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Geometry of a Mapping Satellite

Calculations have shown that it is possible to achieve satisfactory tracking of detectors of a proposed stereo mapping satellite that employs linear detector arrays.

Abstract: *The proposed mapping satellite Mapsat is to consist of fixed fore, vertical, and aft linear detector arrays, any two of which may be used simultaneously to obtain digital images for one-dimensional stereo correlation. The satellite attitude may be varied according to Fourier series to enable a given detector on one array to follow closely the groundtrack sensed by the corresponding detector on another array throughout the orbit. These tracking errors are negligible for a satellite stable within anticipated ranges. The required computations have been programmed in* FORTRAN IV.

Introduction

Mapsat is a mapping satellite that the U.S. Geological Survey (Colvocoresses, 1979) has proposed as a means of obtaining continuous stereo images from space. Instead of providing conventional overlapping photography, Mapsat is designed to obtain digital stereo-images using two selected linear detector arrays out of three available. The arrays are fixed, one looking downward nearly vertically, one looking forward at an angle of 23° from the vertical array, and one looking 23° aft. Each array at any given instant images a swath on the Earth basically perpendicular to the direction of satellite heading. This concept is illustrated in an accompanying paper (Colvocoresses, 1982).

Imaging occurs for two cases:

- Case 1: with the vertical and either the fore or aft arrays in operation for moderate-to-steep terrain (base/height ratio about 0.5).
- Case 2: with the fore and aft arrays in operation for relatively flat terrain (base/height ratio about 1.0).

Although the arrays are fixed, the attitude of the spacecraft may be varied in yaw, pitch, and roll to optimize tracking between corresponding detectors on the two arrays chosen. In order to correlate a string of radiometric signals from a single detector on one array with a string from a single corresponding detector on the other array used, the two tracks should nearly coincide, except for the stereo effects resulting from changing elevations. Such correlation may be called one-dimensional. The mathematical approach described below was developed to compute coefficients for Fourier series by which the satellite attitude may be varied to permit tracking to within a fraction of a 10-metre pixel.

Equations for Point Sensed by Detector

Determining the point on the ground sensed by a given detector on the satellite, as well as finding the inverse, involves fairly straightforward rotational matrices and algebra. The author and Itek Corporation personnel undertook independent development of formulas to serve as a mutual check. The author's first approach was much more cumbersome, using direction cosines and angles between lines in space. Essentially the same answers were obtained by the author with much less computer time, using the matrix approach described by Itek Corp. (1981), but applying this approach (as described below) to a somewhat different means of calculating series coefficients. While Itek Corp. used a least-squares analysis of numerous points along the array to minimize overall tracking errors within one quadrant of the orbit, the following approach is applied to one position at a time along the array, but minimizing errors at that position along the entire orbit. Small secular terms were included in the series developed from

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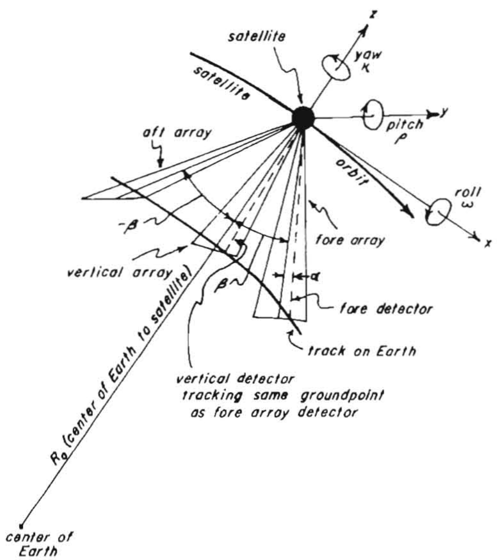

FIG. 1. Coordinate system, attitude, and arrays for satellite. Arrows on ellipses indicate direction of increasing yaw, etc. Sample tracking detectors are shown dashed for Case 1 (fore and vertical). Not to scale.

the evaluation of a single quadrant. Only periodic terms occur if the full orbit is considered.

The formulas given are based on the following symbols and conventions (see Figures 1 and 2), using the same satellite orbit as that of Landsat 1, 2, and 3. Let

