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Algorithmic Aspects in On-Line Triangulation

An efficient sequential algorithm and appropriate methods for the assessment of results are key elements of a successful on-line triangulation procedure.

Abstract: *Evaluation criteria for the quality and efficiency of on-line triangulation systems are based on response time, required computer facilities, methods of analysis of results, and degree and comfort of interactivity. This article discusses some of the major components that influence the on-line triangulation procedure. After a short historical review the goals of on-line triangulation are formulated and problems are isolated. In addition to a brief investigation into the estimation model, estimation principle, and method of data analysis, the main emphasis is placed on the computational algorithm. Three sequential algorithms—the Kalman Covariance Update, the Triangular Factor Update with Gauss/Cholesky decompositions, and the Givens Transformations Update—are particularly emphasized. Some operational aspects are also addressed.*

Introduction

Photogrammetric on-line triangulation is currently considered as a data acquisition and quasi real-time data processing procedure that allows the operator to control blunders and other model errors, and to remove false observations or add new observations at an early stage of block processing. Such a capability increases significantly the speed of execution and the reliability of results of the overall triangulation procedure. A supporting on-line triangulation software package can turn out to be extremely beneficial, particularly if the measurements are performed on an analytical plotter.

The sequential nature of the measurement process lends itself nicely to the application of sequential computational techniques. Because the response times of an on-line triangulation system are critical performance parameters, algorithmic aspects of the computational procedure are isolated in this paper as key problems.

A brief historical account of the development of on-line triangulation techniques over the last 20 years is given in the second section of this paper in order to extract the major areas of concern. The third section is intended to address the purpose and to establish the goals of on-line triangulation. The on-line procedure as it is understood currently is discussed within the wider frame of a network design and quality control unit. The main components of an on-line triangulation system are shown and some major problems are pinpointed.

In the fourth section some of these components, such as the estimation model, estimation principle, and method of data analysis, are discussed in more detail. Blunder detection and systematic error compensation are emphasized as particularly important issues of the data analysis phase.

The computational algorithm, recognized as the core element in on-line triangulation, is addressed in the final section. Although the existence of a great variety of different sequential algorithms is acknowledged, only those which have recently been suggested for use in photogrammetry—the Kalman Covariance Update, the Triangular Factor Update based on Gauss/Cholesky decompositions, and the Givens Transformations Update—are focused on. Some operational considerations related to these

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truly sequential algorithms are also briefly discussed.

Development of On-Line Triangulation

Traditionally, the term "on-line data processing" refers generally in photogrammetry to computer assisted or computer controlled systems, thus going back to U.V. Helava's first publications on the principles of analytical plotters in the late 1950's. Consequently, on-line data processing encompasses a great variety of data processing techniques, utilizing an ample assortment of different type of equipment, ranging from analytical plotters and computer assisted comparators to computer supported analog instruments. Aspects of on-line point positioning in individual stereo-pairs and associated mapping problems are, however, not the concern of this paper.

An early reference to triangulation with analytical plotters is found in Jaksic (1963). Jaksic emphasizes the possibilities for analytical strip formation, a procedure similar to the familiar triangulation technique on analog plotters. The opportunity for an immediate final strip transformation to the ground control system is also stressed.

In a later article, Jaksic (1967) describes the aerial triangulation as it was done at that time on the National Research Council (NRC) analytical plotter. He clearly suggests that ". . . procedures concerning the execution of aerial triangulation have to follow as closely as possible the analog methods. . . ." So does the NRC analytical plotter triangulation procedure, which is composed of relative orientation, scaling, and subsequent off-line strip or block adjustment. Data editing capabilities had not yet been much developed yet. Interestingly, Jaksic already foresaw future concepts by stating that "This (bundle) type of simultaneous solution will be used only for independent models with sufficient ground control. For aerial triangulation on the NRC Analytical Plotter in its present form, if extended beyond a few models, this solution is out of the question since the computer capacities would be overtaxed by the requirements of such a program. As a matter of fact none of the existing analytical plotters can handle this type of process. Consequently a discussion of its practical applications will have to be postponed until analytical plotters are coupled to more powerful computers."

In 1974, Jaksic made special reference to "on-line triangulation" (Jaksic, 1974). Jaksic already realized some potential advantages of the on-line concept; he mentions particularly the checking of replicated measurements and of residual parallaxes in relative orientation. For these procedures he expects computing times from several seconds to a minute. Besides referring to the possibility of automatic ("analytic") positioning, he emphasizes the advantages of housekeeping routines such as the real-time checking of errors in point identification.

It was not until around 1976 that on-line triangulation was generally and internationally considered a valuable topic for research and development. A working group, "Analytical On-Line Triangulation" (WG III/4), was established at the XIIIth ISP Congress in Helsinki in 1976. At the Symposium of Commission III in Moscow in 1978 three papers included the first timid discussions of on-line triangulation topics. Kratky's Invited Paper of WG III/4 (Kratky, 1980b) indicated also the initial stage of online triangulation during the late 1970's. However, there was already a fixed idea as to what the merits of on-line triangulation could be, but thorough scientific investigations into the technical details and problems were still missing. Although the situation has improved slightly during the period 1980-1984, one is still inclined to consider on-line triangulation and its crucial problem areas as a fairly open field, with no operational systems established and no standard procedures as yet agreed upon. It seems that very rarely has an issue of such importance found such reluctant scientific attention and advocates. The sequel gives a brief account of the development of methods and systems over the last decade.

In retrospect, it turns out that the early article of Mikhail and Helmering (1973) on "Recursive Methods in Data Reduction" obviously determined the viewpoints of researchers involved in this issue for almost a decade. Mikhail and Helmering used a method which modifies the inverse of the normal equations in order to account for the addition or deletion of observational data as well as of model parameters. The authors also touched upon a very critical aspect of sequential estimation in linearized models, namely the problem of proper use and updating of approximate values. Furthermore, they gave a formula for the $\mathbf{Q}_{vv}$ matrix update. The suggested sequential algorithm was used for the relative orientation of a single stereopair. Helmering (1977) had refined the previous algorithm by considering the sparsity pattern of the matrices involved in the bundle solution of a stereopair.

Both articles refer to computer assisted comparators as data acquisition devices. It should be noted that the emphasis is on point positioning in small systems rather than on block triangulation (Helmering (1977, p. 470) states "For computer-assisted comparators, the analytical photogrammetric model need be capable only of processing data from a small number of photographs, usually no more than two or three"). The primary intention is to update sequentially the parameter vector (exterior orientation parameters, object point coordinates).

