.[6]

The latter two are direct methods used for locating such objects as experimental equipment, machine tools, and so on. They place an arm tip or an ultrasonic transducer directly on the object. The first method, however, is an indirect method. It has been developed in the photogrammetric field, and applied to those objects on which it is impossible to directly place a sensor tip. It is widely applied to the analysis of electron microscopic photographs, bubble chamber photographs, aerial photographs, and so on. The basis of the method is triangulation; that is, the position of a point on the object is determined as the intersection of two lines, those lines being defined by corresponding image points on a pair of photographs and the camera lens nodal points. Therefore, it is possible to determine the coordinates of only those points whose corresponding images are clearly discernible on both photographs. Thus, using this method, it is difficult to determine the position of a line which contains no distinct points, or marks. Furthermore, if the stereo angle is small, the accuracy of the finally determined location depends largely on that of the measurement of camera axes orientations. In particular, it is difficult to obtain information along the depth direction in the vicinity of the plane determined by the centers of camera lenses and the object point.

In order to complement these defects of triangulation, we have proposed a new method to determine the position of a linear object containing no marks by using multiple photographs taken from different positions.[7-8] The idea of the method is that the coordinates of an object point can be determined based on the optimizing algorithms[9] by utilizing several lines (data lines) simultaneously, which are defined by sampled points (data points) on photographs and by the nodal points of the lenses. The coordinates are determined to minimize a certain cost function defined by these data lines and the location of the object point. The present paper describes the principle of

Abstract: *A method to determine the positions of three-dimensional linear objects that have neither marks nor unique points, by using multiple projections, is proposed. Several practical computation algorithms based on the method are also formulated for such simple objects as points and straight and curved lines. The least-squares method is applied in order that the effects of the all measurement errors may be minimized. Several experiments performed by using those algorithms are also presented.*

Photogrammetric Engineering and Remote Sensing,
Vol. 48, No. 1, January 1982, pp. 73-79.
0099-1112/82/4801-0073$02.25/0

the method and formulates it for such objects as points and straight and curved lines.

Least-Squares Location of an Object by Using Multi-Projections

There are two types of projection, parallel and convergent. The latter is a projection of objects by divergent rays from a point source, while the former is defined by parallel rays and is a special case of the latter.

The method to determine the least-squares position of an object in three-dimensional space by using multiple convergent projections is first described, and then the method is reformulated for the case of parallel projections.

Principles

Consider the problem to determine the position of an object Q in Figure 1 as accurately as possible by using N projections. In the figure, $\mathbf{S}^{(i)}$ and $\mathbf{n}^{(i)}$ are the position vector of the i^{th} source and the normal vector of the i^{th} projection plane, respectively. Available data are $\mathbf{S}^{(i)}$, $\mathbf{n}^{(i)}$, and coordinates of a set of the points (data points) sampled independently on the image of the object for each projection. Denote the coordinates of the j^{th} data point on the i^{th} projection by $\mathbf{p}_j^{(i)}$; $(x_j^{(i)}, y_j^{(i)}, 0)$, where the i^{th} projection plane and its $\mathbf{n}^{(i)}$ are given the i^{th} projection coordinate $x^{(i)}$-$y^{(i)}$ and $z^{(i)}$, respectively.

Now, given the normal vector $\mathbf{n}^{(i)}$ and the position of the origin $\mathbf{o}^{(i)}$ of the i^{th} projection coordinate with respect to the reference coordinate system X-Y-Z, $\mathbf{p}_j^{(i)}$ can be transformed into $\mathbf{P}_j^{(i)}$ $(X_j^{(i)}, Y_j^{(i)}, Z_j^{(i)})$ in the reference system as follows:

$$\mathbf{P}_j^{(i)} - \mathbf{o}^{(i)} = \mathbf{T}^{(i)}\mathbf{p}_j^{(i)} \tag{1}$$

where $\mathbf{o}^{(i)}$ and $\mathbf{T}^{(i)}$ are the principal point $(A^{(i)}, B^{(i)}, C^{(i)})$ and the orientation matrix of the i^{th} projection coordinate in the reference system, respectively. A line determined by the data point $\mathbf{P}_j^{(i)}$ is called a data line $\mathbf{L}_j^{(i)}$. Using the curvilinear coordinate r, any point on the line is given by