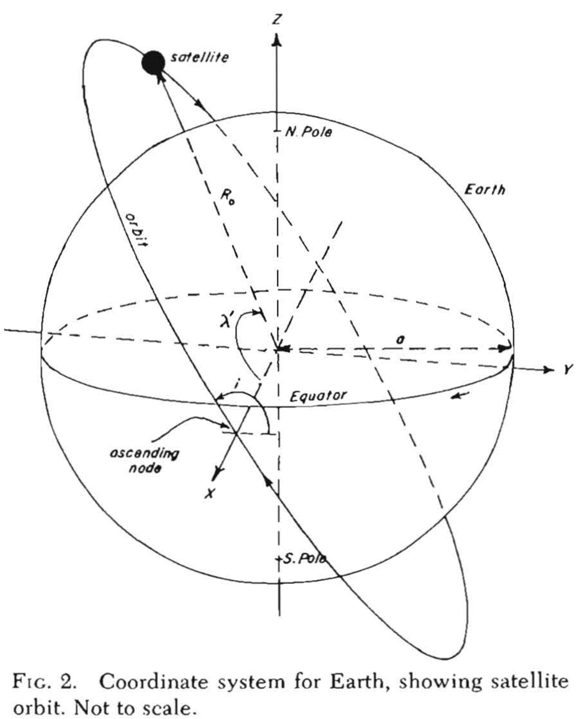

FIG. 2. Coordinate system for Earth, showing satellite orbit. Not to scale.

R_0 = radius of the circular orbit (7,294,690 m).

i = inclination of the orbit, counterclockwise from the equator as viewed at the ascending node (99.092°).

P_2 = period of revolution of the satellite (103.267 min).

P_1 = period of rotation of the Earth between ascending nodes (1440 min for the sun-synchronous Landsat orbit).

For the Earth,

a = semi-major axis of the Earth ellipsoid (the Clarke 1866 ellipsoid is arbitrarily used for the calculations here: a = 6,378,206.4 m).

e = eccentricity of the Earth ellipsoid (e^2 = 0.006 768 658 for the Clarke 1866 ellipsoid).

ϕ = geodetic latitude on the Earth's surface of a point seen by a detector.

λ = geodetic longitude of the point, relative to the ascending node which just precedes the satellite position under consideration.

h = height at (ϕ, λ) of the surface of the Earth above the surface of the reference ellipsoid.

(X,Y,Z) = rectangular coordinates of the Earth surface at (ϕ, λ, h), with the origin at the center of the Earth, the X-axis increasing toward the ascending node, and the Z-axis increasing toward the North Pole.

For the satellite position,

λ' = position of the satellite along the orbit, as the angle from the ascending node, viewed from the center, and directly proportional to time. The ascending node occurs on the dark side of the Earth.

κ = yaw of the satellite, positive counterclockwise when viewed from overhead.

ρ = pitch of the satellite, positive with the nose down (although ϕ is used in photogrammetric work for pitch, it is not used here to avoid confusion with latitude).

ω = roll of the satellite, positive counterclockwise when viewed from the nose.

β = angle of the optical axis for the array (fore is 23°, vertical 0°, and aft is −23°).

α = off-axis angle of the detector on a given array from the optical axis (0° to 5.5° for the approximate range of sensing, positive counterclockwise when viewed from the nose).

(X_s,Y_s,Z_s) = rectangular coordinates of the satellite in the Earth-coordinate system (see X,Y,Z above for the reference system).

(x,y,z) = rectangular coordinates in the satellite-coordinate system, with the origin at the center of the satellite, the x-axis positive toward the direction of motion, and the z-axis positive away from the center of the Earth.

The center detector ($\alpha = 0$) of the vertical array ($\beta = 0$) looks toward the center of the Earth along the radius vector, if pitch and roll are both zero.

The forward formulas, to find the point (ϕ,λ) on the ground sensed by a given detector, may be developed as follows:

Given λ', α, β, h, and the satellite attitude, first the unit vector $\bar{a}$ of the detector ray in the frame of the satellite coordinates is calculated for a satellite without yaw, pitch, and roll:

$$a_x = \sin\beta \cos\alpha \tag{1}$$

$$a_y = \sin\alpha \tag{2}$$

$$a_z = -\cos\beta \cos\alpha \tag{3}$$

If the satellite is rotated with yaw, pitch, and roll, in that order, the vector $\bar{a}$ now has coordinates $\bar{a}'$, still in the satellite-coordinate system, where

$$\bar{a}' = \boldsymbol{\kappa}\, \boldsymbol{\rho}\, \boldsymbol{\omega}\, \bar{a} \tag{4}$$

$$\boldsymbol{\kappa} = \begin{pmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{rotating about the } z\text{-axis for yaw} \tag{5}$$

$$\boldsymbol{\rho} = \begin{pmatrix} \cos\rho & 0 & \sin\rho \\ 0 & 1 & 0 \\ -\sin\rho & 0 & \cos\rho \end{pmatrix} \text{rotating about the } y\text{-axis for pitch} \tag{6}$$

$$\boldsymbol{\omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix} \text{rotating about the } x\text{-axis for roll} \tag{7}$$

