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Array Algebra Estimation in Signal Processing*

Filtering concepts of digital signal processing are interpreted in terms of the new generalized adjustment calculus, numerical analysis, and mathematical statistics of array algebra estimation.

Abstract: *A general theory of linear estimators, called array algebra estimation, is interpreted in terms of multidimensional digital signal processing, mathematical statistics, and numerical analysis. The theory has emerged during the past decade from the new field of a unified vector, matrix, and tensor algebra called array algebra. The broad concepts of array algebra and its estimation theory cover several modern computerized sciences and technologies, converting their notations and terminology into one common language. In this paper some concepts of digital signal processing are adopted into this language after a review of the principles of array algebra estimation and its predecessors in the adjustment calculus of mathematical surveying sciences.*

Introduction

Digital signal processing is a relatively new field (IEEE, 1975; Mitra and Ekstrom, 1978). It experienced intense development during the past ten years due to phenomenal advances in computer technology. The major emphasis of multidimensional digital signal processing has been on the two-dimensional case because of its wide-spread applications, especially in image processing. An important part of the research in photogrammetric image processing and remote sensing is presently conducted using the concepts originated in the field of signal processing.

In comparison to the mathematical surveying sciences, multi-dimensional digital signal processing is a very specialized field with severe restrictions in its design techniques and data structures prohibiting its use in general problems. Similar difficulties were experienced in the first applications of the more general array algebra counterpart of signal processing. Recent developments in array algebra have removed most of these restrictions, thereby calling for a reformulation of the computational problems of many sciences and technologies of linear algebra in terms of the more general digital signal processing of array algebra estimation. Thus, the rethinking process caused by the introduction of the fast transforms and filtering concepts of digital signal processing will be further expanded to the more general problems of linear algebra.

The basic concepts of array algebra (Rauhala, 1972, 1974) are introduced as a general one-dimensional estimation technique, which is then converted into fast multi-dimensional computational rules for handling gridded data. The multidimensional array estimation technique can be interpreted as a generalized algebraic form of the "fast" transform technology (Good, 1958; Greville, 1961; Cooley and Tukey, 1965) as shown by Rauhala (1976, 1977, 1980a).

The first applications of array algebra dealt with separable space domain modeling, releasing the gridding requirement for input arrays and introducing fast multi-dimensional convolution operators for regular grids (Rauhala, 1978, 1979; Kratky, 1978). The remaining restrictions of array algebra were released in the array algebra and sig-

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nal processing formulation of the problem of finite element filtering (Rauhala, 1980b). The present paper combines these general signal processing concepts of array algebra into the general linear estimation theory formulated by Rauhala (1981b).

Linear Estimators of Signal Processing

The Best Linear Unbiased Estimator (BLUE) of mathematical statistics can be interpreted in terms of digital signal processing as the output $\hat{\mathbf{L}}_0 = \mathbf{HL}$, representing the minimum variance unbiased estimator of functions $\mathbf{L}_0 = \mathbf{A}_0\mathbf{X}$, from input $\mathbf{L}$ satisfying the Gauss-Markov model

$$E\underset{m,1}{(\mathbf{L})} = \underset{mn}{\mathbf{A}}\underset{n,1}{\mathbf{X}} \leftrightarrow \underset{mn}{\mathbf{A}}\underset{n,1}{\mathbf{X}} = \underset{m,1}{\mathbf{L}} - \underset{m,1}{\mathbf{V}}, E(\mathbf{V}) = \mathbf{0}. \tag{1}$$

Symbol E denotes the expectation operator and $\mathbf{X}$ consists of the modeling parameters to define the problem of linear algebra to be solved. The design of modeling parameters $\mathbf{X}$ and the choice of functions $\mathbf{AX}$ to be sampled in the observable space domain determine the design matrix $\mathbf{A}$. A judicious choice of the output functions $\underset{p,1}{\mathbf{L}_0} = \underset{pn}{\mathbf{A}_0}\underset{n,1}{\mathbf{X}}$ "solves the problem." In terms of general linear estimators the "problem solution" can be defined as an establishment of the BLUE $\hat{\mathbf{L}}_0 = \mathbf{HL}$ of such basis functions $\mathbf{A}_0\mathbf{X}$ that span the whole unbiasedly estimable space of the problem, or focussed parts of it (Rauhala, 1981b).

For $\hat{\mathbf{L}}_0 = \mathbf{HL}$ to be the BLUE of $\mathbf{A}_0\mathbf{X}$, the following two conditions have to be satisfied:

$$\mathbf{HA} = \mathbf{A}_0 \leftrightarrow \mathrm{E}(\hat{\mathbf{L}}_0) = \mathbf{L}_0 \tag{2a}$$

$$\mathbf{H} = \mathbf{A}_0\,(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{g}\,\mathbf{A}^{\mathrm{T}} \leftrightarrow \mathrm{trace}\,(\mathbf{HH}^{\mathrm{T}}) = \mathrm{min}. \tag{3a}$$

where $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{g}$ is a generalized inverse of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ (Bjerhammar, 1955). In the general estimation theory the concept of a generalized matrix inverse is extended to "estimator" inverse $\mathbf{G}$ of matrix $\mathbf{A}$ in $\mathbf{H} = \mathbf{A}_0\mathbf{G}$ so that the general BLUE $\hat{\mathbf{L}}_0 = \mathbf{A}_0\mathbf{GL}$ satisfies Equations 2a and 3a by

$$\mathbf{A}_0\mathbf{GA} = \mathbf{A}_0 \tag{2b}$$

$$\mathbf{A}_0\mathbf{G} = \mathbf{A}_0\,(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{g}\mathbf{A}^{\mathrm{T}}. \tag{3b}$$

In the general estimation theory $\mathbf{G}$ can be any operator if $\underset{pn}{\mathbf{A}_0}$, $p <$ rank $\underset{mn}{(\mathbf{A})}$, is chosen to span only a subspace of the row space of $\mathbf{A}$ (Rauhala, 1981b).

In the following sections the principles of numerical analysis are reviewed for computing $\mathbf{HL}$. After that, the operator $\mathbf{H}$ is specialized to yield the filtering concepts of digital signal processing.