Dorrer (1978) picked up this approach of updating the normal equation inverse and applied it conceptually to sequential strip triangulation with independent models. He formulated the estimation problem in terms of standard problem I (condition equations), thus treating also the parameters as random variables. He referred to this type of updating of the inverse as the "Kalman form".

Kratky (1979) reported on some principles of on-line solutions and described the stage of on-line triangulation at the NRC analytical plotter. He favored on-line solutions within small block subsystems and subsequent simultaneous off-line block adjustment as opposed to the total on-line approach for the whole block. He felt that a striptriplet was of sufficient and appropriate size for a subsystem. The NRC analytical plotter software is available for on-line model formation and scale transfer along the strip direction. Data editing modules for point deletion, addition, and replacement are operational. The algorithms used are not of sequential type.

A sequential algorithm for on-line bundle triangulation was suggested by Dowideit (1980). The underlying estimation model has a fixed number of parameters. These parameters are all treated as random variables. The associated weight coefficient matrices are originally derived from very coarse approximate value computations at an initial stage of the sequential process. These matrices are sequentially updated using the Kalman-form approach, as image coordinate observations become available. In an experiment Dowideit has simulated this on-line approach in an off-line computer program. Although the sequential estimation approach used here might draw some criticism as outlined in the fourth section, it is for the first time that a subblock is used in a sequential model (six photographs), rather than only two or three strip photographs.

In a follow-up paper to his 1979 article, Kratky (1980a) described the stage and potential of aerial triangulation on the NRC analytical plotter. The on-line bridging mode had now been refined by using a stricter scaling procedure. Kratky explains in much detail some beneficial operational features of analytical plotter triangulation, including

- savings of pass points if the subsequent plotting is done on analytical plotters;
- computer selection and positioning of tie points in strip direction;
- automatic numbering;
- computer positioning and measurement of tie points across the strip direction; and
- treatment of pass points if required for subsequent analog plotting.

Although Kratky is very much in favor of across-strip measurements, he does not support across-strip on-line computations. His train of thought suggests, however, that he is only considering 20 percent sidelap blocks.

In a review paper, Kratky (1980b) reported on the present status of on-line triangulation. He pays particular attention to the methodology of on-line solutions. Two different aspects are isolated here, the solution algorithm and the blunder detection procedure. It is acknowledged that, if the size of the normal equations exceeds a certain limit, the dynamic character of the observational scheme must find its counterbalance in a recursive adjustment in order to provide for tolerable response times. Again, the Kalman form of normal equations inverse and parameter vector updating is considered appropriate. The data snooping technique is suggested for blunder detection. Hence, for the first time three critical elements for a successful execution of modern on-line triangulation are addressed:

- the recursive nature of near real-time computations,
- the use of computational units which clearly exceed two photographs, and
- the need for a sophisticated blunder detection technique.

Rosculet (1980) also recommends the normal equation inverse updating approach for recursive adjustment in the form of a variable size parameter vector. Furthermore, he suggests that the whole block be treated in a sequential mode, and that even surveying observables be included.

The year 1980 and the Hamburg Congress mark an end to a period of early developments and concepts in on-line triangulation. Although the basic advantages of the on-line approach with respect to a more efficient triangulation procedure are acknowledged, one is not ready yet for a fully analytically oriented methodology in data acquisition and processing. The measurement process follows essentially the analog instrument pattern, and the computational strategy still models the analog procedure.

A number of papers have been published in recent years which accentuate a more analytically oriented approach to on-line triangulation. Some authors clearly stress that the recursive treatment of the data should not be confined to two or three photographs at most. The use of larger block-subsystems or even the total on-line concept for the whole block are envisioned and related sequential algorithms are formulated.

Heindl (1981) suggests that data editing (he considers only deletion of observations and object points) be done in the form of a sequential least-squares adjustment, whereby the prereduced normal equations can be directly updated. Interestingly, he recommends his approach for data editing (blunder detection) in off-line block adjustment. Heindl's functional model also includes additional parameters for systematic error modeling.

For the sequential processing of blocks, Molenaar (1981c) suggests splitting up the computing process into two major steps: strip formation by means of relative orientation and triplet formation, plus subsequent connection of the strip to ground control. Thus, Molenaar's computing process still follows the analog pattern. For both adjustments the estimation model is formulated in terms of standard problem 1 (condition equations). Testing procedures for blunder detection are recommended, and the problem of systematic error compensation is discussed.

Blais (1983) suggested Givens transformations for linear least-squares computations in general and for sequential estimation in particular. Although his publication does not indicate any quantitative performance measures, it seems that Givens transformations can be a strong candidate for an efficient sequential estimation algorithm in on-line triangulation.

Another sequential computational approach was recommended by Gruen (1982). His "Triangular Factor Update," based on Gauss or Cholesky factorization, updates directly the prereduced or, in many cases, even the further reduced matrix of normal equations. Wyatt (1982) and Gruen and Wyatt (1983) thoroughly compared the method with the Kalman-form of inverse updating, using computational speed and storage requirements as performance criteria.

Kratky (1982) and Kratky and El-Hakim (1983) reported upon the latest stage of development of the NRC analytical plotter on-line triangulation features. Computer positioning across the strip direction is now operational, and data snooping is used in relative orientation and scale transfer along the strip.

There is a variety of other publications to which the author had access and which are worth mentioning here. These works are not described in more detail because they either are not fully dedicated to on-line triangulation, or they cover just some detail or side aspect, e.g., Kratky (1976), Salmenperae and Vehkaperae (1976), Foerstner (1979), Hobbie (1978), Dorrer (1981), Seymour (1982), Hoehle *et al.* (1982) and Radwan *et al.* (1982).

As it turns out, the most recent developments are increasingly using a rather strict estimation model, based on the bundle solution, and are very much concerned with the design of a fast sequential algorithm.

An account and evaluation of recent achievements must be preceded by a definition of the term "on-line triangulation," in order to come to an understanding as to what this technique is supposed to accomplish. The following section is intended to define the goals of on-line triangulation systems.

Definition and Goals of On-Line Triangulation

In general terms, "on-line triangulation" is the procedure of measuring and immediately processing data for point positioning purposes. By defining "on-line" as "being in direct communication with a computer," the triangulation procedure draws advantage from the fact that the measurements can be processed immediately after being acquired. The related benefits were expressed in Resolution T III/3 of the XIVth Congress of the ISPRS in Hamburg in 1980, where the Congress ". . . recognizes, that the on-line capability to measure and immediately process data increases the speed and reliability of photogrammetric triangulation and may significantly improve the organization of routinely performed work. . . ." An increase in speed and reliability of the overall triangulation procedure can be achieved primarily through the opportunity to perform quality control at the early stage of data acquisition. In addition to the near real-time control of the measurements and their agreement with the selected estimation model, the measurement process as such might be controlled, especially if the system provides for computer feedback in the closed-loop sense (analytical plotter). These latter, more practical aspects, such as the problem of analytical tie point transfer, have been covered by Dorrer (1984), another invited paper of WG III/2.