The satellite-coordinate system is next rotated about the vector (1,1,1) so that the x-, y-, and z-axes exchange places with the z-, x-, y-axes, respectively. This new coordinate system is then rotated about its z-axis (and translated) so that its x-axis coincides with the X-axis of the Earth itself. Finally, the y- and z-axes of this system are rotated about the x-axis to coincide with the Y- and Z-axes, respectively, of the Earth. These transformations are accomplished with three more rotational matrices $\mathbf{P}$, $\boldsymbol{\lambda}'$, and $\mathbf{i}$, respectively:

$$\bar{\mathbf{b}} = \mathbf{i}\, \boldsymbol{\lambda}'\, \mathbf{P}\, \bar{a}' \tag{8}$$

where $\bar{\mathbf{b}} = (b_X, b_Y, b_Z)$, the unit vector of the detector ray in the Earth-coordinate frame, and

$$\mathbf{i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \tag{9}$$

$$\boldsymbol{\lambda}' = \begin{pmatrix} \cos\lambda' & -\sin\lambda' & 0 \\ \sin\lambda' & \cos\lambda' & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{11}$$

The coordinates $\bar{\mathbf{X}}_s = (X_s, Y_s, Z_s)$ of the satellite in the Earth-coordinate frame are, similarly,

$$\bar{\mathbf{X}}_s = \mathbf{i}\, \boldsymbol{\lambda}' \begin{pmatrix} R_0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

or

$$X_s = R_0 \cos\lambda' \tag{13}$$

$$Y_s = R_0 \cos i \sin\lambda' \tag{14}$$

$$Z_s = R_0 \sin i \sin\lambda' \tag{15}$$

The coordinates $\bar{\mathbf{X}} = (X, Y, Z)$ of the point on the Earth's surface are related to $\bar{\mathbf{X}}_s$ and $\bar{\mathbf{b}}$ as follows:

$$\bar{\mathbf{X}} = \bar{\mathbf{X}}_s + L\, \bar{\mathbf{b}} \tag{16}$$

where L is the distance from the satellite to the point $\bar{\mathbf{X}}$.

To find L, we may begin with standard formulas for $\bar{\mathbf{X}}$ in terms of ϕ, λ, h. When the satellite has reached λ', however, the meridians have advanced, due to rotation of the Earth, by an angle $(P_2/P_1)\,\lambda'$, so that longitude λ relative to the ascending node is increased by this angle. Thus

$$X = (N + h) \cos\phi \cos\left[\lambda + (P_2/P_1)\,\lambda'\right] \tag{17}$$

$$Y = (N + h) \cos\phi \sin\left[\lambda + (P_2/P_1)\,\lambda'\right] \tag{18}$$

$$Z = \left[N\,(1 - e^2) + h\right] \sin\phi \tag{19}$$

where

$$N = a/(1 - e^2 \sin^2\phi)^{1/2} \tag{20}$$

Squaring Equations 17 and 18 and adding, the λ terms are eliminated. Solving this sum for $\cos^2\phi$, solving Equation 19 for $\sin^2\phi$, and adding, the final sum is 1: that is,

$$(X^2 + Y^2)/(N + h)^2 + Z^2/\left[N\,(1 - e^2) + h\right]^2 = 1 \tag{21}$$

Substituting from Equation 16 for X, Y, and Z into Equation 21, a quadratic in L is obtained. Solving for L in the usual algebraic manner and applying the sign to the radical to obtain the smaller of the two values of L,

$$L = -[B + (B^2 - 4AC)^{1/2}]\,/2A \tag{22}$$

$$A = N_1\,(b_X^2 + b_Y^2) + N_2\, b_Z^2 \tag{23}$$

$$B = 2\left[N_1\,(b_X X_s + b_Y Y_s) + N_2 b_Z Z_s\right] \quad (24)$$

$$C = N_1\,(X_s^2 + Y_s^2 - N_2) + N_2\,Z_s^2 \quad (25)$$

$$N_1 = \left[N\,(1 - e^2) + h\right]^2 \quad (26)$$

$$N_2 = \left[N + h\right]^2 \quad (27)$$

While $\bar{\mathbf{b}}$ and $\bar{\mathbf{X}}_s$ are known from Equations 1 through 15, ϕ and h are not, but initial assumptions may be $h = 0$ and $\phi = \arcsin\,(\sin i \sin \lambda')$ from Equations 15 and 19 applied to the surface of a sphere.

To find ϕ and λ of the point sensed by the detector, then, first $\bar{\mathbf{b}}$ and $\bar{\mathbf{X}}_s$ are found from Equations 1 through 15. An initial L is found from Equations 22 through 27, and an initial $\bar{\mathbf{X}}$ is determined from Equation 16.