Copernicus-Gauss Philosophy

In the conventional least-squares theory the basis functions $\mathbf{L}_0 = \mathbf{A}_0\mathbf{X}$ spanning the unbiasedly estimable space consist of the modeling parameters $\mathbf{X}$ themselves. Thus, $\mathbf{L}_0 = \mathbf{X}$, $\mathbf{A}_0 = \mathbf{I}$, and the design matrix $\mathbf{A}$ is brought to have full rank, yielding

$$\hat{\mathbf{L}}_0 = \hat{\mathbf{X}} = \mathbf{A}^{l}\mathbf{L},\ \mathbf{A}^{l} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}. \tag{4}$$

The BLUE of all other estimable functions $\mathbf{FX}$, satisfying $\mathbf{FA}^{g}\mathbf{A} = \mathbf{F}$, can be found by $\mathbf{F}\hat{\mathbf{X}} = \mathbf{FA}^{l}\mathbf{L}$.

Advances in the least-squares problems of surveying sciences were made by improving the accuracy of the observed quantities $\mathbf{L}$ and following the philosophy of Copernicus, by developing better modeling parameters $\mathbf{X}$ explaining the physical phenomenon associated with $\mathbf{L}$ by $\mathbf{AX} = \mathbf{L}$ for the "true" values of $\mathbf{X}$, $\mathbf{L}$. The development of this so-called adjustment calculus begun by Gauss over a hundred years ago, was drastically accelerated in the 1950's when computers and matrix algebra became the essential tools of the problem analyst. The highlights of modern adjustment calculus, before the introduction of array algebra estimation in the early 1970's, can be outlined as follows:

Inefficient and fully correlated elements of $\mathbf{X}$, causing ill-conditioning or singularity of the normal equation matrix, can be empirically discarded during the elimination process (Brown, 1955, pp. 9-10). Thus, initially the math model is encouraged to contain any physical parameters that reduce the residuals $\hat{\mathbf{V}} = \mathbf{L} - \mathbf{A}\hat{\mathbf{X}}$. Essentially the same philosophy is behind the theory of general matrix inverses (Bjerhammar, 1955; Rao and Mitra, 1971, pp. vii, 212).

An important practical modeling tool is the explicit introduction of additional physical or empirical modeling parameters $\mathbf{Y}$ for elimination of small unmodeled systematic errors or "signal." To avoid ill-conditioning of the resulting normal equations a-priori statistical constraints $\mathbf{L}_c$ for the additional parameters $\mathbf{Y}$ are introduced (Brown, 1958)

$$\begin{aligned}\mathbf{AX} + \mathbf{BY} &= \mathbf{L} - \mathbf{V}\\ \mathbf{CY} &= \mathbf{L}_c - \mathbf{V}_c.\end{aligned} \tag{6}$$

For serially correlated data the space domain response or "signal" $\mathbf{S} = \mathbf{BY}$ of the systematic errors can be considered as an additional parameter for each observed value. In the case of one-dimensional evenly distributed observations the inverse covariance matrix $\mathbf{\Lambda}^{-1}$ of the signal can be conveniently derived from the process of autocorrelation, yielding the solution, (Brown, 1955; Brown, *et al.*, 1963; Brown and Trotter, 1969; Helstrom, 1967; Moritz, 1978; Rauhala, 1976, p. 118; Strang van Hees, 1981)

$$\hat{\mathbf{X}} = (\mathbf{A}^{\mathrm{T}}\mathbf{\Lambda}^{-1}\mathbf{A})^{-1}\,\mathbf{A}^{\mathrm{T}}\mathbf{\Lambda}^{-1}\mathbf{L}. \tag{7}$$

During the 1960's the computational algorithms for a rigorous least-squares solution of tens of thousands of modeling parameters $\mathbf{X}$ in Equation 7 were designed for the emerging minicomputers.

The algorithms typically utilized zero block matrices for deriving the reduced normal equations of Pranish-Praniewich type. The reduced system exhibits a banded-border structure in the problems of analytical photogrammetry and satellite geodesy (Brown and Trotter, 1969). More detailed reviews of the development of modern adjustment calculus can be found in Brown (1974), Pope (1974), and Moritz (1978).

In the 1970's computer technology flooded the mathematical sciences with an enormous amount of digital data. The next section describes how array algebra extends the Copernicus-Gauss philosophy into empirical multilinear space domain modeling for the computational treatment of large data arrays.

ARRAY ALGEBRA ESTIMATION

The idea of array algebra relaxes the choice of basis functions $\mathbf{L}_0 = \mathbf{A}_0\mathbf{X}$ of the unbiasedly estimable space beyond $\mathbf{A}_0 = \mathbf{I}$ of the above Copernicus-Gauss philosophy. A new fast numerical analysis is introduced for handling multilinear gridded data arrays $\mathbf{L}_0$, $\mathbf{L}$.

For illustration of array algebra estimation, consider the examples of Moritz (1978, formula 72, p. 427) and Rauhala (1972, pp. 18-22; 1978, p. 8) written as the observation equations

$$\underset{mn}{\mathbf{A}}\ \underset{n,1}{\mathbf{X}} = \underset{m,1}{\mathbf{L}} - \underset{m,1}{\mathbf{V}},\ E(\mathbf{V}) = \mathbf{0},\ E(\mathbf{V}\mathbf{V}^{\mathrm{T}}) = \mathbf{I}.$$
$$n \to \infty.$$

The "physical" modeling parameters $\mathbf{X}$ represent $n \to \infty$ coefficients of global spherical harmonics expansion of the Earth's potential function for modeling a large amount of observable functions $\mathbf{AX}$ (gravity anomalies, deflections of vertical, etc.) of a local (say 1° by 1°) area. Matrix $\mathbf{A}$ is singular or very ill-conditioned so that neither $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$ nor $(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}$ exist. This means that the modeling parameters $\mathbf{X}$ are not unbiasedly estimable because $\mathbf{IGA} \neq \mathbf{I}$ for any $\mathbf{G}$ in Equation 2b (Bossler, 1973; Grafarend and Schaffrin, 1974). Thus, we have to find another basis spanning the unbiasedly estimable space.

It follows from Equation 2b that all observed functions $\mathbf{AX}$ are always unbiasedly estimable because $\mathbf{AA}^g\mathbf{A} = \mathbf{A}$, $\mathbf{G} \in \mathbf{A}^g$. Thus, any choice of $p \simeq \mathrm{rank}(\mathbf{A})$ independent $\mathbf{A}_0\mathbf{X}$ among $\mathbf{AX}$ offers the desired basis. In the two-dimensional example $\mathbf{L}$ is assumed to be sampled using the Nyquist rule. In practice, large data sets $\underset{m,1}{\mathbf{L}}$, $m = m_1 m_2$, at grid points are organized (by direct sampling or preprocessing) into array $\underset{m_1 m_2}{\mathbf{L}}$. Then ideally, array $\underset{p_1 p_2}{\mathbf{L}_0}$, $p = p_1 p_2$, is chosen to consist of every second row and column of array $\mathbf{L}$.