$$\mathbf{L}_j^{(i)} = \mathbf{S}^{(i)} + r\mathbf{e}_j^{(i)}, \tag{2}$$

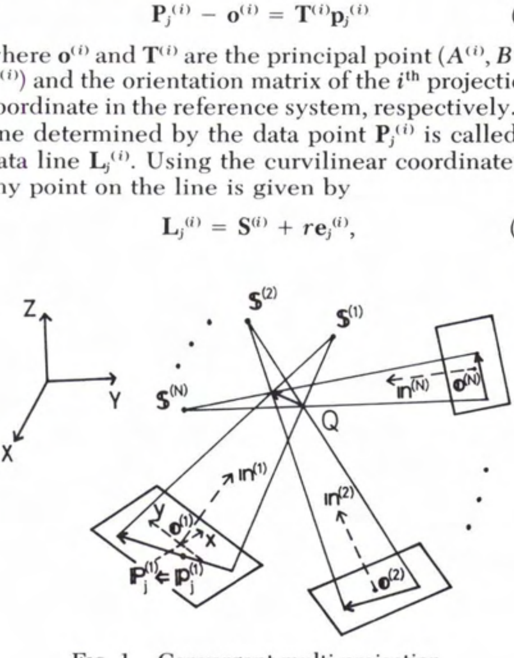


Fig. 1. Convergent multi-projection.

where $\mathbf{S}^{(i)}$ is the position vector $(a^{(i)}, b^{(i)}, c^{(i)})$ of the i^{th} source, and $\mathbf{e}_j^{(i)}$ is the tangential vector $(l_j^{(i)}, m_j^{(i)}, n_j^{(i)})$ of the data line $\mathbf{L}_j^{(i)}$, $\mathbf{e}_j^{(i)} = (\mathbf{P}_j^{(i)} - \mathbf{S}^{(i)})/|\mathbf{P}_j^{(i)} - \mathbf{S}^{(i)}|$.

Ideally, the object must be located on every data line for $i = 1,2, \ldots ,N; j = 1,2, \ldots ,M$. In practice, however, each data line is located a little apart from the object, as shown in Figure 2, due to various errors in the measured values of $\mathbf{S}^{(i)}$, $\mathbf{o}^{(i)}$, $\mathbf{n}^{(i)}$, and $\mathbf{p}_j^{(i)}$.

It is desirable to determine the position of the object as accurately as possible based on those data lines given by Equation 2. Let vector $\mathbf{V}$ denote the parameters describing the position of the object; then the distance $d_j^{(i)}$ between the object Q and the data line $\mathbf{L}_j^{(i)}$ is a function of $\mathbf{V}$. We can estimate the position of the object by $\mathbf{V}^*$, which minimizes the cost function,

$$f(\mathbf{V}) = \sum_i \sum_j d_j^{(i)\,2}(\mathbf{V}). \tag{3}$$

Based on all the data points, the solution $\mathbf{V}^*$ is given and is optimum in the sense of least squares. The minimization problem of Equation 3 can be solved by the use of the simplex method,[9] which is well known to be effective for the optimization problem of the unimodal function.[10]

Accuracy

The accuracy of the position determined by the method depends both on the relative values and on errors in the measurements of $\mathbf{S}^{(i)}$, $\mathbf{T}^{(i)}$, and $\mathbf{p}_j^{(i)}$. For simplicity, consider the case in which the position of a single point is to be determined by the method.

Assume the Euclidean distance between the object point and a data line for the distance $d^{(i)}$ in Equation 3; then $d^{(i)}$ is given as follows:

$$d^{(i)}(\mathbf{V}) = \frac{|(\mathbf{V} - \mathbf{S}^{(i)}) \times (\mathbf{P}^{(i)} - \mathbf{S}^{(i)})|}{|\mathbf{P}^{(i)} - \mathbf{S}^{(i)}|} \tag{4}$$

where $\mathbf{V}$ is the position vector (X,Y,Z) of the object point. The least-squares position can be determined by solving the equations $\partial f(\mathbf{V})/\partial \mathbf{V} (= \partial f/\partial X, \partial f/\partial Y, \partial f/\partial Z) = 0$. Since the solution $\mathbf{V}^*$ can be regarded as a function of those measured quantities $(\mathbf{S}^{(i)}, \mathbf{T}^{(i)}, \mathbf{p}^{(i)})$ and it is possible to measure $\mathbf{p}^{(i)}$

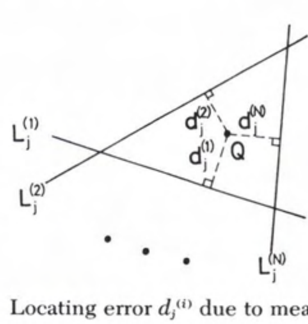


Fig. 2. Locating error $d_j^{(i)}$ due to measurement.