A new trial ϕ is found from an iterative inversion of Equations 19 and 20: that is,

$$\phi = \arcsin\left\{Z/\left[a\,(1 - e^2)/(1 - e^2 \sin^2\phi)^{1/2} + h\right]\right\} \quad (28)$$

in which ϕ is found by successive substitution starting with the previous value of ϕ (a quartic equation could be used without iteration, but it is much more involved). An initial λ is found by dividing Equation 18 by 17: that is,

$$\lambda = \arctan\,(Y/X) - (P_2/P_1)\,\lambda'. \quad (29)$$

(If X is negative, $\pm$ 180° is added to λ.) For this ϕ, λ, a second trial height h can be determined from knowledge of local terrain, and L can be recalculated from Equations 22 through 27. New values of $\bar{\mathbf{X}}$, ϕ, and λ may then be calculated from Equations 16, 28, and 29. Now h may be revised based on the new ϕ, λ. The revised ϕ, λ, h are then used in Equations 22 through 27, 16, 28, and 29 for another iteration, and the process is repeated until sufficient convergence occurs.

For the *inverse formulas,* to find the position λ' of the satellite and α of the detector on the desired array when sensing a given ground point, the forward equations may be fairly easily inverted, resulting in the following algorithm.

Given ϕ, λ, h, β, and the satellite attitude as a function of the unknown λ', rectangular coordinates $\bar{\mathbf{X}}$ and $\bar{\mathbf{X}}_s$ are found from Equations 17 through 20 and 13 through 15, respectively, using a trial λ'. If ϕ, λ, h have been found from the forward formulas using a given array β and λ', the first trial λ' for inverse calculations can be 3.5° more than the forward λ' if the array is changed from fore to vertical or vertical to aft, 3.5° less if the change is reversed, 7.0° more in changing from fore to aft, and so forth. Otherwise, a less precise estimate is made, leading to additional iteration steps.

The Pythagorean relationship is used to find L and $\bar{\mathbf{b}}$: that is,

$$L = \left[(X - X_s)^2 + (Y - Y_s)^2 + (Z - Z_s)^2\right]^{1/2} \quad (30)$$

$$\bar{\mathbf{b}} = (\bar{\mathbf{X}} - \bar{\mathbf{X}}_s)/L \quad (31)$$

The unit vector $\bar{\mathbf{a}}$ of the detector ray is found from the inverse of Equations 4 and 8, or

$$\bar{\mathbf{a}} = \boldsymbol{\omega}^{-1}\quad \boldsymbol{\rho}^{-1}\quad \boldsymbol{\kappa}^{-1}\quad \mathbf{P}^{-1}\quad \boldsymbol{\lambda}'^{-1}\quad \mathbf{i}^{-1}\quad \bar{\mathbf{b}} \quad (32)$$

The matrices in Equations 5, 6, 7, 9, 10, and 11 are orthogonal; thus, each of the inverses equals the transpose of the corresponding matrix.

From $\bar{\mathbf{a}}$, the α and β corresponding to the trial λ' may be calculated:

$$\alpha = \arcsin a_y \quad (33)$$

$$\beta = -\arctan\,(a_x/a_z) \quad (34)$$

Unless the trial λ' is the correct value, this calculated β will not equal the given β. For the second iteration, if the calculated β is too large, λ' may be increased by 0.1°, or decreased if β is too small, and calculation is repeated with Equations 13 through 15, and 30 through 34. The two calculated values of β are compared with that desired, and λ' is readjusted proportionately for a new iteration. The resulting discrepancy in β is used for another adjustment in λ', and so forth, until sufficient convergence occurs. This is essentially a Newton-Raphson iteration using finite differences rather than differentials. The final λ' is the satellite position, and α is the angle of the detector viewing ϕ, λ, h.

Tracking Detectors on Two Arrays

The foregoing equations may now be applied to the fundamental problem of determining the extent to which a given detector on one array continues to track a given detector on another array throughout the orbit. To accomplish this, calculations using the forward equations are followed by calculations using the inverse. From the variation in inverse detector angle required to track a given forward detector, coefficients may be determined for series to generate a cyclical satellite attitude to minimize the variation in detector angle.

Specifically, the approach used is as follows: In case 1, β is first made 0° (vertical array), with $\alpha = -5.5°$ (near an edge of the image), and $h = 0$. The satellite attitude is established with yaw, pitch, and roll either arbitrarily applied to represent possible instabilities or initially using nominal yaw and zero pitch and roll. Subsequent attitude corrections may be calculated from the Fourier series described below. For a chosen base position along the orbit, such as $\lambda' = 0$, the forward formulas are used to determine the ϕ, λ sensed by detector α, β. The inverse formulas are then used to determine a position α called α_1 (about $-5°$) for the detector on the fore array ($\beta = 23°$) sensing the same ϕ, λ. Once this α_1 is established, the forward equations for $\beta = 0°$, $\alpha = -5.5°$, and with λ' varying at 10° intervals from 90° to 270° (the daylight portion of the orbit) are used to determine ϕ, λ at a given h. The

