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The Model Construction Problem Using the Collinearity Condition

The construction problem of a stereo model congruent to the object is discussed algebraically using the collinearity condition between a model point and its measured image coordinates.

Abstract: *The construction of a stereo model congruent to the object plays a significant role in close-range photogrammetry, because absolute positioning with respect to a reference coordinate system is not a matter of major importance. Thus, the orientation problem of overlapped photographs taken with non-metric cameras is discussed for the case where the object space control is established only in terms of distances. The orientation calculation is formulated by using the collinearity condition between a model point and the corresponding measured image coordinates, because the algebraic treatment of photographs is much easier than the conventional geometric approach by means of the coplanarity and similarity conditions. Non-central projective parameters defining the non-linear part of the disturbing feature of non-metric cameras are considered separately so as to make the discussions simple.*

Introduction

Object space control is usually established in terms of coordinates of control points. However, other geodetic measurement such as angles and distances could also be used as a part of object space control (Ghosh, 1962; Wong, 1972, 1975; Kenefick *et al.*, 1978; El-Hakim and Faig, 1981; Wester-Ebbinghaus, 1981; and others). In particular, establishing object space control in terms of distances has a great practical appeal (Atkinson, 1980), because a distance is invariant regardless of the selected coordinate system and its measurement is comparatively easily performed.

In many close-range applications, absolute positioning with respect to a reference coordinate system is not a matter of major importance. Of greater importance is the relative positioning accuracy of points on the surface to be mapped. Also, this purpose is achieved by constructing a stereo model congruent to the object using only distances as the object space control (Okamoto, 1981a, 1982a). In this paper, the construction problem of a stereo model congruent to the object is discussed not only for the general case where a photograph has 11 independent central projective parameters but also for the usual case in close-range photogrammetry, where the geometry of a picture is determined with nine independent elements (the three interior and six exterior orientation parameters). Furthermore, the collinearity condition relating a model point and the corresponding measured image coordinates is employed as the fundamental equations, because the algebraic treatment of photographs (direct linear transformation (DLT) approach (Abdel-Aziz and Karara, 1971; Marzan and Karara, 1975; Okamoto, 1982b)) is much easier than the geometric approach by means of the coplanarity and similarity conditions (Okamoto, 1981a, 1982a). Non-central projective parameters defining non-linear lens distortion and non-linear film deformation are ignored in the first discussion, because we can grasp the mathematical meanings of the DLT approach to the model construction problem more clearly without these

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elements and also because they can be determined only from the coplanarity condition of corresponding rays.

Model Construction Problem Using the Collinearity Condition for the General Case

In this section, the general construction problem of a stereo model congruent to the object is discussed not only for the case of a stereopair of photographs but also for that of three pictures overlapped. The collinearity equations between a model point and its measured image point are expressed in algebraic form (in the DLT form) and the object space control is established only in terms of distances. First, constraints between the coefficients of the collinearity equations are explored and then the orientation calculation is described treating space coordinates of model points as unknowns.

Necessary Constraints Among DLT Coefficients

In metric photogrammetry, a stereo model similar to the object is constructed by means of five relative exterior orientation parameters ($\phi_1,\kappa_1,\phi_2,\omega_2,\kappa_2$ (rotation elements only)) or ($\phi_2,\omega_2,\kappa_2,BY_2,BZ_2$ (five exterior orientation parameters of the right photograph)) of a stereopair of pictures. A stereo model congruent to the object can be formed by determining the model scale in addition to the five relative orientation elements. This means that six exterior orientation parameters ($\phi_2,\omega_2,\kappa_2,BX_2,BY_2,BZ_2$) of the right picture can also be selected for this purpose under the assumption that those of the left photograph are equal to zero.

In non-metric photogrammetry, however, we must further determine all "independent" interior orientation parameters of a stereopair of pictures for the construction of a stereo model congruent to the object. By designating measured image coordinates as (x_{c1},y_{c1}) and (x_{c2},y_{c2}) for the left and right photographs, respectively, and the stereo model space as (X_M,Y_M,Z_M), the collinearity condition relating a model point $P_M(X_M,Y_M,Z_M)$ and its measured image point $p_c(x_c,y_c)$ will be discussed for the left and right pictures, respectively, as follows (see Figure 1).