The fictitious error-free observables of the chosen basis functions

$$\mathbf{A}_0\mathbf{X} = \mathbf{L}_0 \tag{8}$$

have a full-rank design matrix $\underset{pn}{\mathbf{A}_0}$, $\mathrm{rank}(\mathbf{A}_0) = p \leq n$.

Thus, the parameter transformation can be performed by solving for Equation 8

$$\mathbf{X} = \mathbf{A}_0^{\mathrm{m}}\mathbf{L}_0,\ \mathbf{A}_0^{\mathrm{m}} = \mathbf{A}_0^{\mathrm{T}}(\mathbf{A}_0\mathbf{A}_0^{\mathrm{T}})^{-1}, \tag{9}$$

which inserted in the original ill-conditioned observation equations in parameters $\mathbf{X}$ yields the full-rank observation equations in parameters $\mathbf{L}_0$ by

$$\underset{mp}{(\mathbf{AA}_0^{\mathrm{m}})}\ \underset{p,1}{\mathbf{L}_0} = \underset{m,1}{\mathbf{L}} - \underset{m,1}{\mathbf{V}},\ E(\mathbf{V}) = \mathbf{0},\ E(\mathbf{V}\mathbf{V}^{\mathrm{T}}) = \mathbf{I},$$

$$\mathrm{rank}\ \underset{mp}{(\mathbf{AA}_0^{\mathrm{m}})} = p. \tag{10a}$$

The BLUE of $\mathbf{L}_0$ is the least-squares solution of Equation 10a

$$\begin{aligned}\underset{p,1}{\hat{\mathbf{L}}_0} &= \underset{pm}{(\mathbf{AA}_0^{\mathrm{m}})^{l}}\ \underset{m,1}{\mathbf{L}} \leftrightarrow \|\mathbf{L} - \mathbf{AA}_0^{\mathrm{m}}\hat{\mathbf{L}}_0\| = \min. \\ &= \mathbf{HL} \\ &= \mathbf{A}_0\mathbf{GL} \qquad \leftrightarrow \mathbf{G} = \mathbf{A}_0^{\mathrm{m}}(\mathbf{AA}_0^{\mathrm{m}})^{l}\end{aligned} \tag{11}$$

The choices for gridded $\mathbf{L},\mathbf{L}_0$ in Equation 11 usually cannot be fully utilized because of the "physical" modeling parameters determining the design matrix $\mathbf{A}$. A new philosophy for the space domain modeling is introduced in array algebra to take advantage of the gridded structure of arrays $\mathbf{L},\mathbf{L}_0$.

In the example, the system (Equation 10a) partitions as

$$\begin{bmatrix}\mathbf{I}\\ \mathbf{K}_E\end{bmatrix}\mathbf{L}_0 = \begin{bmatrix}\mathbf{L}_{om}\\ \mathbf{L}_E\end{bmatrix} - \begin{bmatrix}\mathbf{V}_0\\ \mathbf{V}_E\end{bmatrix},\ \mathbf{A} = \begin{bmatrix}\mathbf{A}_0\\ \mathbf{A}_E\end{bmatrix},$$

$$\mathbf{AA}_0^{\mathrm{m}} = \begin{bmatrix}\mathbf{I}\\ \mathbf{K}_E\end{bmatrix},\ \mathbf{K}_E = \mathbf{A}_E\mathbf{A}_0^{\mathrm{m}}, \tag{10b}$$

since $\mathbf{L}_0$ is coincided with p observed functions. The redundant observed functions $\mathbf{A}_E\mathbf{X}$ are expressed as interpolations $\mathbf{K}_E\mathbf{L}_0$ from the chosen basis functions $\mathbf{L}_0 = \mathbf{A}_0\mathbf{X}$. The crucial idea of the new space domain modeling technique of array algebra estimation is to replace the "Copernicus-Gauss interpolation" $\mathbf{K}_E\mathbf{L}_0$ in Equation 10b by an empirical one, resulting in a fast space domain solution $\hat{\mathbf{L}}_0$; in the first step we only separate the noise $\hat{\mathbf{V}}$ from the discrete observed values $\mathbf{L}$ using the efficient filtering rules of array algebra and digital signal processing for densely distributed $\hat{\mathbf{L}}_0$. In the second step we analyze the behavior of $\hat{\mathbf{L}}_0$ in order to design parameters $\mathbf{X}$ properly. Computation of $\hat{\mathbf{X}}$ from $\hat{\mathbf{L}}_0$ is facilitated since $\hat{\mathbf{L}}_0$ and the associated empirical interpolation rule represent a non-discrete function which can be evaluated at any strategic and computationally desirable variable locations.

In the example the above philosophy can be interpreted as follows: The measuring density of gridded $\underset{m_1 m_2}{\mathbf{L}}$ is in practice far higher than the resolution capability of "physical" modeling parameters $\underset{n,1}{\mathbf{X}}$ even if in our case $n \rightarrow \infty$. It is customary to determine the sampling rate or resolution of $\underset{m_1 m_2}{\mathbf{L}}$ with respect to the "worst case" to capture local abrupt behavior of the observable space. Thus, in our example, the resolution of both $\underset{m_1 m_2}{\mathbf{L}}$ and $\underset{p_1 p_2}{\mathbf{L}_0}$ far exceeds that of the physically explainable parameters $\mathbf{X}$. Therefore, it is only natural to replace the low resolution "Copernicus-Gauss interpolation" $\mathbf{K}_E\mathbf{L}_0$ of Equation 10b by any feasible and local separable interpolations from few (usually 4 by 4) closest unknown values of the chosen basis $\underset{p_1 p_2}{\mathbf{L}_0}$. Separable interpolation is characterized by $F(y,x) = \underset{1,p_1}{\mathbf{k}_1(y)}\ \underset{p_1 p_2}{\mathbf{L}_0}\ \underset{p_2,1}{\mathbf{k}_2^T(x)}$ where $\underset{1,p_1}{\mathbf{k}_1(y)}\ \underset{p_1 p_2}{\mathbf{L}_0}$ interpolates the values $F(y,j)$, $j = 1, 2, \ldots, p_2$ along the columns of $\mathbf{L}_0$ and the subsequent interpolation of the resulted values along the row-direction of $\mathbf{L}_0$ by $\underset{1,p_2}{F(y,j)}\ \underset{p_2,1}{\mathbf{k}_2^T}(x)$ yields $F(y,x)$.