Molenaar (1981b) has identified some problem areas in the field of quality control. He particularly refers to blunder detection, systematic deformations, unified testing procedures, analysis of the stochastical model, connection of photogrammetric data to ground control, variance component estimation, and accuracy measures in practice. If an on-line triangulation model has to act as a quality controller, it should be capable of responding to all those critical questions. However, besides operating as a passive quality control unit, a very advanced version of an on-line triangulation procedure might even perform the functions of an active network design module of first, second, and third order type. Such a module could advise the operator or the automatic correlator where to place points and observations in order to achieve a prespecified precision/accuracy structure in the object space. Clearly, such a system is far from being realized today. But we have to acknowledge the fact that the hardware is available, and, to a certain extent, the methods are developed which would allow us to design, install, and operate such a system.

Because in on-line triangulation we have to deal with redundant observations, our problem at hand is that of optimal estimation. This involves the problems of proper model set-up, estimation approach, and assessment of results. Although the model might be drawn up differently, as long as those different versions are formulated at the same level of rigour, and if the same estimation approach is used (e.g., unbiased, minimum variance), we may expect identical results. The model formulation, however, influences significantly the computational algorithm. Because the primary evaluation criteria for an algorithm are computational speed (response time) and storage requirements, it clearly matters, for instance, which standard problem of least-squares adjustment is used.

Furthermore, the algorithm depends on the type of results required. The task of point estimation delivers the parameter vector $\mathbf{x}$ and the residual vector $\mathbf{v}$, interval estimation requires the associate weight coefficient matrices $\mathbf{Q}_{xx}$ and $\mathbf{Q}_{vv}$ together with an estimate of the variance factor σ_0^2. The algorithm might also be influenced by the method of data acquisition, e.g., whether groups of observations or single observations are added, and whether the size of the parameter vector is kept constant or varies. Also influential are the data editing procedure (addition, deletion, or replacement of observations) and method of data analysis (numerical, graphical, or combinations; operator controlled or automatic; level of statistical strength).

Figure 1 depicts the major components of an on-line triangulation system. The evaluation of such a system should consider the following parameters:

- computer characteristics (computing speed, random access memory, I/O times)
- admissible response time
- block size to be used in near real-time mode (numbers of photographs and object points)
- data acquisition mode
- type and number of required results
- method of assessment of results
- data editing procedure
- estimation model
- estimation approach
- computational algorithm

The key to a successful on-line triangulation technique is the optimal adjustment of these parameters. Notwithstanding the previously formulated high demands on a future on-line triangulation system, it is currently generally agreed that the major task of on-line triangulation is the early check and correction of observations (Kratky, 1980b). This aspect of quality control involves the check of the agreement of the acquired observations with a suitable estimation model. Any distortion between observations and adopted model must entail either a reacquisition or redefinition of the observations (remeasurement, renumbering, etc.), or an alteration of the estimation model.

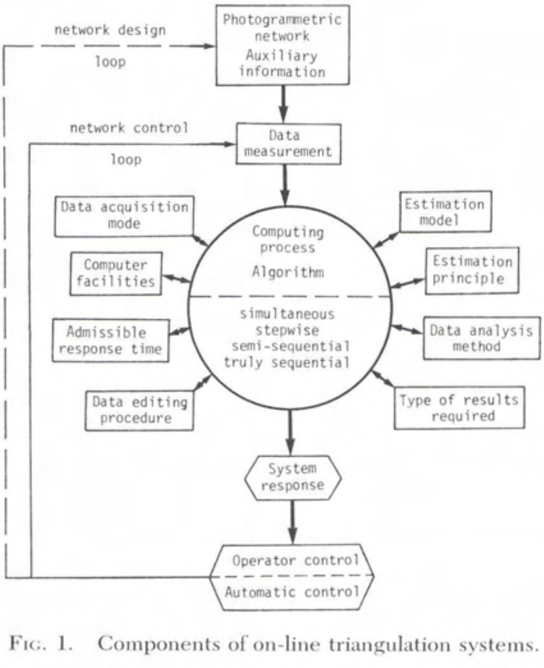


FIG. 1. Components of on-line triangulation systems.

Because on-line triangulation is considered a highly interactive process with information flowing back and forth among operator, computer, and measurement device, near real-time responses of the system to an operator request are very crucial. Hence, the time factor becomes a very important element, and any computational methods for estimation and assessment of the results must stand up to a critical time performance and evaluation. The functional and stochastical parts of estimation models and the associated computational algorithms have to be tailored such that they meet this time requirement.

The next section investigates some of the most critical parameters that determine the quality of an on-line triangulation system.

Evaluation of Some On-Line Triangulation Components

Figure 1 pictures the parameters which determine the design of an on-line triangulation procedure. Some of those components, which cover more practical aspects, such as admissible response time, computer facilities, data acquisition mode, and data editing procedures, are not further addressed in this paper. Closer attention is paid to the estimation model and principle, and the method for data analysis. The core of on-line triangulation, the computational algorithm, is addressed in the final section of this paper.

ESTIMATION MODEL AND ESTIMATION PRINCIPLE

Problems associated with the proper choice and evaluation of an estimation model have been increasingly discussed in the geodetic literature in recent years, e.g., Koch (1980) and Schaffrin (1983). In the following reference is made only to those concepts which have been used in photogrammetric on-line triangulation so far.

Standard Approach. The Gauss-Markov model is the estimation model most widely used in photogrammetric linear or linearized estimation problems.

An observation vector $\mathbf{l}$ of dimension $n \times 1$ is functionally related to a $u \times 1$ parameter vector $\mathbf{x}$ through

$$\mathbf{l} - \mathbf{e} = \mathbf{A}\mathbf{x}. \tag{1}$$

The design matrix $\mathbf{A}$ is an $n \times u$ matrix with $n \geq u$ and the rank Rank(A) $= u$. There is no need to work with rank-deficient design matrices in on-line triangulation. Rank deficient systems, caused by missing observations, generally do not allow for a comprehensive model check. Observations should be accumulated until the system is regular and can be solved using standard techniques. For rank deficiency caused by incomplete datum, see the section on Operational Considerations. Sequential least-squares estimation with pseudoinverses is very costly (compare Boullion and Odell, 1971, p. 50 ff).