Regarding the right photograph, the general col-

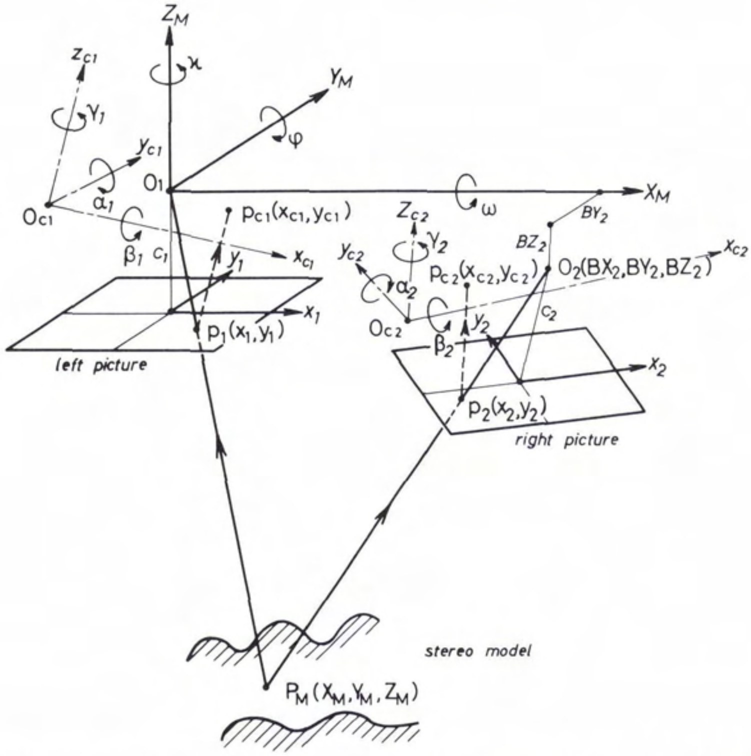

FIG. 1. The collinearity condition between a model point and the corresponding measured image coordinates for the left and right pictures. The projection center of the left photograph is given as (0,0,0) in the model space coordinate system (X_M,Y_M,Z_M) and as $(x_{co1},y_{co1},z_{co1})$ in the left comparator coordinate system (x_{c1},y_{c1},z_{c1}). That of the right photograph has the space coordinates (BX_2,BY_2,BZ_2) in the model space coordinate system and those $(x_{co2},y_{co2},z_{co2})$ in the right comparator coordinate system (x_{c2},y_{c2},z_{c2}). The relationship between the picture plane and the comparator coordinate system considered here is fictitious, because two rotation elements defining this relationship are functions of all parameters describing linear systematic errors of photo coordinates.

linearity equations are valid between $P_M(X_M, Y_M, Z_M)$ and $p_{c2}(x_{c2}, y_{c2})$ because the right picture has more than 11 unknown parameters (six exterior orientation elements $(\phi_2, \omega_2, \kappa_2, BX_2, BY_2, BZ_2)$, three interior orientation parameters (x_{H2}, y_{H2}, c_2), those defining linear film deformation and linear systematic measurement errors, and so on), eleven of which are central-projectively independent. However, the relationship between a model point $P_M(X_M, Y_M, Z_M)$ and its measured image point $p_{c1}(x_{c1}, y_{c1})$ for the left picture does not yield the general collinearity equations, which means that the left picture has less than 11 independent central projective elements and thus constraints are generated between the 11 coefficients, if the collinearity condition is described in the DLT form. These constraints will be found in the following way. Without loss of generality, the general collinearity condition can be described in terms of 11 photogrammetric parameters of a picture (Faig, 1975; Moniwa, 1976, 1981; Okamoto, 1981a, 1982a). Relating this fact to the collinearity equations for the left photograph, we can see that they are expressed only in terms of five central projective parameters (the planimetric coordinates (x_{co1}, y_{co1}) of the projection center referred to the comparator coordinate system (x_{c1}, y_{c1}, z_{c1}), the principal distance, c_1, and the two rotation parameters α_1 and β_1 about the comparator coordinate axes y_{c1} and x_{c1}, respectively), because the six exterior orientation elements are taken to zero. Therefore, we have six constraints between the 11 coefficients of the collinearity equations in algebraic form. Next, this problem will be discussed more precisely. The subscript 1, indicating the left picture, is omitted for simplicity.