with sufficient accuracy, the locating error ($\Delta\mathbf{V}$) is affected mainly by the errors ($\Delta\mathbf{S}^{(i)}$, $\Delta\mathbf{e}^{(i)}$) of ($\mathbf{S}^{(i)}$, $\mathbf{e}^{(i)}$) and is given as follows:

$$\Delta\mathbf{V} = \frac{\partial \mathbf{V}}{\partial \mathbf{S}^{(i)}} \cdot \Delta \mathbf{S}^{(i)} + \frac{\partial \mathbf{V}}{\partial \mathbf{e}^{(i)}} \cdot \Delta \mathbf{e}^{(i)}, \tag{5}$$

where $\partial\mathbf{V}/\partial\mathbf{S}^{(i)} = (\partial\mathbf{V}/\partial a^{(i)}, \partial\mathbf{V}/\partial b^{(i)}, \partial\mathbf{V}/\partial c^{(i)})$, and $\partial\mathbf{V}/\partial\mathbf{e}^{(i)} = (\partial\mathbf{V}/\partial l^{(i)}, \partial\mathbf{V}/\partial m^{(i)}, \partial\mathbf{V}/\partial n^{(i)})$.

Tables 1 and 2 show the values of $\partial\mathbf{V}/\partial\mathbf{S}^{(i)}$ and $\partial\mathbf{V}/\partial\mathbf{e}^{(i)}$ obtained by Equation 5 with experimental data described later. The maximum errors of the experiment are as follows:

- Source (or the center of camera lens) locating error, $\Delta\mathbf{S}^{(i)}$, ±1cm;
- Projective orientation error, $\Delta\mathbf{e}^{(i)}$, ± 0.5°; and
- Data point coordinate error, $\Delta\mathbf{p}^{(i)}$, ± 1mm.

The maximum estimates of $\Delta\mathbf{V}$ due to $\Delta\mathbf{S}^{(i)}$ and $\Delta\mathbf{n}^{(i)}$ are 3 cm and 20 cm, respectively, obtained by using Equation 5 and the values in Table 1 and 2. So the effect of $\Delta\mathbf{n}^{(i)}$ is larger than that of $\Delta\mathbf{S}^{(i)}$ in the experiment. In the case of methods based on the triangulation, generally, the effect of the projective orientation error is the largest.

The most favorable means for minimizing the orientation error is to use the three projections with orientations perpendicular to each other, so as to minimize $\partial\mathbf{V}/\partial\mathbf{e}^{(i)}$. However, such projection conditions are not always possible, and a calibrating method is necessary for the projective orientation so as to minimize errors.

CALIBRATION OF THE PROJECTIVE ORIENTATIONS

The projective orientations data are contaminated not only by errors in the measurement but also by errors in the enlarging process. It is desirable for the accurate locating of the object that the projective orientations should be calibrated by using several reference points whose images are definitely recognized in every projection.

Assume that M reference points are gained for every projection; then the least-squares position of the reference point $\mathbf{V}_j^*$ ($j = 1,2, \ldots, M$) with respect to orientation matrices $\{\mathbf{T}^{(i)}\}$ ($i = 1,2, \ldots, N$) is obtained as a solution of the equation $\partial f(\mathbf{V}_j)/\partial\mathbf{V}_j = 0$, and the solution $\mathbf{V}_j^*$ is a function with respect to $\mathbf{T}^{(i)}$. Therefore, $\mathbf{T}^{(i)}$ can be calibrated, by minimizing the sum of the cost function $f(\mathbf{V}_j)$ for the all reference points. That is, the orientation matrices $\{\mathbf{T}^{(i)*}\}$ which minimize the following function are optimum in the sense of least squares:

$$F(\mathbf{T}^{(1)}, \mathbf{T}^{(2)}, \ldots, \mathbf{T}^{(N)}) = \sum_{j=1}^{M} f(\mathbf{V}_j). \tag{6}$$

The solution $\{\mathbf{T}^{(i)*}\}$ minimizing Equation 6 can be obtained with an iterative method.

SOME NOTES FOR THE PARALLEL PROJECTION

In the case of the parallel projection, the directions of data lines are equal to the projective orientations. Therefore, assuming that the projective plane is perpendicular to the projective orientation, the distance $d_j^{(i)}$ between the object Q and the data line $\mathbf{L}_j^{(i)}$ is equal to the distance between the image q and the data point $\mathbf{p}_j^{(i)}$ in the projective plane (see Figure 3), where the image q is the projection of Q on the plane.