The vector **e** represents the true errors. With the expectation E(**e**) = 0 and the dispersion operator D, we get

$$E(\mathbf{l}) = \mathbf{Ax}, \tag{2a}$$

$$D(\mathbf{l}) = \mathbf{C}_{ll} = \sigma_0^2\mathbf{P}^{-1}, \text{ and} \tag{2b}$$

$$D(\mathbf{e}) = \mathbf{C}_{ee} = \mathbf{C}_{ll}. \tag{2c}$$

The estimation of **x** and σ_0^2 is usually attempted as unbiased, minimum variance estimation, performed by means of least squares, and results in

$$\text{parameter vector } \hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{PA})^{-1}\mathbf{A}^T\mathbf{Pl}, \tag{3a}$$

$$\text{residual vector } \mathbf{v} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{l}, \tag{3b}$$

$$\text{variance factor } \hat{\sigma}_0^2 = \frac{\mathbf{v}^T\mathbf{Pv}}{r}, \; r = n - u. \tag{3c}$$

The architecture of **A** is determined by the type of triangulation method used. Although some authors still favor approximate methods for on-line triangulation, there is a strong indication that the bundle method is becoming a standard technique for systems which generate data in the form of image coordinates (Salmenperae and Vehkaperae, 1976; Helmering, 1977; Dowideit, 1980; Rosculet, 1980; Dorrer, 1981; Heindl, 1981; Molenaar, 1981c; Gruen, 1982).

For a bundle adjustment, Equation 1 can be written as

$$-\mathbf{e} = \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{t} - \mathbf{l};\ \mathbf{P} \tag{4a}$$

where

- **x** is the vector of object point coordinates;
- **t** is the vector of orientation elements;
- $\mathbf{A}_1$ and $\mathbf{A}_2$ are the associated design matrices; and
- **e**, **l**, and **P** are the true error vector, constant vector, and weight matrix for image point observations, respectively.

x and **t** are considered here as unconstrained (free) parameters. If observations are available for some or all of the object point coordinates, a second system of observation equations is added; that is,

$$-\mathbf{e}_c = \mathbf{Ix} - \mathbf{l}_c;\ \mathbf{P}_c. \tag{4b}$$

Similarly, observations for the orientation elements would add

$$-\mathbf{e}_t = \mathbf{It} - \mathbf{l}_t;\ \mathbf{P}_t. \tag{4c}$$

The combined system of Equations 4a, 4b, and 4c was formally used by Dowideit (1980). It should be noted, however, that Dowideit had no real observations for **x** and **t** available. Although Equation 4a would have been the proper description for the type of information at hand, he instead used Equations 4a, 4b, and 4c in order to be able to operate within a constant size state vector system for sequential estimation. Covariance matrices were assigned to **x** and **t**, generated through the process of approximate value computation for bundle adjustment. Such a procedure can pose severe convergency problems.

For high accuracy data processing, additional parameters for systematic error modeling should become an indispensable part of the estimation model. The inclusion of an additional parameter vector **z** in Equation 4a results in

$$-\mathbf{e} = \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{t} + \mathbf{A}_3\mathbf{z} - \mathbf{l};\ \mathbf{P}. \tag{5a}$$

If the additional parameters can be considered observed quantities, one would have to add

$$-\mathbf{e}_z = \mathbf{Iz} - \mathbf{l}_z;\ \mathbf{P}_z. \tag{5b}$$

The least-squares principle in Equations 5a, 5b, 4b, and 4c now leads to the combined minimum

$$\mathbf{v}^T\mathbf{Pv} + \mathbf{v}_c^T\mathbf{P}_c\mathbf{v}_c + \mathbf{v}_t^T\mathbf{P}_t\mathbf{v}_t + \mathbf{v}_z^T\mathbf{P}_z\mathbf{v}_z \rightarrow \text{Min.} \tag{6}$$

Because Equations 5a, 5b, 4b, and 4c still represent a Gauss-Markov model, we refer in the sequel to Equation 4a for simplification and without loss of generality.

Variations. Using Tienstra's terminology, the previous model (Equation 1) is referred to as "standard problem II" (least-squares adjustment with observation equations). Molenaar (1981c) has shown that the bundle adjustment (Equation 4a) can also be formulated as "standard problem I" (least-squares adjustment with condition equations). Because the conditions are based on strictly perspective relations, any disturbance therefrom, such as systematic errors, cannot be formulated within the functional portion of this model. There is the possibility to consider systematic errors in the stochastical model, which requires, however, a definite knowledge of the systematic deformation pattern prior to the execution of the triangulation, if expensive weight estimation is to be avoided.

Mixed forms of these standard problems (nos. III and IV) are also in use. They can always be reduced to either standard problem I or II. If the rotation elements ϕ, ω, and κ are not measured, Molenaar's original set-up of equations for relative orientation and triplet formation is actually given in terms of standard problem IV. He reduces the set to standard problem I before entering the final solution. This procedure is designed virtually from the very beginning as a sequential approach, so it does not require a specific sequential computational algorithm. The procedure is presented as a "one-way" technique, building up models and strips, and transforming the strips to ground control. There is no reference as to how this procedure should be used optimally if observations and parameters have to be deleted/added at a stage prior to the latest stage of measurement and block generation.

Dorrer (1978) formulates the strip triangulation in terms of standard problem I, considering both the real observations and the usual parameters as stochastical variables. He demonstrates his approach with strip triangulation of independent models, and operates with transfer and control conditions. The sequential updating procedure applies the Kalman form of covariance updating.

The unbiased, minimum variance estimation approach by means of least squares leading to Equations 3a, 3b, and 3c is by far the most popular estimation principle in photogrammetry. It is a widely familiar and thoroughly investigated approach, simple and inexpensive. A current general trend in statistics uses a slightly biased parameter vector, which is, however, expected to be with higher probability closer to the true value than its unbiased counterpart. Reported disadvantages of the least-squares technique with respect to the detection of model errors, in particular blunders, support this trend. Robust estimation (Andrews, 1974; Huber, 1981), ridge regression (Hoerl and Kennard, 1982), James-Stein estimation (Draper and van Nostrand, 1979), etc., are some of the more recent techniques. The Danish method (Kubik, 1982) and the Median method (Fuchs, 1981; Fuchs and Leberl, 1982) fit right into the category of "robust estimation." The Danish method, which can be interpreted as an iterative reweighted least-squares estimation, has found some advocates in photogrammetry (El-Hakim, 1982; Larsson, 1982; Sarjakoski, 1982). Other than the works of El-Hakim (1982) and Kilpelae *et al.* (1982), comparative studies with ordinary least-squares estimation are not yet available to the author.