The relationship relating a picture point (ideal) $p(x,y)$ and its measured image point $p_c(x_c, y_c)$ is described by an orthogonal transformation of the form

$$\begin{aligned} x_c - x_{co} &= d_{11}x + d_{12}y - d_{13}c \\ y_c - y_{co} &= d_{21}x + d_{22}y - d_{23}c \end{aligned} \tag{1}$$

in which the coefficients $d_{ij}(i = 1, 2; j = 1, 2, 3)$ are functions of the rotation parameters α and β about the comparator coordinates axes y_c and x_c as follows:

$$\mathbf{D}_\alpha \mathbf{D}_\beta = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha\sin\beta & \sin\alpha\cos\beta \\ 0 & \cos\beta & -\sin\beta \\ -\sin\alpha & \cos\alpha\sin\beta & \cos\alpha\cos\beta \end{pmatrix}. \tag{2}$$

Equation 1 can also be rewritten as

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & a_3 \\ d_{21} & d_{22} & a_6 \end{pmatrix} \begin{pmatrix} x \\ y \\ -c \end{pmatrix} \tag{3}$$

in which

$$a_3 = d_{13} - \frac{x_{co}}{c}, \; a_6 = d_{23} - \frac{y_{co}}{c}.$$

On the other hand, the relationship between a model point $P_M(X_M, Y_M, Z_M)$ and the picture point $p(x,y)$ is given by

$$\begin{pmatrix} x \\ y \\ -c \end{pmatrix} = -c \begin{pmatrix} \dfrac{X_M}{Z_M} \\ \dfrac{Y_M}{Z_M} \\ 1 \end{pmatrix} \tag{4}$$

because the six exterior orientation parameters of the left picture are taken to be zero. Then, we substitute Equation 4 into Equation 3 to obtain

$$\begin{aligned} x_c &= \frac{-cd_{11}X_M - cd_{12}Y_M - ca_3Z_M}{Z_M} \\ &= \frac{-cd_{11}X_M - cd_{12}Y_M - ca_3Z'_M - ca_3}{Z'_M + 1} \\ y_c &= \frac{-cd_{21}X_M - cd_{22}Y_M - ca_6Z_M}{Z_M} \\ &= \frac{-cd_{21}X_M - cd_{22}Y_M - ca_6Z'_M - ca_6}{Z'_M + 1} \end{aligned} \tag{5}$$

in which

$$Z'_M = Z_M - 1.$$

Describing Equation 5 in the DLT form

$$\begin{aligned} x_c &= \frac{B_1X_M + B_2Y_M + B_3Z'_M + B_4}{B_9X_M + B_{10}Y_M + B_{11}Z'_M + 1} \\ y_c &= \frac{B_5X_M + B_6Y_M + B_7Z'_M + B_8}{B_9X_M + B_{10}Y_M + B_{11}Z'_M + 1} \end{aligned} \tag{6}$$

the following six constraints are generated between the 11 coefficients $B_i(i = 1, \ldots, 11)$, i.e.,

$$\begin{aligned} B_5 &= B_9 = B_{10} = 0, \\ B_{11} &= 1, \\ B_3 &= B_4, \; B_7 = B_8 \end{aligned} \tag{7}$$

in which $B_5 = 0$ is obtained from the fact that one matrix element d_{21} in Equation 2 equals zero.