When the object is a point, the position $\mathbf{v}$ of the image q is given as

$$\mathbf{v} = \mathbf{T}^{(i)T}\mathbf{V},$$

where $\mathbf{V}$ is the reference coordinates (X,Y,Z) of Q, and $\mathbf{T}^{(i)T}$ is the transpose of $\mathbf{T}^{(i)}$ in Equation 1.

Distance $d_j^{(i)}(\mathbf{V})$ in the cost function is given by

$$d_j^{(i)}(\mathbf{V}) = |\mathbf{v} - \mathbf{p}_j^{(i)}| = \sqrt{(x - x_j^{(i)})^2 + (y - y_j^{(i)})^2},$$

where both $\mathbf{p}_j^{(i)}$ and $\mathbf{v}$ are of the projection coordinates, so that $d_j^{(i)}(\mathbf{V})$ for the parallel projection is simpler than that for convergent projection.

FORMULATION FOR SIMPLE OBJECTS

POINT

The Parameters $\mathbf{V}$, which determine the position of a point, are coordinates (X,Y,Z) of the point.

TABLE 1. SENSITIVITY OF $\mathbf{V}$ TO $a^{(i)}$.

i	$\frac{\partial X}{\partial a^{(i)}}$	$\frac{\partial Y}{\partial a^{(i)}}$	$\frac{\partial Z}{\partial a^{(i)}}$
1	0.224	0.320	0.254
2	1.33	−0.558	0.369
3	−0.545	1.27	0.376

TABLE 2. SENSITIVITY OF $\mathbf{V}$ TO ($l^{(i)}$, $m^{(i)}$, $n^{(i)}$).

$\mathbf{V}$	$\frac{\partial \mathbf{V}}{\partial l^{(i)}}$	$\frac{\partial \mathbf{V}}{\partial m^{(i)}}$	$\frac{\partial \mathbf{V}}{\partial n^{(i)}}$
X	-0.244×10^4	-0.200×10^4	-0.112×10^3
Y	-0.313×10^4	-0.289×10^4	-0.179×10^3
Z	-0.386×10^3	-0.332×10^3	-0.131×10^3

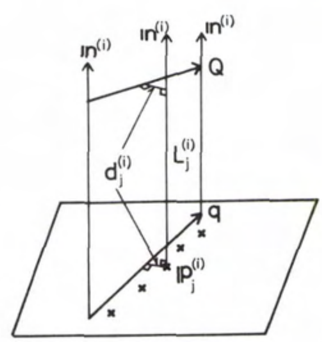


FIG. 3. Locating error $d_j^{(i)}$ projected on to the i^{th} projection plane in parallel projection.

Assume that only one data point is obtained for a projection, and that the total number of data lines given by Equation 2 is N. The cost function $f(\mathbf{V})$, defined by substituting $d^{(i)}(\mathbf{V})$ of Equation 4 into Equation 3, becomes a unimodal function with respect to each parameter. For a point object, therefore, the least-squares solution is always obtained by the simplex method.

Even if the objects contain plural points which are indistinguishable on each projection, the least-squares position of each point can be obtained by searching the best combination of data points that makes the cost function $f(\mathbf{V})$ a minimum. However, when the number of points contained in the objects increases, the necessary computation in order to minimize $f(\mathbf{V})$ grows too large to be practical, and for such cases some preprocessing to decrease the number may be required.

STRAIGHT LINE WITHOUT MARKS

Assume that a straight line has no marks or discernible points on it and that there exists no correspondence among data points sampled on each projection, that is, data points are sampled independently on each projection. Then a line can be expressed as

$$\mathbf{L}_0 = \mathbf{R}_0 + r\mathbf{e}_0, \quad (7)$$

where $\mathbf{R}_0$ is a certain point (X_0,Y_0,Z_0) on the line, $\mathbf{e}_0$ is the tangential vector (l_0,m_0,n_0) of that point, and r is the curvilinear coordinate.

The distance $d_j^{(i)}(\mathbf{V})$ between the object line and a data line is given as

$$d_j^{(i)}(\mathbf{V}) = |(\mathbf{e}_0 \times \mathbf{e}_j^{(i)})\cdot(\mathbf{R}_0 - \mathbf{S}^{(i)})|. \quad (8)$$

Since it is always possible to set $X_0 = 0$ and $l_0 = \sqrt{1 - m_0^2 - n_0^2}$, the vector $\mathbf{V}$ for a straight line consists of four independent parameters (Y_0,Z_0,m_0,n_0).