The drastic computational consequence of the above separable space domain modeling for gridded $\mathbf{L},\mathbf{L}_0$ allows Equations 10a and 10b to be rewritten as the array algebra observation equations

$$\underset{m_1 p_1}{\mathbf{K}_1}\ \underset{p_1 p_2}{\mathbf{L}_0}\ \underset{p_2 m_2}{\mathbf{K}_2^T} = \underset{m_1 m_2}{\mathbf{L}} - \underset{m_1 m_2}{\mathbf{V}} \tag{12a}$$

using formally the approximation

$$\mathbf{AA}_0^m = \mathbf{K}_1 \otimes \mathbf{K}_2 = \{(\mathrm{k}_1)_{r_1 j_1}(\mathrm{k}_2)_{r_2 j_2}\} \tag{12b}$$

where $\otimes$ denotes the Kronecker or tensor product. In Equation 12a matrix multiplication $\mathbf{K}_1\mathbf{L}_0$ interpolates the row values of the unknown grid $\mathbf{L}_0$ into the row locations of the measured grid $\mathbf{L}$. The column values of the resulting grid are then interpolated to the column locations of grid $\mathbf{L}$ by $(\mathbf{K}_1\mathbf{L}_0)\mathbf{K}_2^T$.

The fast least-squares array solution of Equation 12a is found by Rauhala (1972, p. 10; 1974-1981b),

$$\underset{p_1 p_2}{\hat{\mathbf{L}}_0} = \underset{p_1 m_1}{\mathbf{H}_1}\ \underset{m_1 m_2}{\mathbf{L}}\ \underset{m_2 p_2}{\mathbf{H}_2^T} \leftrightarrow \left|\left|\mathbf{L} - \mathbf{K}_1\hat{\mathbf{L}}_0\mathbf{K}_2^T\right|\right| = \text{min.} \tag{13}$$

$$\begin{aligned}\mathbf{H}_1 &= \mathbf{K}_1^l = (\mathbf{K}_1^T\mathbf{K}_1)^{-1}\mathbf{K}_1^T \\ \mathbf{H}_2 &= \mathbf{K}_2^l = (\mathbf{K}_2^T\mathbf{K}_2)^{-1}\mathbf{K}_2^T.\end{aligned} \tag{14}$$

In three and higher dimensions the notational systems of matrix and tensor calculus would fail for a convenient shorthand expression of Equations 12a and 13, and array algebra was designed to fill this gap. In a more general case there are several batches of separable or non-separable observation equations and the least-squares solution (Equation 13) becomes non-separable as shown next by interpreting the above approach in terms of digital signal processing.

LSV Filtering

The non-separable estimator of Equation 11

$$(\hat{l}_0)_{i,j} = \sum_q \sum_r (h_{q,r})_{i,j}\, l_{q,r} \tag{15}$$

or the separable estimator of Equation 13

$$(\hat{l}_0)_{i,j} = \sum_q (h_q)_i \sum_r (h_r)_j\, l_{q,r} \tag{16}$$

usually have LSV (Linear Shift Variant) operators $h_{q,r}$, h_q, h_r called impulse responses in digital signal processing, i.e., each combination of i,j has its own impulse response different from the others. For the purpose of illustrating the explicit connection of array algebra estimation with the filter design of digital signal processing, the least-squares solution of sequential array equations will be derived next.

The sequential separable observation equations

$$\begin{aligned}\mathbf{K}_1\ \mathbf{L}_0\ \mathbf{J}_1^T &= \mathbf{L}_1 - \mathbf{V}_1 \\ \mathbf{K}_2\ \mathbf{L}_0\ \mathbf{J}_2^T &= \mathbf{L}_2 - \mathbf{V}_2 \\ &\vdots \\ \mathbf{K}_s\ \mathbf{L}_0\ \mathbf{J}_s^T &= \mathbf{L}_s - \mathbf{V}_s\end{aligned} \tag{17}$$

result in the normal equations

$$\begin{aligned}&\mathbf{N}_1\mathbf{L}_0\mathbf{M}_1 + \mathbf{N}_2\mathbf{L}_0\mathbf{M}_2 + \ldots + \mathbf{N}_s\mathbf{L}_0\mathbf{M}_s = \mathbf{U} \\ &\mathbf{U} = \sum_{t=1}^{s} \mathbf{K}_t^T\mathbf{L}_t\mathbf{J}_t \\ &\mathbf{N}_t = \mathbf{K}_t^T\mathbf{K}_t,\ \mathbf{M}_t = \mathbf{J}_t^T\mathbf{J}_t,\ t = 1, 2, \ldots, s.\end{aligned} \tag{18}$$

In terms of two-dimensional unitary transform coefficients (Pratt, 1972),

$$\begin{aligned}&\mathbf{Z} = \mathbf{S}_1\mathbf{L}_0\mathbf{S}_2^T \\ &\mathbf{S}_1^T\mathbf{S}_1 = \mathbf{S}_1\mathbf{S}_1^T = \mathbf{I},\ \mathbf{S}_2^T\mathbf{S}_2 = \mathbf{S}_2\mathbf{S}_2^T = \mathbf{I},\end{aligned} \tag{19}$$

the normal equations (Equation 18) read by premultiplications with $\mathbf{S}_1$, and postmultiplications with $\mathbf{S}_2^T$ as

$$\mathbf{Q}_1\mathbf{Z}\mathbf{R}_1 + \mathbf{Q}_2\mathbf{Z}\mathbf{R}_2 + \ldots + \mathbf{Q}_s\mathbf{Z}\mathbf{R}_s = \mathbf{S}_1\mathbf{U}\mathbf{S}_2^T. \tag{20}$$

The purpose of the parameter transformation (Equation 19) is to bring a special structure to the normal equations (Equation 20) so that a very efficient solution for $\hat{\mathbf{Z}}$ from Equation 20 exists. The important practical case of diagonal matrices

$$\begin{aligned}\mathbf{Q}_t &= \mathbf{S}_1\mathbf{N}_t\mathbf{S}_1^T \\ \mathbf{R}_t &= \mathbf{S}_2\mathbf{M}_t\mathbf{S}_2^T\end{aligned}\quad t = 1, 2, \ldots, s$$

dominates the filter design of digital signal processing. Now the diagonal operator

$$(\mathbf{H}*)_{i,j} = 1/\sum_{t=1}^{s} (q_{i,i}\, r_{j,j})_t, \quad \begin{matrix} i = 1,2, \ldots, p1 \\ j = 1,2, \ldots, p2 \end{matrix} \qquad (21)$$

is called "transfer function." In this special case the solution of the diagonalized system (Equation 20) is simply

$$\begin{aligned} \hat{\mathbf{Z}} &= \mathbf{H}* (\mathbf{S}_1 \mathbf{U} \mathbf{S}_2^{\mathrm{T}}) \\ (\hat{\mathbf{Z}})_{i,j} &= (\mathbf{H}*)_{i,j}\, (\mathbf{S}_1 \mathbf{U} \mathbf{S}_2^{\mathrm{T}})_{i,j} \end{aligned} \qquad (22)$$

where $*$ denotes elementwise (dot) multiplications.