Although some benefits with respect to blunder detection are expected from robust estimation, those expectations could not be quantified to date in photogrammetry. Of crucial importance for proper functioning of this method is the weighting scheme and, in particular, the choice of starting values. As summarized in Hocking (1983), a great variety of weighting schemes is available, which generates the danger of arbitrary action. Starting from least-squares residuals is not recommended, because those already reflect strongly the masking and swamping effect of blunders. Robust estimators should be used only with great care, and by knowledgeable personnel. A black box approach is not advisable. This aspect does not qualify robust estimation for on-line triangulation, which is supposed to be a highly automated technique, controlled by photogrammetric operators. The intrinsic iterative scheme of robust estimation and the associated computational load questions even further its suitability.

METHODS OF DATA ANALYSIS

The overall on-line triangulation concept, especially its computational approach, is very much determined by the method of data analysis. The on-line triangulation technique suggested in Gruen (1982), for instance, is primarily based on the particular internal reliability structures of photogrammetric networks, and on the related conditions for blunder detection with Baarda's data-snooping or modified techniques (Gruen, 1980, 1981).

As it was emphasized earlier, the major short range goal of current on-line triangulation efforts is the detection of model errors at an early stage. This allows for inexpensive remeasurements and provides a fairly clean data set for the final execution of the simultaneous block adjustment. Conventionally, model errors are classified into blunders, systematic errors, and stochastical errors. This classification scheme has tended to be abandoned in recent years. Stochastical errors are sometimes treated intentionally as systematic errors and vice versa, and no distinction is made anymore between blunders and systematic errors, because both cause a deficiency in the functional part of the estimation model. In this paper the author uses the conventional classification, because there is not much experience available yet with respect to the joint treatment of different types of errors. In addition, those different errors do have a distinctly different genesis, although their effect might be similar, and because, in on-line triangulation one is closer to the original source of errors and is better capable of correcting errors rather than compensating them or removing the related observations, the conventional classification seems to fit better to the on-line triangulation situation.

The stochastical errors in image coordinates have not yet received too much attention in photogrammetry (Schroth (1982) and related references). It seems that the usual assumption of equally weighted, uncorrelated image coordinates needs to be abandoned only in case of extremely high model refinement requirements.

Blunder Detection and Location. Much has been written in recent years in the statistical and geodetic literature about blunders and their detection. Photogrammetrists in particular have always longed for efficient blunder detection techniques. The knowledge and the solutions gained in off-line triangulation need to be transferred now to on-line systems.

An excellent review paper on blunder detection was recently published by Beckman and Cook (1983). Two basic standard methods for the treatment of blunders have evolved over the years: identification and accommodation. Following the notions of Barnett and Lewis, (1978), identification of a blunder may lead to (a) its rejection, (b) important new information contained in concomitant variables that would otherwise have gone unnoticed, (c) its incorporation through a revision of the model or method of estimation, or (d) a recognition of an inherent weakness in the data and thus further experimentation. Accommodation of blunders is achieved through suitable modifications of model and/or estimation procedure and/or method of analysis. Robust methods are regarded as omnibus methods for accommodation.

Identification and accommodation can be related to two different notions of a blunder, the "mean shift model" and the "variance inflation model," respectively. In the mean shift model, the marginal normal distribution of a blunder is considered to be $n(\mu + \lambda, \sigma^2)$ as compared to the distribution $n(\mu, \sigma^2)$ of all other errors. The variance inflation model considers the blunder to follow the normal distribution $n(\mu, a^2\sigma^2)$, $a^2 > 1$, while all other errors follow again the normal law $n(\mu, \sigma^2)$. Thus, the mean shift model interprets a blunder as an error in the functional model for estimation, whereas the variance inflation model refers to a blunder as a stochastical error in the estimation model. Identification and accommodation coincide with the two most popular approaches for blunder treatment in photogrammetry, Baarda's data-snooping and the Danish method. Identification is generally judged to be more fundamental, avoiding the obscurrence of essential information, which is a basic drawback of accommodation. As explained earlier, identification is preferable to accommodation in on-line triangulation, for computational as well as for cognitive reasons.

Problems of masking, swamping, and multiple outlier tests are also addressed by Beckman and Cook, (1983). It is obvious that photogrammetrists still have a fairly long way to go in their efforts to test and compare other approaches to blunder detection than those that are already in use.

Graphical methods, for instance, could turn out to be very potent, particularly in multiple blunder situations, and lend themselves favorably to on-line triangulation, if a graphical screen is available as part of the system.

Baarda's data-snooping has been suggested for on-line triangulation by Foerstner (1979), Kratky (1980b), Molenaar (1981c), Dowideit (1982), and Gruen (1982). The method requires only the diagonal elements of the $\mathbf{Q}_{vv}$ matrix, is thus computationally manageable, considering the sparsity of the matrices involved, and has proved its potential for the one-blunder case in many applications. A remarkable computational speed-up of this technique can be achieved with the method of "unit observation vector," described by Gruen (1982), if only a few observations have to be tested at a time and only the related diagonal elements of the $\mathbf{Q}_{vv}$ matrix are required. This is likely to be the case in on-line triangulation, where observations acquired at earlier stages of the sequential process have already been finally accepted.

Even multiple-blunder situations can be satisfactorily dealt with by the use of the data-snooping technique, if the $\mathbf{Q}_{vv}$ $\mathbf{P}$ matrix is structured such that no significant masking and swamping occurs.

Masking and swamping were tried to retort by applying multi-dimensionally derived test criteria (Stefanovic, 1980). Tests conducted with this approach by Elious (1983) were not as successful as expected. Because these test criteria include inverses of submatrices of $\mathbf{Q}_{vv}$, the problem of singularity of those submatrices has to be dealt with. Altogether, this method requires a tremendous organizational and computational overhead, which devalues it for application in on-line triangulation. Other modifications and variations of Baarda's original data-snooping are reported upon in Pope (1976), Clerici and Harris (1980), Gruen (1980), Molenaar (1981a), Benciolini *et al.* (1982), and El-Hakim (1982). Initial tests in close-range networks with a procedure that is designed to recover masking and swamping effects in a recursive mode are reported on in Madani (1984). This method has some interesting features for on-line triangulation, it is computationally attractive, and it displays decent blunder location properties. It needs further refinement and more practical testing before it can be recommended as operational, though. A thorough, effective, and fast specific blunder location procedure for on-line triangulation has yet to be developed.