The cost function $f(\mathbf{V})$ obtained by substituting Equation 8 into Equation 3 becomes a multimodal function with respect to each parameter. Therefore, it is required, for the simplex method used to minimize $f(\mathbf{V})$, that the initial simplex $\mathbf{V}_h(h = 1,2, \ldots ,5)$ must be selected within the domain in which the minimum value of $f(\mathbf{V})$ is obtained.

Though the general solution for the above problem has not been known in the general optimization problem, the following simple treatment is effective for a straight line:

- Search the most distant pair $\mathbf{p}_S^{(i)}$, $\mathbf{p}_F^{(i)}$ among all data points on each projection;
- Define a plane $PL^{(i)}$ by three points, $\mathbf{p}_S^{(i)}$, $\mathbf{p}_F^{(i)}$, and $\mathbf{S}^{(i)}$, for $i = 1,2, \ldots ,N$; and
- Define a line by the intersection of two planes chosen arbitrarily among $\{PL^{(i)}\}$; $(i = 1,2, \ldots ,N)$ and use its parameters for the initial simplex.

It is reasonable to assume that the locating of the object line determined by the all data points is sufficiently close to those lines obtained by the above steps. Therefore, with the initial simplex $\mathbf{V}_h$; $(h = 1,2, \ldots ,5)$, the cost function $f(\mathbf{V})$ is well minimized. In other words, the least-squares position of the line may be determined by using the simplex method.

MULTI-LINES APPROXIMATION FOR A CURVED OBJECT

Since it is possible to approximate a curve to any desirable accuracy by employing multiple straight lines, then the approximate location of the curve is also made possible by applying the method formulated in the last section to the locating of those multiple lines. Assume that the curve is approximated by K lines and that K sets of the parameters given by Equation 7 are used. The total number of parameters used to locate the curve is $4K$ $(Y_{01},Z_{01},m_{01},n_{01}, \ldots , Y_{0K},Z_{0K},m_{0K},n_{0K})$. For the distance measure $d_j^{(i)}$, the least distance among the K distances between the data line and K approximate lines is available. That is, $d_j^{(i)}(\mathbf{V})$ for a curve is defined as

$$d_j^{(i)}(\mathbf{V}) = \min\{d_{kj}^{(i)}(\mathbf{V})\} \quad k = 1,2, \ldots ,K \quad (9)$$

where $d_{kj}^{(i)}(\mathbf{V})$ is the distance between the data line $\mathbf{L}_j^{(i)}$ and the k^{th} approximate line, and is given by Equation 8 with $\mathbf{e}_0 = (l_{0k},m_{0k},n_{0k})$ and $\mathbf{R}_0 = (0,y_{0k},z_{0k})$. The cost function $f(\mathbf{V})$, obtained by substituting Equation 9 into Equation 3, also becomes a multi-modal function and it is difficult to select the initial simplex. For a relatively simple curve, however, the following treatment is effective for selecting the initial simplex $\mathbf{V}_h(h = 1,2, \ldots ,4K + 1)$ within the domain in which the minimum value of $f(\mathbf{V})$ is obtained:

- Find a pair of data points $\mathbf{p}_S^{(i)}$, $\mathbf{p}_F^{(i)}$ located at both ends of the projected curve in each projection;
- Arrange the residual data points from $\mathbf{p}_S^{(i)}$ to $\mathbf{p}_F^{(i)}$ in a string, for each projection, so that the distance between neighboring points is minimum among distances between any two points;
- Choose $K - 1$ data points $\mathbf{p}_{Cg}^{(i)}$ $(g = 1,2, \ldots ,K - 1)$ which divide the data points string into K parts, so that an equal number of points are selected in each part;
- Define K planes $PL_j^{(i)}$ $(i = 1,2, \ldots ,N;\ j = 1, 2, \ldots ,K)$ for each projection with respect to the point source $\mathbf{S}^{(i)}$ and each two data points $(\mathbf{p}_s^{(i)}, \mathbf{p}_{C1}^{(i)})$, $(\mathbf{p}_{C1}^{(i)}, \mathbf{p}_{C2}^{(i)}) \ldots (\mathbf{p}_{C(K-1)}^{(i)}, \mathbf{p}_F^{(i)})$; and
- Define the initial simplex of the j^{th} approximate line by the intersection of the two arbitrary planes selected among $\{PL_j^{(i)}\}$; $(i = 1,2, \ldots ,N)$.

The final approximate lines are considered to be located near the initial approximate lines obtained by the above process. In this case, however, when the number of data points per one approximate line is as small as two or three, the simplex often fails to converge to the least-squares solution due to errors contained in the initial simplex. Therefore, the larger the number of approximate lines is, the more data points are required.