If Equation 20 is non-singular, estimator $\hat{\mathbf{Z}}$ is BLUE qualifying as the "problem solution" and representing a basis in the "filter transform domain" of $\mathbf{Z}$. Estimators $\hat{\mathbf{L}}_0$ of the space domain and $\hat{\mathbf{X}}$ of "design transform domain" are found by linear transformations

$$\underset{p_1 p_2}{\hat{\mathbf{L}}_0} = \underset{p_1 p_1}{\mathbf{S}_1^{\mathrm{T}}}\ \underset{p_1 p_2}{\hat{\mathbf{Z}}}\ \underset{p_2 p_2}{\mathbf{S}_2} \qquad (23)$$

$$\underset{n,1}{\hat{\mathbf{X}}} = \underset{np}{\mathbf{A}_0^{\mathrm{m}}}\ \underset{p,1}{\hat{\mathbf{L}}_0},\ p = p_1 p_2. \qquad (24)$$

Estimator $\hat{\mathbf{X}}$ is biased if $p < n$. In practice $\hat{\mathbf{X}}$ is best computed by evaluating $\hat{\mathbf{L}}_0$ through the empirical local separable interpolations as $\hat{\bar{\mathbf{L}}}_0$ at fictitious points where $\bar{\mathbf{L}}_0 = \bar{\mathbf{A}}_0\, \mathbf{X}$. The fictitious design matrix $\bar{\mathbf{A}}_0$ is brought to exhibit some special structure so that $\hat{\mathbf{X}} = \bar{\mathbf{A}}_0^{\mathrm{m}} \hat{\bar{\mathbf{L}}}_0 \simeq \mathbf{A}_0^{\mathrm{m}}\, \hat{\mathbf{L}}_0$ can be efficiently realized.

According to the custom of adjustment calculus, the variance-covariance matrices of estimators $\hat{\mathbf{Z}}$, $\hat{\mathbf{L}}_0$, $\hat{\mathbf{X}}$ can be computed as

$$\operatorname{cov} \underset{p,1}{(\hat{\mathbf{Z}})} = \hat{\delta}^2 (\mathbf{Q}_1 \otimes \mathbf{R}_1 + \mathbf{Q}_2 \otimes \mathbf{R}_2 + \ldots + \mathbf{Q}_s \otimes \mathbf{R}_s)^{-1}$$

$$\operatorname{cov} \underset{p,1}{(\hat{\mathbf{L}}_0)} = \underset{pp}{\mathbf{S}^{\mathrm{T}}}\ \underset{pp}{\operatorname{cov}}\ (\hat{\mathbf{Z}})\ \underset{pp}{\mathbf{S}},\ \mathbf{S} = \mathbf{S}_1 \otimes \mathbf{S}_2,$$

$$\operatorname{cov} \underset{n,1}{(\hat{\mathbf{X}})} = \mathbf{A}_0^{\mathrm{m}} \operatorname{cov} \hat{\mathbf{L}}_0)\ \mathbf{A}_0^{\mathrm{mT}}$$

$$\hat{\delta}^2 = \sum_s v^2/\text{d.f.} \qquad \text{d.f.} = \text{degree of freedom}$$

$$\sum v^2 = \sum_{t=1}^{s} \|\mathbf{L}_t\|^2 - \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} \underset{p_1 p_2}{(\hat{\mathbf{Z}}} * \underset{p_1 p_2}{\mathbf{S}_1 \mathbf{U} \mathbf{S}_2^{\mathrm{T}})_{i,j}}. \qquad (25)$$

The above solution is treated in more detail by Rauhala (1980b) and specialized to Karhunen-Loeve transform of a tridiagonal matrix called the FET (Fast Eigen Transform or Finite Element Transform), closely related to the FST (Fast Sinus Transform) and the FFT (Fast Fourier Transform). The application of the above solution technique in Rauhala (1980b) consisted of array algebra modification of finite element filtering (Junkins, 1976; Ebner and Reiss, 1978; Meissl, 1980).

The space domain modeling of finite elements can be treated by the introduction of additional transform domain parameters. Assume that the "design transform domain" parameters $\mathbf{X}$ represent the physical "Copernicus-Gauss" parameters of structural finite elements, $\underset{p_1 p_2}{\mathbf{L}_0}$ is a regular array of their space domain response, and $\mathbf{Z} = \mathbf{S}_1\, \mathbf{L}_0\, \mathbf{S}_2^{\mathrm{T}}$ represents the "filter transform domain." For the empirical space domain modeling it is often convenient to introduce additional "spatial transform domain" parameters $\mathbf{Y}$ by linear transform $\underset{p_1 p_2}{\mathbf{L}_0} = \underset{p_1 p_1}{\mathbf{B}_1}\ \underset{p_1 p_2}{\mathbf{Y}}\ \underset{p_2 p_2}{\mathbf{B}_2^{\mathrm{T}}}$. An example of such parameters are the spline functions (Greville, 1961; Kubik, 1971; DeMasson d'Autume, 1976, 1978). It is convenient to choose $\mathbf{B}_1$, $\mathbf{B}_2$ in finite element filtering such that they are diagonalized by the same FET unitary transforms as $\mathbf{Q}$, $\mathbf{R}$ in Equation 20 (Rauhala, 1980b, pp. 25-26).