Systematic Error Compensation. Like blunders, systematic errors can be interpreted as a deficiency of the functional portion of the estimation model. Unlike blunders, which can only be detected within the Gauss-Markov model (Equation 1) from post adjustment data (residuals or functions of them), systematic errors can effectively be modeled in the estimation model prior to the adjustment. The *a priori* available knowledge about both error types leads to the basic difference in their treatment. While there is no *a priori* knowledge about blunders, neither about their location nor about their size, we have a fairly well understood and documented knowledge about possible systematic errors that are likely to occur in the data. This allows us to model those anticipated systematic errors as additional parameters in the estimation model. This procedure is known as "self-calibration." Only the type of error needs to be anticipated, not its size. Self-calibration has proved its potential in off-line triangulation. If on-line triangulation is performed on a high accuracy level, self-calibration is also indispensable. Self-calibration is particularly crucial if "small" blunders (up to two times the size of the just detectable blunder) are chased. Systematic errors would otherwise interfere inadmissibly with the blunder detection procedure. This would cause observations which contain just systematic errors to be tagged as grossly erroneous, and true blunders could be masked. Heindl (1981) included additional parameters in his system, and Gruen (1982) has recommended their use. Molenaar's approach (Molenaar, 1981c) does not allow the use of additional parameters in the functional model, but it does in the stochastical model.

A widely accepted strategic approach in the treatment of additional parameters in off-line triangulation is to include a fairly large set in order to be sure to cover all possible errors. Because this creates the danger of overparameterization, a procedure for the deletion of nondeterminable additional parameters must be incorporated. The major information regarding the detectability comes from sources such as number and location of control points, overlap, and flight direction arrangement.

In on-line triangulation the control points are sometimes not considered at all (Gruen, 1982), or systems are built up along a strip (Dorrer, 1978; Molenaar, 1981c; etc.), and generally the process starts with small units, e.g., one stereomodel. These conditions do not favor a solid determination of a comprehensive set of additional parameters. In order to avoid running into too many non determinable additional parameters and spending too much time with the clearance procedure, one should better operate in on-line triangulation with a fairly small set of additional parameters, considering only the most important ones which are known to be determinable in those relevant arrangements. This restriction is not critical because non determinable systematic errors do not have an effect on the residuals of the adjustment (Gruen, 1978b) and do not interfere with blunders. A computationally fast and reliable procedure for the handling of additional parameters (checking on determinability and possibly significance) has still to be adopted for on-line triangulation (see related off-line approaches in Gruen (1978a, 1983a), Ackermann (1980), Foerstner (1981) and Jacobsen (1982)).

The dynamic estimation of additional parameters in on-line triangulation allows one to monitor a possible change of the systematic error pattern throughout the block and provides for valuable information regarding the use of block variant additional parameters in the final off-line adjustment.

Algorithmic Aspects

An algorithm is defined as a sequence of computational instructions to solve a certain problem. In on-line triangulation this term can be applied to both the organization of the overall computational procedure and the specific approach selected for the computation of variables (solution vector, residuals, etc). We assign the term "computing process" to the former while "algorithm" is used for the latter procedure.

As Figure 1 indicates, the computing process is the most critical element in on-line triangulation. It determines the response times, and its performance makes a procedure operational or causes its failure in practical application. There is a great variety of computing processes and algorithms available to solve Equation 1 or Equations 5a, 5b, 4b, and 4c either strictly or approximately. The fundamental choice to be made in on-line triangulation is between the sequential and simultaneous process, but, as Figure 2 indicates, there are also mixtures in use.

Figure 3 shows the flow of the computations and the type of interference in non-simultaneous processes. The situation is best explained with the example of strip treatment. Assume that the appropriate simultaneous solution for strip triangulation is the off-line bundle adjustment. Stepwise procedures are followed by Salmenperae and Vehkaperae (1976), Foerstner (1979), Hobbie (1978) and Seymour (1982) where the strip is formed by sequen-

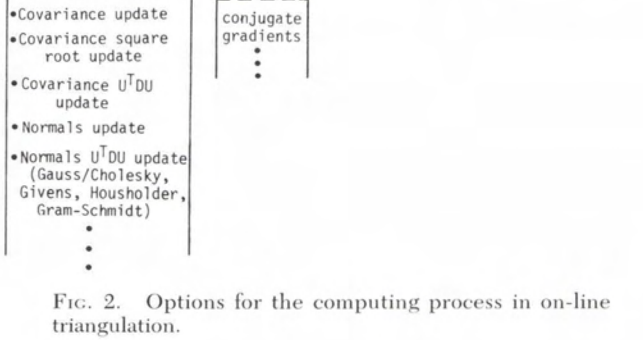


Fig. 2. Options for the computing process in on-line triangulation.

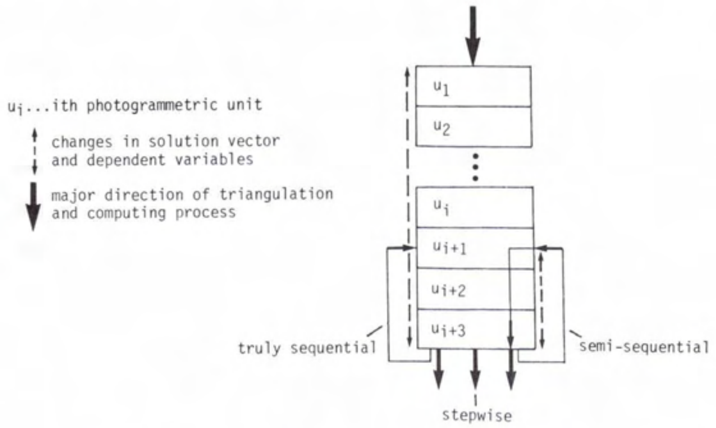


FIG. 3. Flow chart for non-simultaneous computing processes.

tially adding photos, models, and/or triplets. Editing of data (deletion, addition, replacement) is only possible at the latest stage. The final solution is not strictly equivalent to a simultaneous bundle solution, but the differences may not matter for practical purposes.

A semi-sequential procedure would allow us to extend the data editing back to earlier stages, but would consider changes in the solution vector and dependent variables only between the point of interference and the latest stage (Kratky, 1982). With some reorganization of the computing process, all previously mentioned stepwise approaches could possibly be used in a semi-sequential version.

The truly sequential process allows for interference at any location, and considers all subsequent alterations in the solution vector, etc., no matter how small they are, so that the final results correspond exactly with the simultaneous bundle solution. The procedures of Mikhail and Helmering (1973), Helmering (1977), Rosculet (1980), Heindl (1981), Dowideit (1982), Gruen (1982), and Blais (1983) fit conceptually into that category. Molenaar's procedure (Molenaar, 1981c) is somehow a mixture between a stepwise and a truly sequential computing process. His adding of observations at the latest stage is organized as a stepwise technique, while the editing of observations that were earlier incorporated follows a truly sequential concept. Dependent on the blocksize, this latter approach can be very costly.