Application to Close-Range Photogrammetry

The method was applied to several photogrammetric examples. Positions of points and lines on an iron frame structure and a curved cord were determined by using three convergent photographs. The accuracy of the position determined by this method was checked by comparing the position with its true value.

Analysis with photographs taken by a camera

If an ordinary camera is idealized to a pin hole camera, then a set of photographs of an object are convergent projections and the object can be located by them. Now, the center of camera lenses and a photograph correspond to a point source and a projection, respectively. The necessary data are the position and orientation of the camera for each projection, the coordinates of the data points on the projection, and each distance between the source and the projection plane.

The projection system is illustrated in Figure 4. Let the projective orientation, that is, the direction of the line defined by the center $\mathbf{o}^{(i)}$ of the i^{th} projection plane (principal point) and the pin hole $\mathbf{S}^{(i)}$, be the $z^{(i)}$–axis. The projection coordinate system, x-y-z, can be transformed to the reference system, X-Y-Z, by Equation 1.

Let $\theta^{(i)}$, $\psi^{(i)}$ be angles between the $y^{(i)}$-axis, $z^{(i)}$-axis of the i^{th} projective system and the Y-axis, Z-axis of the reference system, respectively; then, Equation 1 is reduced to

$$\begin{pmatrix} X_j^{(i)} - A^{(i)} \\ Y_j^{(i)} - B^{(i)} \\ Z_j^{(i)} - C^{(i)} \end{pmatrix} \tag{10}$$

$$= \begin{pmatrix} \cos\theta^{(i)}\cos\psi^{(i)} & \sin\theta^{(i)} & -\cos\theta^{(i)}\sin\psi^{(i)} \\ -\sin\theta^{(i)}\cos\psi^{(i)} & \cos\theta^{(i)} & \sin\theta^{(i)}\sin\psi^{(i)} \\ \sin\psi^{(i)} & 0 & \cos\psi^{(i)} \end{pmatrix} \begin{pmatrix} x_j^{(i)} \\ y_j^{(i)} \\ f \end{pmatrix},$$

where $(A^{(i)}, B^{(i)}, C^{(i)})$ are the reference coordinates of the i^{th} pin hole, and f is the focal length of the camera lens.

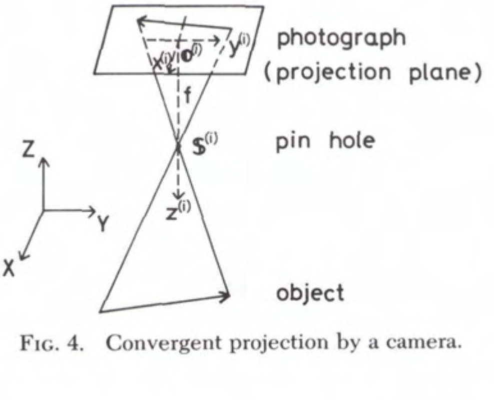


Fig. 4. Convergent projection by a camera.

Experiments for points and a straight line

An iron frame structure was analyzed from three photographs (Figures 5a, 5b, and 5c) taken from different positions about 5 m away from it. Four points, A, B, C and D, on the frame were selected for the objects, and their three-dimensional coordinates were determined by the present method.

Table 3 shows distances calculated from the coordinates of each point, in comparison with those

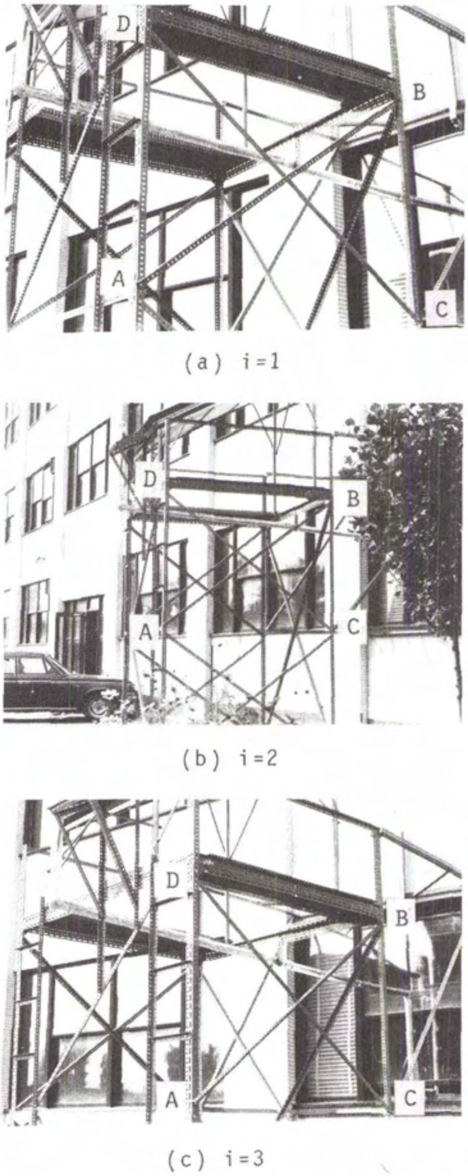


Fig. 5. Three photographs of an iron frame structure to be analyzed.