FOR THE CASE OF A STEREOPAIR OF PHOTOGRAPHS

The collinearity equations between a model point $P_M(X_M, Y_M, Z_M)$ and the measured image coordinates (x_{c1}, y_{c1}) for the left picture can be simplified by substituting Equation 7 into Equation 6 as

$$\begin{aligned} x_{c1} &= \frac{{}_1A_1X_M + {}_1A_2Y_M + {}_1A_3Z_M}{Z_M} \\ y_{c1} &= \frac{{}_1A_6Y_M + {}_1A_7Z_M}{Z_M} \end{aligned} \tag{8}$$

Also, the equations for the right photograph are in the same form of the general collinearity equations: i.e.,

$$x_{c2} = \frac{{}_2A_1X_M + {}_2A_2Y_M + {}_2A_3Z_M + {}_2A_4}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}$$
$$y_{c2} = \frac{{}_2A_5X_M + {}_2A_6Y_M + {}_2A_7Z_M + {}_2A_8}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}. \quad (9)$$

Equations 8 and 9 yield

$$x_{c1} = \frac{{}_1A_1X_M + {}_1A_2Y_M + {}_1A_3Z_M}{Z_M}$$
$$y_{c1} = \frac{{}_1A_6Y_M + {}_1A_7Z_M}{Z_M}$$
$$x_{c2} = \frac{{}_2A_1X_M + {}_2A_2Y_M + {}_2A_3Z_M + {}_2A_4}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}$$
$$y_{c2} = \frac{{}_2A_5X_M + {}_2A_6Y_M + {}_2A_7Z_M + {}_2A_8}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}. \quad (10)$$

Mathematically, Equation 10 includes one equation equivalent to the coplanarity condition of corresponding rays. Also, the coplanarity condition provides seven independent orientation parameters (Okamoto, 1981a). Thus, seven unknown coefficients among the 16 independent ones of Equation 10 can be uniquely determined by setting up Equation 10 for seven model points. However, a stereo model formed only by means of the coplanarity condition of corresponding rays is not similar to the object. In order to make the stereo model congruent to the object, the following expression,

$$L_{rs} = \sqrt{(X_{Mr} - X_{Ms})^2 + (Y_{Mr} - Y_{Ms})^2 + (Z_{Mr} - Z_{Ms})^2}, \quad (11)$$

must be valid for nine distances given as the object space control (Okamoto, 1981a). In the above expression, L_{rs} denotes a distance from point r to point s in the object space, and the right hand side is the corresponding length expressed in terms of the model space coordinate system.

The 16 unknown coefficients of Equation 10 may be determined as follows. We set up Equation 10 for 18 model points at both ends of nine lengths given as the object space control, and Equation 11 for the nine distances. Also, space coordinates of the 18 model points are treated as unknowns. Then, we obtain 81 equations with respect to 70 unknowns (the 16 coefficients plus 54 coordinates of the 18 model points). This is an overdetermined system which is caused by the fact that we have used 18 coplanarity equations, although the coplanarity condition for the general case provides only seven independent orientation elements. However, a least-squares solution can overcome this problem.

FOR THE CASE OF THREE PHOTOGRAPHS OVERLAPPED

We will assume that three pictures were taken with an overlap. According to Okamoto (1981b), a united stereo model can be constructed with three such photographs. Also, the united stereo model can be made congruent to the object by using nine distances as control. This procedure will be outlined using the collinearity equations as the determination equations as follows.

Equation 10 is valid for points photographed in common on the first and second pictures. The same can be constructed for points in the overlapped part between the second and third photographs; i.e.,

$$x_{c2} = \frac{{}_2A_1X_M + {}_2A_2Y_M + {}_2A_3Z_M + {}_2A_4}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}$$
$$y_{c2} = \frac{{}_2A_5X_M + {}_2A_6Y_M + {}_2A_7Z_M + {}_2A_8}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}$$
$$x_{c3} = \frac{{}_3A_1X_M + {}_3A_2Y_M + {}_3A_3Z_M + {}_3A_4}{{}_3A_9X_M + {}_3A_{10}Y_M + {}_3A_{11}Z_M + 1}$$
$$y_{c3} = \frac{{}_3A_5X_M + {}_3A_6Y_M + {}_3A_7Z_M + {}_3A_8}{{}_3A_9X_M + {}_3A_{10}Y_M + {}_3A_{11}Z_M + 1} \quad (12)$$