Truly sequential solutions have the greatest potential for future on-line triangulation applications. They give rigorous solutions, which is of importance for the detection and location of small blunders, while possibly providing for manageable computing times.

TRULY SEQUENTIAL ALGORITHMS

As indicated by Figure 2, we distinguish here between direct and indirect solutions. Indirect solutions, such as the conjugate gradient method, are easy to mechanize for simultaneous applications, but because convergence problems and unfavorable computing times have been reported in some places, those indirect methods have never found very much support among photogrammetrists. For sequential estimation purposes, however, the use of indirect methods should be reevaluated.

This paper will focus on direct methods, particularly on those which have been already suggested for on-line triangulation. Bierman (1977) gives an account of direct sequential estimation algorithms that have emerged over the past two decades and that have been used primarily for orbit determination. He emphasizes particularly the Kalman Covariance Update, including its square root covariance factorization and its $\mathbf{U}^T\mathbf{D}\mathbf{U}$ covariance factorization modifications, the Housholder Update, and the SRIF (square root information filter) algorithm. Our discussions will center around the Kalman Update, Givens Update, and TFU Update.

Basically, a sequential update can be formulated at any stage of the least-squares computing process, i.e., at the normals, the partially or fully decomposed normals, or at the inverse. The following list gives a short explanation of the currently most popular direct sequential algorithms:

- Kalman Covariance Update; updates covariance matrix of solution vector.
- Square Root Covariance Update; updates square root factor of covariance matrix, used in order to improve the stability of the Kalman Update, e.g., Potter's algorithm (Bierman, 1977).

- $\mathbf{U}^T\mathbf{DU}$ Covariance Update; updates unit upper triangle $\mathbf{U}$ of covariance matrix factors; special case: Cholesky decomposition ($\mathbf{C}^T\mathbf{C}$ covariance update)
- SRIF (square root information filter); updates the upper triangular factor of the decomposed normal equations; can be based on orthogonal decompositions (Housholder, Givens, Gram-Schmidt) or on Gauss/Cholesky factorization.

Three sequential algorithms have been suggested so far for use in photogrammetric triangulation:

- Kalman Covariance Update,
- Triangular Factor Update (TFU) of the factorized normals with Gauss/Cholesky, and
- Givens Update of the factorized normals.

For a more detailed description of these algorithms, we refer in the sequel to Equation 4a, which was written as

$$-\mathbf{e} = \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{t} - \mathbf{l};\ \mathbf{P}. \tag{4a}$$

The resulting normal equations, which are further assumed to be regular, are of the form

$$\mathbf{N}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{t}}\end{bmatrix} = \begin{bmatrix}\mathbf{N}_{xx} & \mathbf{N}_{xt}\\ \mathbf{N}_{xt}^T & \mathbf{N}_{tt}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{t}}\end{bmatrix} = \begin{bmatrix}\mathbf{l}_x\\ \mathbf{l}_t\end{bmatrix}, \tag{7}$$

with

$$\begin{aligned}\mathbf{N}_{xx} &= \mathbf{A}_1^T\mathbf{P}\mathbf{A}_1,\ \mathbf{l}_x = \mathbf{A}_1^T\mathbf{P}\mathbf{l}\\ \mathbf{N}_{xt} &= \mathbf{A}_1^T\mathbf{P}\mathbf{A}_2,\ \mathbf{l}_t = \mathbf{A}_2^T\mathbf{P}\mathbf{l}\\ \mathbf{N}_{tt} &= \mathbf{A}_2^T\mathbf{P}\mathbf{A}_2\end{aligned}$$

$\mathbf{N}$ is further assumed to be regular. In an off-line environment Equation 7 is usually solved by applying Gauss or Cholesky factorization. The former can formally be described as an $\mathbf{LU}$ factorization, decomposing $\mathbf{N}$ into a product of lower and upper triangular matrices $\mathbf{L}$ and $\mathbf{U}$, i.e.,

$$\mathbf{L}\,\mathbf{U}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{t}}\end{bmatrix} = \begin{bmatrix}\mathbf{l}_x\\ \mathbf{l}_t\end{bmatrix}, \tag{8}$$

or, with $\mathbf{L} = \mathbf{U}^T\mathbf{D}$ ($\mathbf{D}$ is a diagonal matrix), in the alternate formulation

$$\mathbf{U}^T\mathbf{DU}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{t}}\end{bmatrix} = \begin{bmatrix}\mathbf{l}_x\\ \mathbf{l}_t\end{bmatrix}. \tag{9}$$

After the reduction of the right hand side, the solution vector is computed from

$$\mathbf{U}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{t}}\end{bmatrix} = \mathbf{L}^{-1}\begin{bmatrix}\mathbf{l}_x\\ \mathbf{l}_t\end{bmatrix} \tag{10}$$

by back-substitution.

In photogrammetric triangulation the factorization is usually done as a stepwise procedure, stopping the reduction of $\mathbf{N}$ right before it enters what was originally the $\mathbf{N}_{tt}$ matrix. This procedure leads to the prereduced normals $\mathbf{N}_R$, i.e.,

$$\mathbf{N}_R\hat{\mathbf{t}} = \mathbf{l}_R \tag{11}$$

with $\mathbf{N}_R = \mathbf{N}_{tt} - \mathbf{N}_{xt}^T\mathbf{N}_{xx}^{-1}\mathbf{N}_{xt}$ and
$\mathbf{l}_R = \mathbf{l}_t - \mathbf{N}_{xt}^T\mathbf{N}_{xx}^{-1}\mathbf{l}_x$.

$\mathbf{N}_R$ is finally factorized to an upper triangle $\mathbf{N}_{RR}$ and $\mathbf{t}$ is obtained by back-substitution from

$$\mathbf{N}_{RR}\hat{\mathbf{t}} = \mathbf{l}_{RR}. \tag{12}$$

The mechanization of this off-line factorization algorithm takes advantage of the fact that $\mathbf{N}_{xx}$ is a blockdiagonal matrix with 3×3 submatrices along the diagonal. Therefore, the reduction of the point coordinates can be done on a "point by point" basis, leaving the structure of the $\mathbf{N}_{xx}$ and $\mathbf{N}_{xt}$ matrices unchanged, i.e., producing no new fill-ins in those matrices. This particular feature, based on the structure of $\mathbf{N}_{xx}$, is the key to a successful application of the Triangular Factor Update technique in on-line triangulation.