TABLE 3. DISTANCES BETWEEN OBJECT POINTS (A, B, C, AND D) ON THE FRAME IN cm.

	$A-C$	$A-D$	$B-C$	$B-D$
True Value	171.0	127.5	132.5	171.0
Calculated Value	172.6	127.0	131.1	171.1
Difference	1.6	−0.5	−1.4	0.1

measured by a precision tape scale. The accuracy of the positions determined by the present method can be estimated based on those results. It can be seen from the table that the accuracy of the distance determined by the method is better than 1 per cent for any two points.

Since the projective direction errors in this experiment were accidentally $\pm 0.5^{0}$, the errors of the determined positions were more than 4 cm for the distance of 5 m when employing ordinary triangulation. The optimization procedure in the present method decreased those erros to one half or one quarter. The standard deviation of the distances between the object and data lines was 2.8 cm.

The line AB on the frame was then analyzed with the same photographs. Sampling eight data points on the line image for each photograph randomly, the parameters (Y_0,Z_0,m_0,n_0) of the line AB were determined based on those 24 total data points.

The resultant parameters are shown in Table 4 in comparison with those calculated by the least-squares positions of points A and B obtained by the last experiment. Those parameters obtained by the two different methods almost agree with each other. Therefore, the parameters of a line may be determined with the same accuracy as those of a point. It can be seen also from the table that the accuracies of Y_0, m_0, for the Y-axis are less than those of Z_0,n_0, for the Z-axis, because the parallax for the Z-axis is less than that for the Y-axis. The standard deviation of the distances between the object and data lines, in this case, was 2.5 cm, and it was also on the same order as that for a point.

EXPERIMENTS FOR A CURVED LINE

The algorithm described for the multi-lines approximation for a curved object was applied to the determination of the approximate position of a curved line by using K straight lines (K = 1,2,3,4).

A curved cord, suspended in a span as shown in Figure 6, was analyzed by employing three photographs which were taken with a camera placed at three different positions about 3 m distant from the cord. Fifty-four data points (18 points for each photograph) on the images of the curved line were randomly sampled from these photographs and the parameters of the approximate lines were determined from those data.

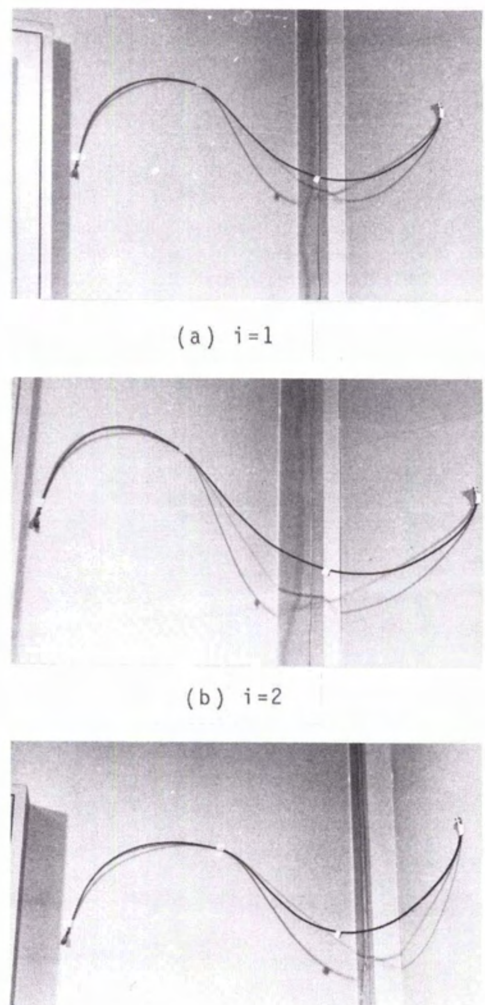
Table 5 shows the RMS values D of $d_j^{(i)}$.

$$D = \sqrt{\frac{1}{M_t} \sum \sum d_j^{(i)2}}, \tag{11}$$

where M_t is the total number of data points. A

TABLE 4. PARAMETERS OF THE LINE AB IN cm.