Furthermore, the collinearity equations for the three pictures are written down together as

$$x_{c1} = \frac{{}_1A_1X_M + {}_1A_2Y_M + {}_1A_3Z_M}{Z_M}$$
$$y_{c1} = \frac{{}_1A_6Y_M + {}_1A_7Z_M}{Z_M}$$
$$x_{c2} = \frac{{}_2A_1X_M + {}_2A_2Y_M + {}_2A_3Z_M + {}_2A_4}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}$$
$$y_{c2} = \frac{{}_2A_5X_M + {}_2A_6Y_M + {}_2A_7Z_M + {}_2A_8}{{}_2A_9X_M + {}_2A_{10}Y_M + {}_2A_{11}Z_M + 1}$$
$$x_{c3} = \frac{{}_3A_1X_M + {}_3A_2Y_M + {}_3A_3Z_M + {}_3A_4}{{}_3A_9X_M + {}_3A_{10}Y_M + {}_3A_{11}Z_M + 1}$$
$$y_{c3} = \frac{{}_3A_5X_M + {}_3A_6Y_M + {}_3A_7Z_M + {}_3A_8}{{}_3A_9X_M + {}_3A_{10}Y_M + {}_3A_{11}Z_M + 1} \quad (13)$$

which is valid for points taken on the three photographs at the same time. Equation 13 containts one equation which corresponds to the coplanarity condition for the first and second pictures, one equation equivalent to that for the second and third photographs, and one equation for making the second stereo model coincide with the first one. Also, the coplanarity condition provides seven independent orientation elements, and the transformation of the second stereo model space into the first one determines four independent orientation parameters (Okamoto, 1981b).

Mathematically, Equation 10 is valid for three points in the first stereo model space, Equation 12 for three points in the second one, and Equation 13 for four points in the space which belongs to both model spaces in common. Also, we can apply Equation 11 for nine distances given as the object space control. However, in order to employ directly Equation 10 to 13 as the determination equations, we may set up Equation 10, 12, and 13 for 18 model points at both ends of the nine lengths. Accordingly, we obtain more equations than those required for the unique determination of the unknowns, namely, the 27 coefficients of the collinearity equations for the three photographs and the 54 coordinates of the 18 end points. Consequently, the orientation calculation must be carried out by using a least-squares adjustment, even when we have the minimum number of control distances.

THE USUAL CASE

NECESSARY CONSTRAINTS AMONG DLT COEFFICIENTS

In the usual case in close-range photogrammetry, where the geometry of a picture can be determined by nine independent central projective parameters (the six exterior and three interior orientation elements), a stereo model congruent to the object can be constructed by determining six exterior orientation elements ($\phi_2, \omega_2, \kappa_2, BX_2, BY_2, BZ_2$) of the right picture of a stereopair of photographs and six interior orientation parameters ($x_{H1}, y_{H1}, c_1, x_{H2}, y_{H2}, c_2$) of the two photographs. In order to solve this problem by means of the collinearity condition expressed in the DLT form, the following discussions must be preceded (see Figure 2).

With regard to the left picture, the collinearity equations relating a model point $P_M(X_M, Y_M, Z_M)$ and the corresponding measured image point $p_{c1}(x_{c1}, y_{c1})$ are described as

$$\begin{aligned} x_c = x_H - c\frac{X_M}{Z_M} &= \frac{-cX_M + x_H Z_M}{Z_M} \\ &= \frac{-cX_M + x_H Z'_M + x_H}{Z'_M + 1} \\ y_c = y_H - c\frac{Y_M}{Z_M} &= \frac{-cY_M + y_H Z_M}{Z_M} \\ &= \frac{-cY_M + y_H Z'_M + y_H}{Z'_M + 1} \end{aligned} \qquad (14)$$

in which $Z'_M = Z_M - 1$, and the subscript 1, indicating the left picture, is omitted for simplicity. Expressing Equation 14 in the DLT form (Equation 6), we have two more constraints in addition to the six (Equation 7) in the previous section, i.e.,