TABLE 1. TRACKING DISCREPANCIES, MAPSAT, USING LANDSAT ORBIT. STABLE ATTITUDE

Satellite position λ'	Case 1: Fore and Vertical						Aft and Vertical					
	Yaw κ	Pitch ρ	Tracking Discrepancies (m)				Yaw κ	Pitch ρ	Tracking Discrepancies (m)			
			D_1	D_2	D_3	D_4			D_5	D_6	D_7	D_8
90°	−0.253°	0.040°	0.00	−1.29	0.00	−2.14	0.253°	−0.040°	−0.01	−1.25	−0.01	−2.10
120	−2.219	0.028	0.01	−1.09	0.02	−1.80	−1.785	−0.032	0.00	−1.02	0.00	−1.78
150	−3.589	0.008	0.00	−0.62	0.00	−1.00	−3.342	−0.012	0.03	−0.49	0.03	−0.95
180	−4.000	0.000	−0.01	−0.02	0.03	0.00	−4.000	0.000	0.02	0.07	0.06	0.12
210	−3.342	0.012	0.01	0.53	0.01	1.10	−3.589	−0.008	0.04	0.66	0.04	1.18
240	−1.785	0.032	0.00	1.02	0.00	1.99	−2.219	−0.028	−0.02	1.09	−0.01	2.07
270	−0.253	0.040	0.01	1.25	0.01	2.35	−0.253	−0.040	0.00	1.29	0.00	2.53

Symbols:
λ' position of satellite in circular orbit for vertical array, geocentric angle from ascending node, degrees.
κ total yaw, degrees (positive is counterclockwise from overhead), for satellite at λ'.
ρ pitch, degrees (positive is nose down), for satellite at λ'.
Tracking discrepancy, meters (height h above ellipsoid is zero, except for D_4 and D_8):
D_1: detector −5.5° from axis on vertical array (north of groundtrack)
D_2: detector 0.0° from axis on vertical array
D_3: detector 5.5° from axis on vertical array (south of groundtrack)
$\Big\}$ fore and vertical arrays
D_4: same as D_1 but height h = 1,000 metres
D_5: same as D_1
D_6: same as D_2
D_7: same as D_3
$\Big\}$ but aft and vertical arrays
D_8: same as D_5 but height h = 1,000 metres

inverse formulas are then used with $\beta = 23°$ to determine the α sensing the same ϕ, λ, h. Multiplying the differences $(\alpha - \alpha_1)$ in radians by L, linear discrepancies are obtained. These are not the true ground distances, but they are very close and reach zero when the ground discrepancy is zero. The process is repeated for $\alpha = 5.5°$ on the vertical array (the other side of the image), and later for both sides of the image but using the vertical and aft arrays.

The results are summarized in Table 1, where it is shown that during the descending ("daylight") portion of the satellite orbit a yaw of up to 4° and a pitch of up to only 0.04° can limit the distance between the tracks sensed by the central detectors on the vertical array and the fore (D_2) or aft (D_6) arrays to less than 1.3 metres when $h = 0$. At the same time, outer detectors (D_1, D_3, D_5, and D_7) are tracking within 0.06 m. The same yaw and pitch lead to a tracking discrepancy of over 2.5 m at the outer detectors if the height above the ellipsoid is 1000 m (D_4 and D_8).

For Case 2, the approach is similar to Case 1, except that values of $\beta = 23°$, $\alpha = -5.0°$ and a chosen λ' are used initially with the forward formulas, and a β of −23° is used with the inverse to determine the matching α_1 at the ascending node. Proceeding around the orbit, β's of 23° and −23°, respectively, are used for the tracking instead of 0° and 23°. The other side of the image ($\alpha = 5.0°$) is also checked. In Table 2, for Case 2 the tracking discrepancies are seen to be less than 0.5 m for central or outer detectors, with yaw varying about 4° during the descending orbit. The height above the ellipsoid may reach 1000 m before this tolerance is exceeded.

These tracking checks as described above are based on an ideal satellite which does not deviate from the orbit or attitude prescribed. In practice, the satellite has limited stability and will vary from the prescribed attitude. Based on limits stated by the National Aeronautics and Space Administration, the tracking ability was checked with a fixed error of 10 seconds of arc in each of the three components of the attitude, with a constant change of 10^{-6} degree per second in each component, and with ten times these instabilities. These effects were evaluated separately and are listed in Table 3.