Parameter	Y_0	Z_0	m_0	n_0
Determined by the least-squares positions of A and B	304.1	26.6	0.093	0.606
Determined by Equation 8	302.8	26.7	0.109	0.607
Difference	1.3	0.1	0.016	0.001



(a) i=1

(b) i=2

(c) i=3

FIG. 6. Three photographs of a curved cord to be analyzed.

TABLE 5. RMS VALUES D OF $d_j^{(i)}$ FOR K = 1,2,3,4 IN cm.

K	1	2	3	4
D	12.4	10.5	3.9	3.0

small value of D indicates the good approximation. It can be seen from the table that D decreases as K increases. When the number of the approximate lines K is increased from two to three, D decreases considerably, that is, the approximation is greatly improved. However, when K was increased from three to four, D did not decrease so much. Therefore, the cord can be sufficiently well approximated by using three or four lines.

Figure 7 shows the inverse projections of the final approximate lines on each projection plane overlapped onto data points. These figures also show that sufficiently accurate approximations are achieved for K more than 3.

SUMMARY

A new method, determining the least-squares position of a point, a straight line, or a curved line by using plural projections, is proposed. The method gives the optimum position in the sense of least squares.

The features of the method are as follows:

- The influence of the errors on the measurements is minimized, since the result is obtained as the least-squares solution for all data points;
- Relative positions of data points on the object are unnecessary for every projection;
- The value of the cost function f(**V**) at the termination of iterative calculation indicates the variance of the distances between data lines and the least-squares position;
- As more projections and data lines are acquired for higher accuracy, the amount of the calculations increases; and
- In the case of a complex shaped object, it is often difficult to select the initial simplex within the domain of the peak containing the minimum value of f(**V**).

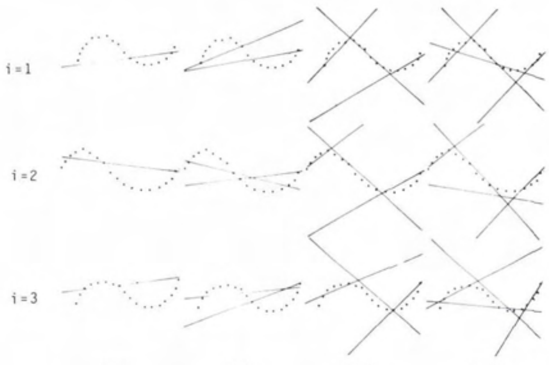


FIG. 7. Data points and inverse projections of the final approximate lines.

REFERENCES

1. Yukio Ozaki, *Photogrammetry*, Morikita Press., 1970 (in Japanese).
2. G. Konecny, Interior Orientation and Convergent Photography, *Photogrammetric Engineering*, vol. 31, no. 4, pp. 625-634, 1965.
3. W. Faig and H. Moniwa, Convergent Photos for Close Range, *Photogrammetric Engineering*, vol. 39, no. 6, pp. 605-610, 1973.
4. K. E. Wong, Mathematical Formulation and Digital Analysis in Close-Range Photogrammetry, *Photogrammetric Engineering and Remote Sensing*, vol. 41, no. 11, pp. 1355-1373, 1975.
5. S. A. Veress and R. S. Tiwari, Fixed-Frame Multiple-Camera System for Close-Range Photogrammetry, *Photogrammetric Engineering and Remote Sensing*, vol. 42, no. 9, pp. 1195-1210, 1976.
6. W. E. Mortz and P. L. Shreve, Analysis of an Ultrasonic Spatial Locating System, *IEEE Trans. Inst. and Meas.*, vol. IM-25, pp. 43-50, 1976.
7. S. Kondo, R. Ohba, and K. Murata, An Optimal Locating Method for 3-D Object and the Accuracy of the Measurement, Preprint, *Conf. 17th Instrumentation and Control*, pp. 561-562, 1978 (in Japanese).
8. S. Kondo, R. Ohba, and K. Murata, Application of Optimal Locating Method to Analysis of Dislocation, Preprint, *Conf. 18th Instrumentation and Control*, pp. 341-342, 1979 (in Japanese).
9. J. Kowalik and M. R. Osborne, *Methods for Unconstrained Optimization Problems*, American Elsevier Publishing Company, Inc. 1968.
10. J. A. Nelder and R. Mead, A Simplex Method for Function Minimization, *Computer J.*, 7, p. 308, 1965.

(Received 25 August 1980; revised and accepted 24 June 1981)