LSI Filtering

The emphasis of the vast literature of signal processing is put on the design of the very special and computationally efficient LSI (Linear Shift Invariant) filtering using FIR (Finite Impulse Response) coefficients $h_{y,r}$ in the convolution of Equations 15 and 16. Now the non-separable LSI filtering of Equation 15 can be specialized to

$$(\hat{l}_0)_{i,j} = \sum_q \sum_r h_{qr}\, u_{i-q,j-r} \qquad (26)$$

reading in the separable case as

$$\begin{aligned} h_{qr} &= (h_1)_q\, (h_2)_r \\ (\hat{l}_0)_{i,j} &= \sum_q (h_1)_q \left(\sum_r (h_2)_r\, u_{i,j-r} \right)_{i-q} \\ &= \sum_r (h_2)_r \left(\sum_q (h_1)_q\, u_{i-q,j} \right)_{j-r}. \end{aligned} \qquad (27)$$

In the "filter transform domain" Equation 26 is expressed as

$$\hat{\mathbf{L}}_0 = \mathbf{S}_1^{\mathrm{T}} (\mathbf{H} * \mathbf{S}_1 \mathbf{U} \mathbf{S}_2^{\mathrm{T}})\, \mathbf{S}_2. \qquad (28)$$

To illustrate the idea of the filter design in Equation 28, assume a perimeter controlled grid of a leveling net of Figure 1 measured along all horizontal and vertical traverses.

The observation equations can be expressed as a specialized array algebra Pranish-Praniewich system of Rauhala (1979) by

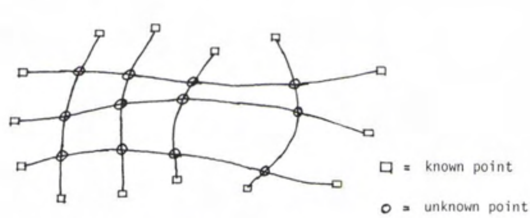

FIG. 1. A perimeter controlled grid of a leveling net measured along all horizontal and vertical traverses.

$$\underset{p_1+1,p_1}{\mathbf{A}_1}\ \underset{p_1p_2}{\mathbf{L}_0} = \underset{p_1+1,p_2}{\mathbf{L}_1} - \mathbf{V}_1$$

$$\underset{p_1p_2}{\mathbf{L}_0}\ \underset{p_2,p_2+1}{\mathbf{A}_2^{\mathrm{T}}} = \underset{p_1,p_2+1}{\mathbf{L}_2} - \mathbf{V}_2$$

$$\sqrt{q}\ \mathbf{L}_0 = \sqrt{q(\mathbf{L}_0^0 - \mathbf{V}_0)} \tag{29}$$

$$\underset{\substack{p_1=3\\p_2=4}}{\mathbf{A}} = \begin{bmatrix} 1 & & 0 \\ -1 & 1 & \\ & -1 & 1 \\ 0 & & \ddots \end{bmatrix}$$

yielding the normal equations

$$q\mathbf{L}_0 + (2\mathbf{I} + \mathbf{B}_1)\,\mathbf{L}_0 + \mathbf{L}_0\,(2\mathbf{I} + \mathbf{B}_2) = \mathbf{U} \tag{30}$$

$$\mathbf{U} = \mathbf{A}_1^{\mathrm{T}}\mathbf{L}_1 + \mathbf{L}_2\mathbf{A}_2^{\mathrm{T}} + q\mathbf{L}_0^0,\ \mathbf{B} = \begin{bmatrix} 0 & -1 & & 0 \\ -1 & 0 & -1 & \\ & -1 & 0 & -1 \\ 0 & & -1 & \ddots \end{bmatrix}$$

Here $\mathbf{L}_0^0$ denotes a priori estimates of the unknowns all associated with the homogeneous a priori weight q.

Singular value decomposition of $\mathbf{B}_1 = \mathbf{S}_1^{\mathrm{T}}\boldsymbol{\lambda}_1\mathbf{S}_1$, $\mathbf{B}_2 = \mathbf{S}_2^{\mathrm{T}}\boldsymbol{\lambda}_2\mathbf{S}_2$ results in the solution, Equation 28 (Rauhala, 1980b),

$$\hat{\mathbf{Z}} = \mathbf{H} * \mathbf{S}_1\mathbf{U}\mathbf{S}_2^{\mathrm{T}},\ \hat{\mathbf{L}}_0 = \mathbf{S}_1^{\mathrm{T}}\hat{\mathbf{Z}}\mathbf{S}_2$$

where $\mathbf{S}_1$, $\mathbf{S}_2$ represent the FET transforms

$$\underset{p_1p_1}{\mathbf{S}_1} = \{s_1 i,j\} = (-1)^i \sin\left[ij\pi/(p_1+1)\right] \div \sqrt{(p_1+1)/2}$$

$$\underset{p_2p_2}{\mathbf{S}_2} = \{s_2 i,j\} = (-1)^i \sin\left[ij\pi/(p_2+1)\right] \div \sqrt{(p_2+1)/2}$$

$$(\mathbf{H}*)_{i,j} = (q + 2 + (\lambda_1)_{i,i} + 2 + (\lambda_2)_{j,j})^{-1}. \tag{31}$$

The beauty of the solution (28), (31) resides in the fact that the derivation of the transfer function (31) actually "solves the problem" by FET domain filtering of the right hand side array $\mathbf{U}$. In the vectorized formulation (28) translates into

$$\underset{p,1}{\hat{\mathbf{L}}_0} = \underset{pp}{\mathbf{N}^{-1}}\ \underset{p,1}{\mathbf{U}},\ p = p_1p_2,$$

$$\underset{pp}{\mathbf{N}^{-1}} = \underset{pp}{(\mathbf{S}_1 \otimes \mathbf{S}_2)^{\mathrm{T}}}\,\mathrm{diag}\,\underset{pp}{(\mathbf{H}*)}\,\underset{pp}{(\mathbf{S}_1 \otimes \mathbf{S}_2)}\,. \tag{32}$$

The inverse $\mathbf{N}^{-1}$ of the normal equation matrix has three common interpretations: (1) It represents the variance-covariance matrix of $\underset{p,1}{\hat{\mathbf{L}}_0}$, (2) the i^{th} row of $\mathbf{N}^{-1}$ is the covariance function of the i^{th} element of $\hat{\mathbf{L}}_0$ (Hirvonen, 1956), known as (3), the impulse response in the terminology of digital signal processing (IEEE, 1975; Mitra and Ekstrom, 1978).