TABLE 2. TRACKING DISCREPANCIES, MAPSAT, USING LANDSAT ORBIT. STABLE ATTITUDE

Satellite Position λ'	Case 2: Fore and Aft				
	Yaw κ	Tracking Discrepancies (m)			
		D_1	D_2	D_3	D_4
90°	0.000°	0.12	−0.01	0.29	−0.26
120	−2.010	−0.23	−0.01	0.41	−0.32
150	−3.479	−0.22	0.02	0.32	0.00
180	−4.015	0.00	−0.02	0.00	0.06
210	−3.479	0.26	−0.02	−0.21	0.48
240	−2.010	0.32	−0.01	−0.09	0.11
270	0.000	0.15	−0.01	0.32	−0.22

Symbols:
λ' position of satellite in circular orbit for fore array, geocentric angle from ascending node, degrees.
κ total yaw, degrees (positive is counterclockwise from overhead), for satellite at λ'.
Tracking discrepancy, metres (height h above ellipsoid is zero for D_1 through D_3):
D_1: detector −5.0° from axis on fore array (north of groundtrack).
D_2: detector 0.0° from axis on fore array.
D_3: detector 5.0° from center on fore array (south of groundtrack).
D_4: same as D_1 but height h = 1,000 metres.

Fourier series. To determine these coefficients, Simpson's rule for numerical integration may be used with 9° intervals in λ', beginning with a small nonzero λ' to avoid indeterminant values at the edge of each quadrant.

For the circular orbit and ellipsoidal Earth, it is found that coefficients are zero except for the following:

$$\kappa = A_1 \sin \lambda' + A_3 \sin 3\lambda' + A_5 \sin 5\lambda' + \ldots + B_1 \cos \lambda' + B_3 \cos 3\lambda' + B_5 \cos 5\lambda' + \ldots \quad (51)$$

where
$$A_n = (1/\pi) \int_0^{2\pi} \kappa \sin n\lambda' \, d\lambda' \quad (52)$$

$$B_n = (1/\pi) \int_0^{2\pi} \kappa \cos n\lambda' \, d\lambda' \quad (53)$$

and κ in Equations 52 and 53 is calculated for a given λ' from Equations 13 through 15 and 35 through 50.

Using values of R_0, i, P_2, P_1, a, and e, as described for Landsat in the list of symbols, the following Fourier coefficients are obtained from the above computations to obtain nominal yaw in degrees for the fore-and-vertical alternate of Case 1,

$$A_1 = -0.2513666$$
$$A_3 = 0.0019329$$
$$A_5 = -0.0000014$$
$$B_1 = 4.0015825$$
$$B_3 = -0.0013384$$
$$B_5 = -0.0000009$$

For the aft-and-vertical alternate of Case 1, the signs of coefficients A_1, A_3, and A_5 are reversed, and for Case 2, coefficients A_1, A_3, and A_5 are set equal to zero. The simplification for Case 2 is not rigorous, but coefficients A_n are zero and coefficients B_n vary only in the seventh decimal place. Even this discrepancy is offset by the calculations below for additional yaw to improve tracking.

Determining Fourier Coefficients for Final Yaw and Pitch for Case 1 Detector Tracking

If the nominal yaw above is applied to Mapsat, using a geometric groundtrack with the above parameters, tracking near the edge of the image will be in error by up to 30 metres in the course of the orbit. (If the groundtrack were based on vertical sensing by the center detector, as in the case of Landsat, the discrepancy would reach 250 m). It is found that the application of an additional yaw adjustment and some pitch can reduce this tracking error to a few hundredths of a metre at any one pair of detectors. The maximum tracking error of all corresponding detectors may be reduced to about one metre by a cyclical yaw and pitch adjustment. The principles and formulas given in this paper may be applied to orbits other than that proposed for Mapsat merely by inserting different values of i, P_2, P_1, and R_0.

Even more than for nominal yaw, the use of Fourier series greatly simplifies the repeated computation of the final yaw and pitch required. To calculate the necessary coefficients, tracking discrepancies, using the nominal yaw coefficients and series (Equation 51) with an initial pitch of zero, are determined for every 9° of λ' around the orbit, beginning at $\lambda' = 0$. The detector position α is made $-5.5°$ and then $+5.5°$ to place it near the edges of the vertical array, on each side of the groundtrack. The corresponding α is found for the forward array, using the tracking technique described above. The tracking discrepancies D_a and D_b in metres for the + and − values, respectively, of α compared with the base α_1 are converted to an approximate additional yaw and pitch adjustment as follows:

$$\Delta\Delta\kappa = \text{additional yaw (radians)} = -\tfrac{1}{2}\,(D_a + D_b)/390000 \quad (54)$$

$$\Delta\rho = \text{additional pitch (radians)} = \tfrac{1}{2}\,(D_a - D_b) \times 0.000015 \quad (55)$$

These formulas are empirically based on the geometry of the effect of yaw and pitch upon ability to track.