$$B_1 = B_6, \text{ and } B_2 = 0. \qquad (15)$$

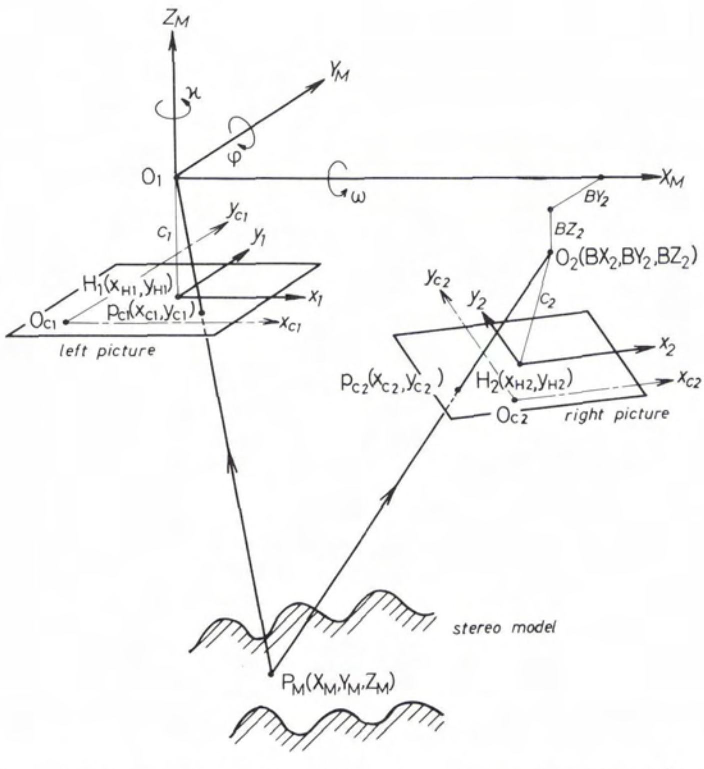

FIG. 2. The collinearity condition between a model point and the corresponding measured image coordinates for the left and right pictures under the assumption that we have no linear systematic errors of photo coordinates.

On the other hand, the right photograph has nine independent orientation parameters ($\phi_2, \omega_2, \kappa_2, BX_2, BY_2, BZ_2, x_{H2}, y_{H2}, c_2$) to be determined. Also, the necessary constraints among the 11 coefficients of the collinearity equations are given by, e.g., Bopp and Krauss (1978) and by others.

FOR THE CASE OF A STEREOPAIR OF PHOTOGRAPHS

The collinearity condition between a model point and its measured image point for the left picture can be described in the form of Equation 8, if we rewrite the two constraints (Equation 15) as follows:

$$ {}_1A_1 = {}_1A_6, \text{ and } {}_1A_2 = 0 \qquad (16)$$

Also, that for the right photograph has the same form of Equation 9 under the constraints between the 11 DLT coefficients: i.e.,

$$\begin{aligned} &{}_2N_{ab} - {}_2N_{ca} \cdot {}_2N_{bc} \cdot {}_2D_c^{-1} = 0, \\ &{}_2D_a - {}_2D_b - ({}_2N_{ca}^2 - {}_2N_{bc}^2){}_2D_c^{-1} = 0 \end{aligned} \qquad (17)$$

in which

$$\begin{aligned} {}_2D_a &= {}_2A_1^2 + {}_2A_2^2 + {}_2A_3^2,\ {}_2D_b = {}_2A_5^2 + {}_2A_6^2 + {}_2A_7^2, \\ {}_2D_c &= {}_2A_9^2 + {}_2A_{10}^2 + {}_2A_{11}^2, \\ {}_2N_{ab} &= {}_2A_1 \cdot {}_2A_5 + {}_2A_2 \cdot {}_2A_6 + {}_2A_3 \cdot {}_2A_7 \\ {}_2N_{bc} &= {}_2A_5 \cdot {}_2A_9 + {}_2A_6 \cdot {}_2A_{10} + {}_2A_7 \cdot {}_2A_{11} \\ {}_2N_{ca} &= {}_2A_9 \cdot {}_2A_1 + {}_2A_{10} \cdot {}_2A_2 + {}_2A_{11} \cdot {}_2A_3. \end{aligned}$$

The orientation calculation to employ the collinearity equations (Equation 10) as the determination equations will be outlined as follows.