The impulse response $\underset{p_1p_2}{\mathbf{H}}{}_{i,j}$ of element $(\hat{\mathbf{L}}_0)_{i,j}$ can be extracted from $\mathbf{N}^{-1}$ by

$$\underset{p_1p_2,1}{\mathbf{H}_{i,j}} = (\mathbf{S}_1 \otimes \mathbf{S}_2)^{\mathrm{T}}\,\mathrm{diag}\,(\mathbf{H}*)\,(\mathbf{S}_1 \otimes \mathbf{S}_2)\,(\mathbf{e}_i^{\mathrm{T}} \otimes \mathbf{e}_j^{\mathrm{T}})$$

$$\underset{p_1p_2}{\mathbf{H}_{i,j}} = \mathbf{S}_1^{\mathrm{T}}\,(\mathbf{H} * \mathbf{S}_1\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_j\mathbf{S}_2^{\mathrm{T}})\mathbf{S}_2$$

$$\begin{aligned} \mathbf{e}_i &= \Big[\overbrace{0, 0, \ldots 0, 1}^{i}, 0 \ldots 0, 0\Big] \\ \mathbf{e}_j &= \Big[\underbrace{0, 0, \ldots 0, 1}_{j}, 0, \ldots 0, 0\Big] \end{aligned} \tag{33}$$

where $\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_j$ contains the unit pulse in location i,j of the input space and $\mathbf{H}_{i,j}$ is the response in the output space. The transfer function $\mathbf{H}*$ determines the amount and type of filtering of the system by acting upon the "filter transform" of the input space through the simple elementwise multiplications.

The special case of LSI filtering is found in this example of leveling net adjustment as the size p_1,p_2 of $\underset{p_1p_2}{\mathbf{L}_0}$ becomes large and as $q >> 0$. With the exception of narrow strips around the boundary points, the impulse responses of the remaining center points become (a) finite, i.e., only a finite number of elements in $\underset{p_1p_2}{\mathbf{H}}{}_{i,j}$ around the element i,j are significantly different from zero, and (b) these coefficients remain the same for all center points i,j. In the example the impulse response is also octavially symmetric. In this case $\mathbf{H}$ is identical to that of finite element filtering with first differences as the continuity constraints (Rauhala, 1980, p. 63). Notice that this case of $q >> 0$ corresponds to the shift invariant error propagation of a rigorous photogrammetric bundle adjustment while the weaker LSV case $q \rightarrow 0$ reflects the error propagation of crossing traversing or leveling closely related to the error model of analog photogrammetry of Meissl (1971).

The solutions of Equation 26 by convolution and Equation 28 through the transform domain are exactly equivalent. Computationally, the convolution approach is usually most advantageous for implementation of the FIR filtering of large arrays using disk oriented minicomputers if the size of the input array exceeds the core capacity. The computation of the impulse response after an analytical derivation of the transfer function has several analogies in the simulation techniques of mathematical surveying sciences. An early example is the "Bolz arrays," consisting of ready tabulated LSV inverse matrices for hand calculation of net adjustments before the computers were available. Also, several methods in digital terrain modeling utilize one-time simulated square or rectangular inverse matrices.

The implementation of separable and nonseparable convolution is extensively studied in the literature of signal processing and array algebra. Some new strategies of excellent computational efficiency are reported by Rauhala (1979, 1980b). One of their leading ideas can be characterized as "filtering by data snooping," which combines the techniques of FIR (Finite Impulse Response) and IIR (Infinite Impulse Response) in a simple manner; the small "normal equation residual" $\hat{\mathbf{V}}_u = \mathbf{L}_0 - \mathbf{U}$ is convolved from $(\hat{d}_{q,r})_{i,j} = U_{q+i,r+j} - U_{i,j}$ using the impulse response $\mathbf{H}_{\hat{v}_u} = \mathbf{e}_i^{\mathrm{T}}\mathbf{e}_j - \mathbf{H}$, drastically reducing the summation limits of indices q,r in the convolution. In some experiments of digital terrain models a 5 by 5 impulse response sufficed for $\hat{\mathbf{V}}_u$ in contrast to the 15 by 15 width for the direct convolution of $\hat{\mathbf{L}}_0$. In the new approach the same $\hat{\mathbf{L}}_0$ is found by adding the output $\hat{\mathbf{V}}_u$ to the original input $\mathbf{U}$. The name data snooping comes from the fact that typically the impulse response $\mathbf{H}_{\hat{v}_u}$ of $\mathbf{L}_0 \in \mathbf{L}$ fully describes the variance-covariance matrix of least-squares residuals used in the control theory of data snooping for extracting blunderous data in a net adjustment (Baarda, 1973).

Applications

Implementation of LSI array filtering using a new computational algorithm of "rotating addressing" made the octavially symmetric non-separable filtering even more efficient than the separable one. In finite element filtering of digital terrain modeling 720,000 nodes were convolved in a CPU time of 58.5 seconds with 22 K bytes allocated for the data in the minicomputer SEL 32/55. Over 200 million nodes were solved to experimentally find out the best math models for processing volumous data arrays at the Defense Mapping Agency Aerospace Center. Also, a new complete family of theories for splitting non-separable modeling into a sequence of separable ones in any dimension was found by Rauhala (1980b) by extending the singular value decomposition technique of Treitel and Shanks (1971). In Rauhala (1978) this splitting technique has been applied for the solution of the boundary value problem in physical geodesy.

To extend the practical applicability of the very fast LSI array filtering, a new theory has been under development aiming at solving any computational problem of properly redefined linear algebra using a new linear solution algorithm. The LSI filtering is an integral part of this futuristic algorithm (Rauhala, 1980b).

The above example of a leveling net has rather straight-forward extensions to the "fast" solutions of self-calibrating inertial networks, readjustment of huge continental or national networks, self-calibrating Global Positioning System, and other fundamental problems of digital surveying sciences. Probably the most exciting avenue opened by array algebra is a realistic computational solution of the fundamental problem of digital photogrammetry.

The ultimate problem of digital photogrammetry consists in converting the overlap areas of several digitized images into a high-resolution orthophoto and digital terrain model. With a pixel size of 10 to 15 μm, a frame may contain 175 to 400 million pixels with 1 to 2 bytes in each pixel. The first task is to rectify the images to account for the effect of interior and exterior orientation. The second stage consists of rigorous "precision" array correlations of every single pixel or blocks of pixels. The powerful concept of array correlation automatically incorporates several pre- and post-processing tasks of the conventional correlation methods (Rauhala, 1977, 1980a,b). The third stage consists of orthorectification and some other futuristic possibilities. To perform all these tasks involving billions of arithmetical operations in a competitive time frame using off-the-shelf hardware costing approximately the same as an analytical plotter is a reasonable challenge to the array algebra software and hardware technology. In this paper this new technology was interpreted as an advanced form of digital signal processing and modern computer sciences.

References

Baarda, W., 1973. S-Transformation and Criterion Matrices, *Netherlands Geodetic Commission,* Vol. 5, No. 1.