These values of $\Delta\Delta\kappa$ and $\Delta\rho$ are added after each iteration to the values of $\Delta\kappa$ and ρ which resulted in these discrepancies, and the totals are summed according to Simpson's rule for numerical integration for two separate sets of Fourier coefficients with integrals as follows:

$$C_0 = (1/2\pi) \int_0^{2\pi} (\Delta\kappa) \, d\lambda' \quad (56)$$

$$C_n = (1/\pi) \int_0^{2\pi} (\Delta\kappa) \cos n\lambda' \, d\lambda' \quad (57)$$

$$J_n = (1/\pi) \int_0^{2\pi} (\Delta\kappa) \sin n\lambda' \, d\lambda' \quad (58)$$

$$E_0 = (1/2\pi) \int_0^{2\pi} \rho \, d\lambda' \quad (59)$$

$$E_n = (1/\pi) \int_0^{2\pi} \rho \cos n\lambda' \, d\lambda' \quad (60)$$

$$F_n = (1/\pi) \int_0^{2\pi} \rho \sin n\lambda' \, d\lambda' \quad (61)$$

After determining these coefficients, $\Delta\kappa$ (which is added to nominal yaw) and ρ are calculated for each λ' used in the numerical integration around the orbit, using the following equations:

$$\Delta\kappa = C_0 + \sum_{n=1}^{n} (C_n \cos n\lambda' + J_n \sin n\lambda') \quad (62)$$

$$\rho = E_0 + \sum_{n=1}^{n} (E_n \cos n\lambda' + F_n \sin n\lambda') \quad (63)$$

A new set of tracking discrepancies D_a and D_b is obtained, followed by new coefficients which are added algebraically to previous coefficients, and a further iteration is made using the revised coefficients for yaw and pitch. Using this approach, a preset number of iterations is used, because ideal convergence cannot be reached because the discrepancies cannot be eliminated altogether. Values of D_a and D_b can be compared for subsequent iterations, and a decision to stop iteration can be reached when no further improvement is made. While this can probably be done more analytically, such an approach was not further investigated, because calculations of coefficients need to be made only once for a given set of orbit parameters.

In practice, about twenty sets of iterations were made. Final significant coefficients found are as follows (using parameters as given in the list of symbols): For $\Delta\kappa$ in degrees (Case 1):

$$C_0 = -0.0000062$$
$$C_1 = 0.0000503$$
$$C_3 = -0.0000553$$
$$J_1 = 0.0005129$$
$$J_3 = -0.0002231$$

For ρ in degrees (Case 1):

$$E_0 = 0.0201504$$
$$E_2 = -0.0201196$$
$$E_3 = -0.0000021$$
$$E_4 = -0.0000129$$
$$F_2 = 0.0025506$$
$$F_4 = 0.0000138$$

The above coefficients apply as shown for the combination of forward and vertical arrays. For vertical and aft arrays, the signs of J_n and E_n are reversed.

Determining Fourier Coefficients for Final Yaw for Case 2 Detector Tracking

If only nominal yaw is applied to Case 2, tracking will be in error by up to 200 metres. A small amount of additional yaw will reduce this to less than a metre. To determine Fourier coefficients for this yaw, a procedure analogous to that for Case 1 is employed. Because only yaw was found to be required, the tracking is determined for the center detector of the fore and aft arrays ($\alpha = 0°$, $\beta = \pm 23°$), and the discrepancy D_c in metres, computed as described for D earlier, is converted to approximate required yaw as follows:

$$\Delta\Delta\kappa = \text{additional yaw (radians)} = D_c/750000 \quad (64)$$

Numerical integration with Simpson's rule at each 9° of λ' leads to coefficients for formulas identical with Equations 56 through 58 except that G and H replace C and J respectively, so that

$$\Delta\kappa = G_0 + \sum_{n=1}^{n} (G_n \cos n\lambda' + H_n \sin n\lambda') \quad (65)$$

With this yaw added to the previous yaw, new discrepancies are calculated, and the resulting new yaw coefficients are added to the previous coefficients. When discrepancies become minimal, the coefficients are considered final. Only six to eight iterations are sufficient, and most coefficients are found to be near enough to zero to be omitted in the final series. Only coefficients G_n, for odd values of n, are finally needed. For $\Delta\kappa$ in degrees (Case 2):

$$G_1 = 0.0154673$$
$$G_3 = -0.0004777$$
$$G_5 = 0.0000036$$

Plotting Imagery from a Mapping Satellite on a Map Projection

The strip nature of continuous mapping from imagery sensed by any one of the arrays of Mapsat or of other mapping satellites suggests use of a conformal map projection on which the geocentric groundtrack remains continually true to scale. The Space Oblique Mercator (SOM), conceptually developed by Colvocoresses (1974), is the only such projection known. In the form in which working equations have been published (Snyder 1978, 1981), there are modifications for the vertical groundtrack of Landsat, but only the geocentric groundtrack is rigorously true to scale. Because the use of the geocentric groundtrack for Mapsat was soon found to present much more satisfactory tracking than the vertical, the SOM in its basic form is well suited. On the other hand, the application of yaw and pitch for improved tracking cause the groundtrack to shift slightly, but its scale remains true on the SOM within a few millionths, as did that of the vertical groundtrack. Attempting to adjust for the shift would unnecessarily complicate the formulas, because the location of the actual line tracked by the center detector of the vertical array is somewhat academic.