The coplanarity condition of corresponding rays mathematically determines six independent orientation parameters. Also, for the construction of a stereo model congruent to the object, six distances must be given in the object space as control (Okamoto, 1981a). As the six lengths have 12 end points, we must first set up Equation 10 for these 12 end points, which provides 48 equations. Further, Equation 11 is valid for the six distances, and the four constraints (Equations 16 and 17) must be satisfied between the 16 unknown coefficients of Equation 10. Accordingly, we obtain 58 equations for the determination of 52 unknowns (the 16 coefficients plus 36 coordinates of the 12 model points). This overdetermined system is due to the fact that we have used 12 coplanarity equations. In addition, we usually have more than six lengths as the object space control. Therefore, this orientation problem may be solved by means of two different ways to consider the four constraints (Equations 16 and 17) between the coefficients of Equation 10: a method to incorporate the constraints in a least-squares adjustment with parameters and additional constraints between these parameters and a technique to regard the constraints as additional observation equations with zero variances (Bopp and Krauss, 1978).

FOR THE CASE OF THREE PHOTOGRAPHS OVERLAPPED

The orientation procedure described above can readily be extended to the usual case with three photographs overlapped, if we further introduce two constraints between the 11 coefficients of the collinearity equations for the third picture, which are described in the same form of Equation 17. Thus, the detailed discussions are omitted here.

CONSIDERATION OF NON-LINEAR DISTORTION

Measured photo coordinates usually include two types of systematic errors: linear deformation and non-linear distortion. We have already explored the properties of the former in the model construction problem by means of the DLT approach. However, the latter has been ignored in order to make the discussion simple and comprehensive. Therefore, this section supplements the consideration for the characteristics of the non-linear errors.

The non-linear distortion of a measured image point is mainly due to lens distortion, which can be modeled with a functional form (see, e.g., Brown (1971): i.e.,

$$\begin{aligned} \Delta x &= x(k_1r^2 + k_2r^4 + k_3r^6) + p_1(r^2 + 2x^2) + 2p_2xy \\ \Delta y &= y(k_1r^2 + k_2r^4 + k_3r^6) + 2p_1xy + p_2(r^2 + 2y^2) \end{aligned} \qquad (18)$$

where x,y indicate ideal photo coordinates and $r^2 = x^2 + y^2$. The coefficients $(k_1, k_2, k_3, p_1, p_2)$ of Δx, Δy are non-central projective elements, because the functional form is non-linear with respect to the ideal photo coordinates. From this fact we can see that the coefficients of Equation 18 can be determined only from the coplanarity condition of corresponding rays (Hallert, 1956; Koelbl, 1972; Faig, 1975; Moniwa, 1976, 1981; Okamoto, 1982a). Therefore, we simply increase the number of the coplanarity equations by that of the non-central projective elements in order to obtain them together with the central projective parameters.

CONCLUDING DISCUSSIONS

The construction problem of a stereo model congruent to the object has been investigated algebraically by using the collinearity condition between a model point and the corresponding measured image coordinates. The object space control is established only in terms of distances. The characteristics of the methods presented are as follows:

- In the general case where a picture has 11 independent central projective parameters, the number of independent orientation parameters to be determined in the model construction process is 16 for a stereopair of photographs and is 27 for three pictures taken with an overlap.
- In the usual case in close-range photogrammetry, where the geometry of a picture can be determined by the nine conventional exterior and interior orientation elements, the number of such parameters

is reduced to 12 for a stereopair of photographs and 21 for three pictures overlapped.

- When expressing the collinearity equations in algebraic form, constraints are generated between the 11 coefficients. However, through careful geometrical consideration of this model construction problem, we can construct the determination equations without such constraints for the general case. Also, in the usual case, only the two constraints formulated by Bopp and Krauss (1978) need be adopted to the right photograph of a stereopair of pictures, or to the second and third photographs of three pictures overlapped.
- Space coordinates of model points (mainly those of end points of distances given as object space control) are treated as unknowns in this orientation calculation.

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