Bjerhammar, A., 1955. En ny matris algebra. *Svensk Lantmäteritidskrift,* 5/6, Stockholm.

Bossler, J. D., 1973. A Note on the Meaning of Generalized Inverse Solutions in Geodesy, *Journal of Geophysical Research* 78:14.

Brown, D. C., 1955. *A Matrix Treatment of the General Problem of Least Squares Considering Correlated Observations,* Ballistic Research Laboratories Report No. 937, Aberdeen, Maryland.

———, 1958. *A Solution to the General Problem of Multiple Station Analytical Stereotriangulation,* AF Missile Test Center Technical Report No. 58-8.

———, 1974. Evolution, Application and Potential of the Bundle Method of Photogrammetric Triangulation. Invited Paper of ISP Symposium, Comm. III, Stuttgart.

Brown, D. C., N. Bush, and J. Sibol, 1963. *Study of the Feasibility of Rocket and Satellite Approaches to the Calibration of Tracking Systems,* AF Cambridge Research Labs, Report No. 63-789.

Brown, D. C., and J. E. Trotter, 1969. *SAGA, A Computer Program for Short Arc Geodetic Adjustment of Satellite Observations,* AFCRL Final Report of Contract F19628-68-C-0093.

Cooley, J., and J. Tukey, 1965. An Algorithm for Machine Computation of Complex Fourier Series, *Math. Comput.* Vol. 19.

DeMasson d'Autume, G., 1976. Interpolation by a Flexible Rod (Spline): Applications to Digital Photogrammetry. Presented Paper, Comm. III, the 13th ISP Congress, Helsinki.

———, 1978. Surface Modeling by Means of an Elastic Grid. Presented paper of ISP Comm. III Symposium, Moscow and Photogrammetria pp. 65-74, 1979.

Ebner, H., and P. Reiss, 1978. Height Interpolation by the Method of Finite Elements, *Proceedings of Digital Terrain Models (DTM) Symposium*, May 9-11, St. Louis. American Society of Photogrammetry, Falls Church, VA.

Good, I. J., 1958. The Interaction Algorithm and Practical Fourier Series, *J. Royal Stat. Soc.*, Ser. B, 20.

Grafarend, E., and B. Schaffrin, 1974. Unbiased Free Net Adjustment, *Survey Review*, XXII, 171.

Greville, T. N. E., 1961. Note on Fitting of Functions of Several Independent Variables, *J. Soc. Indust. Appl. Math.* Vol. 9, No. 1.

Helstrom, C. W., 1967. Image Restoration by the Method of Least Squares, *Opt. Soc. Amer. J.* 57(3).

Hirvonen, R. A., 1956. On the Precision of the Gravimetric Determination of the Geoid, *Trans. Amer. Geophysical Union* 37, 1-8.

IEEE Digital Signal Processing Committee, ASSPS, 1975. *Selected Papers in Digital Signal Processing II*, IEEE Press, N.Y.

Junkins, J. L., 1976. Investigation of Finite-Element Representations of the Geopotential, *AIAA Journal*, Vol. 14, No. 6.

Kratky, V., 1978. DTM Interpolation with Gliding Vectors, *Proceedings of Digital Terrain Models (DTM) Symposium*, May 9-11, St. Louis, American Society of Photogrammetry, Falls Church, VA.

Kubik, K., 1971. The Application of Piecewise Polynomials to Problems of Curve and Surface Approximation. *Rijkswater-Staat Communications 12*, The Hague.

Meissl, P., 1971. Theoretical Analysis of Random Error Propagation in Large Photogrammetric Blocks. *Proceedings of ASP Fall Meeting*, pp. 751-770, San Francisco, Sept. 7-11.

———, 1980. *A Priori Prediction of Roundoff Error Accumulation in the Solution of a Super-Large Geodetic Normal Equation System*, NOAA Professional Paper 12.

Mitra, S. K., and M. P. Ekstrom (Editors), 1978. Two-Dimensional Digital Signal Processing. *Benchmark Papers in Electrical Engineering and Computer Science*, Vol. 20, Dowden, Hutchinson and Ross, Inc., PA.

Moritz, H., 1978. Least-Squares Collocation. *Reviews of Geophysics and Space Physics*, Vol. 16, No. 3.

Pope, A. J., 1974. Modern Trends in Adjustment Calculus. Presented paper in Symposium of North American Geodetic Networks, New Brunswick.

Pratt, W. K., 1972. Generalized Wiener Filtering Computation Techniques, *IEEE Trans. Computers* C-21(7).

Rao, C. R., and S. K. Mitra, 1971. *Generalized Inverse of Matrices and Its Applications*, Wiley, N.Y.

Rauhala, U. A., 1972. Calculus of Matrix Arrays and General Polynomial and Harmonic Interpolation by Least Squares with New Solutions in Photogrammetry and Geodesy, *Fot. Medd.* VI: 4, Department of Photogrammetry, Royal Institute of Technology, Stockholm.

———, 1974. Array Algebra with Applications in Photogrammetry and Geodesy, *Fot. Medd* VI:6, KTH, Stockholm.

———, 1976. A Review of Array Algebra, *Fot. Medd.* 2:38, KTH, Stockholm.

———, 1977. Array Algebra as General Base of Fast Transforms, *Mitteilungen der Geodätischen Institute Der Technischen Universität Graz*, Folge 29, Graz.

———, 1978. Loop Inverses and Array Algebra as Geodetic Tools. Presented Paper of AGU Fall Meeting, San Francisco, December 4-8. Appendix II of Rauhala (1979).

———, 1979. *Development of Experimental Array Algebra Algorithms for Filtering and Compaction of AS-11B-X and Seasat Altimetry Data*, Final report to DMAAC, Contract 700-78-C-0022.

———, 1980a. Introduction to Array Algebra, *Phot. Engineering and Remote Sensing*, Vol. 46, No. 2.

———, 1980b. *Development of Array Algebra Algorithms for Finite Element Filtering*, Final Report to DMAAC, Contract 700-78-C-0022 P 00002.

———, 1981a. Sequential Array Algebra Using Direct Solution of Eigenvectors, *Bull. Geod.* No. 2.

———, 1981b. Note on General Linear Estimators and Matrix Inverses. *Manuscripta Geodaetica*, Vol. 6, pp. 375-386.

Strang van Hees, G. L., 1981. Collocation and Adjustment, *Zeitschrift für Vermessungswesen* 5.

Treitel, S., and J. L. Shanks, 1971. The Design of Multistage Separable Planar Filters, *IEEE Trans. Geosci. Electron*, Vol. GE-9.

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