The formulas for the SOM as published previously are suitable, although the formulas for transformation from geodetic coordinates (ϕ, λ) to "transformed" latitude and longitude (ϕ'', λ'') may also be modified for transformation from rectangular geodetic coordinates (X, Y, Z) to ϕ'', λ'', because ϕ, λ do not need to be calculated. The double prime was used in the SOM formulas for Landsat to distinguish between coordinates relative to the vertical groundtrack (ϕ', λ') and relative to the geocentric groundtrack (ϕ'', λ''). For Mapsat, the latter may be used for a convenient set of intermediate coordinates wherein λ'' relates to an imaginary Landsat-like satellite as λ' relates to Mapsat, and ϕ'' is the angular distance to the left of the geocentric groundtrack as viewed from the satellite but measured from the center of the Earth. The use of rectangular geodetic coordinates does not appear to reduce computer time.

To find ϕ'', λ'' for a given λ', array, and attitude, first $\bar{\mathbf{X}}$ is calculated from many of the Equations 1 through 29 (for $h = 0$). Then, with λ' as the first trial λ'', λ'' is found from Equations 66 through 68 by successive substitution, until λ'' does not change significantly. Calculation of ϕ'' follows completion of the iteration for λ''.

$$\theta = (P_2/P_1)(\lambda'' - \lambda') \tag{66}$$

$$\bar{\mathbf{X}}_r = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{\mathbf{X}} \tag{67}$$

$$\begin{pmatrix} X_r \tan\lambda'' \\ a \sin\phi'' \end{pmatrix} = \begin{pmatrix} \cos i & \sin i \\ -\sin i & \cos i \end{pmatrix} \begin{pmatrix} Y_r \\ Z_r \end{pmatrix} \tag{68}$$

For inverse equations, given ϕ'', λ'', and to find $\bar{\mathbf{X}}$:

$$\begin{pmatrix} Y_r \\ Z_r \end{pmatrix} = \begin{pmatrix} \cos i & -\sin i \\ \sin i & \cos i \end{pmatrix} \begin{pmatrix} X_r \tan\lambda'' \\ a \sin\phi'' \end{pmatrix} \tag{69}$$

where

$$X_r = \left[a/(1-e^2)\right]\left\{-e^2 \sin i \cos i \sin\phi'' \tan\lambda'' \pm (1-e^2)^{1/2}\left[-\sin^2\phi'' \tan^2\lambda'' - \sin^2\phi''(1-e^2\sin^2 i) + \tan^2\lambda''(1-e^2\cos^2 i) + 1 - e^2\right]^{1/2}\right\}/\left[1 - e^2 + \tan^2\lambda''(1-e^2\cos^2 i)\right]. \tag{70}$$

The $\pm$ sign takes the sign of $\cos\lambda''$. Angle θ is then found from Equation 66, and a trial $\bar{\mathbf{X}}$ is found from $\bar{\mathbf{X}}_r$ using the inverse of Equation 67, for which the matrix is transposed. The first trial λ' in Equation 66 is λ'', and subsequent trial values of λ' are obtained from the earlier inverse equations, including Equations 30 through 34, using the trial value of $\bar{X}$ calculated here, instead of Equations 17 through 19. The derivations of Equations 66 through 70 are omitted, as are the formulas for converting ϕ'', λ'' to x,y and vice versa on the SOM. The latter may be found in the references cited above.

All the computations described in this paper have been programmed in FORTRAN IV (H-compiler).

Summary of Tracking Capabilities

While perfect tracking does not appear feasible, it does appear possible to achieve tracking to within a fraction of a 10-metre pixel. This is accomplished by varying yaw and pitch in Case 1 and yaw only in Case 2, using a satellite with its equivalent of an optical axis initially pointing geocentrically rather than vertically. Matching is also satisfactory within the small predicted range of instability. A Fourier series for each of these attitude corrections may be determined after a one-time iteration is performed for a given set of orbit parameters.

It should be noted that the discrepancies in Tables 1, 2, and 3 should be treated qualitatively only. Calculated for a mathematical model, even the "unstable attitude" conditions described in Table 3 are idealistic in that they represent constant offset or a constant rate of change. In practice, instability would follow a wobbly pattern, but it should fall within the ranges shown.

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