The covariance matrix for $\mathbf{x}$ and $\mathbf{t}$ is obtained by either inverting $\mathbf{N}$, i.e.,

$$\mathbf{\Sigma}_{xt} = \sigma_0^2\mathbf{Q}_{xt} = \sigma_0^2\mathbf{N}^{-1}, \tag{13}$$

or better, if $\mathbf{U}$ has already been computed, by deriving $\mathbf{Q}_{xt}$ from $\mathbf{U}$ according to

$$\mathbf{Q}_{xt} = -(\mathbf{DU} - \mathbf{I})\mathbf{Q}_{xt}\ ;+\ \mathbf{DR} \tag{14}$$

where $\mathbf{R} = \mathrm{L}^{-1}$.

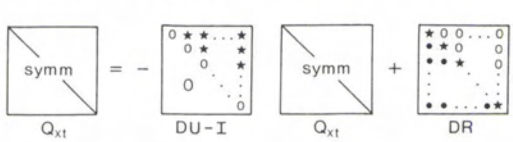
$\mathbf{R}$ is a "reduction matrix" of the form

$$\mathbf{R} = \begin{bmatrix}1 & 0 & 0 & \dots & 0\\ r_{21} & 1 & 0 & & 0\\ r_{31} & r_{32} & 1 & & 0\\ \vdots & & & \ddots & \vdots\\ r_{n1} & r_{n2} & r_{n3} & \dots & 1\end{bmatrix}. \tag{15}$$

The elements $r_{\mu\nu}$ of $\mathbf{R}$ are actually not computed. $\mathbf{R}$ is only formally used to demonstrate the reduction of $\mathbf{N}$ to an upper triangle $\mathbf{U}$

$$\mathbf{U} = \mathbf{RN} = \mathbf{L}^{-1}\mathbf{N}. \tag{16}$$

Equation 14 involves the matrix structures



$\mathbf{Q}_{xt}$ is built up rowwise from the end, starting with its last element q_{nn}. This graphically shows that, because of the symmetry of $\mathbf{Q}_{xt}$, the elements of $\mathbf{DR}$ marked with · are not needed in the computations.

Kalman Covariance Update. A system of observation equations becomes available at a time k; i.e.,

$$-\mathbf{e}_{(k)} = \mathbf{A}_{(k)}\mathbf{x} - \mathbf{l}_{(k)};\ \mathbf{P}_{(k)} \tag{17}$$

The state vector $\mathbf{x}$ is allowed to change from stage to stage according to

$$\mathbf{x}_k = \mathbf{T}_{(k)}\mathbf{x}_{k-1} + \mathbf{w}_{(k)};\ \mathbf{P}_{w(k)}. \tag{18}$$

Equation (18) involves a first order autoregressive process. $\mathbf{T}_{(k)}$ is the "transition matrix." The random vector $\mathbf{w}_{(k)}$ has zero-mean uncorrelated components with the covariance matrix $\mathbf{\Sigma}_{w(k)} = \sigma_0^2\mathbf{Q}_{w(k)} = \sigma_0^2\mathbf{P}_{w(k)}^{-1}$.

Previous to the stage k the estimates $\mathbf{x}_{k-1/k-1}$ of the state vector and the associated weight coefficient matrix $\mathbf{Q}_{k-1/k-1}$ are available. The Kalman updating equations for stage k are given by

$$\mathbf{x}_{k/k-1} = \mathbf{T}_{(k)}\mathbf{x}_{k-1/k-1}, \tag{19a}$$

$$\mathbf{Q}_{k/k-1} = \mathbf{T}_{(k)}\mathbf{Q}_{k-1/k-1}\mathbf{T}^{\mathrm{T}}_{(k)} + \mathbf{Q}_{w(k)}, \tag{19b}$$

$$\mathbf{H}_{(k)} = \mathbf{P}_{(k)} + \mathbf{A}_{(k)}\mathbf{Q}_{k/k-1}\mathbf{A}^{\mathrm{T}}_{(k)}, \tag{19c}$$

$$\mathbf{K}_{(k)} = \mathbf{Q}_{k/k-1}\mathbf{A}^{\mathrm{T}}_{(k)}\mathbf{H}^{-1}_{(k)}, \tag{19d}$$

$$\mathbf{x}_{k/k} = \mathbf{x}_{k/k-1} + \mathbf{K}_{(k)}(\mathbf{l}_{(k)} - \mathbf{A}_{(k)}\mathbf{x}_{k/k-1}), \tag{19e}$$

$$\mathbf{Q}_{k/k} = \mathbf{Q}_{k/k-1} - \mathbf{K}_{(k)}\mathbf{A}_{(k)}\mathbf{Q}_{k/k-1}. \tag{19f}$$

$\mathbf{K}_{(k)}$ is the "Kalman gain" matrix.

In on-line triangulation applications the state vector is not assumed to follow an autoregressive process, that is, it is not supposed to switch stages as a random variable. With the specifications

$$\mathbf{T}_{(k)} = \mathbf{I} \text{ and } \mathbf{Q}_{w(k)} = 0, \tag{20}$$

we get the equations

$$\mathbf{K}_{(k)} = \mathbf{Q}_{k-1/k-1}\mathbf{A}^{\mathrm{T}}_{(k)}(\mathbf{P}_{(k)} + \mathbf{A}_{(k)}\mathbf{Q}_{k-1/k-1}\mathbf{A}^{\mathrm{T}}_{(k)})^{-1}, \tag{21a}$$

$$\mathbf{x}_{k/k} = \mathbf{x}_{k-1/k-1} + \mathbf{K}_{(k)}(\mathbf{l}_{(k)} - \mathbf{A}_{(k)}\mathbf{x}_{k-1/k-1}), \text{ and} \tag{21b}$$

$$\mathbf{Q}_{k/k} = \mathbf{Q}_{k-1/k-1} - \mathbf{K}_{(k)}\mathbf{A}_{(k)}\mathbf{Q}_{k-1/k-1}. \tag{21c}$$

Equations of the form 21a, 21b, and 21c were used by Mikhail and Helmering (1973), Helmering (1977), Dorrer (1978), Dowideit (1980), Rosculet (1980), and Kratky (1980b). Because of their close relation to the original Kalman Equations 19d, 19e, and 19f, they are called the "Kalman form" update equations (Dorrer, 1978). The key element here is the computation of the gain matrix $\mathbf{K}_{(k)}$, which involves the weight coefficient matrix $\mathbf{Q}_{k-1/k-1}$ of the previous stage. $\mathbf{K}_{(k)}$ is used for the updating of the state vector and the associated covariance matrix.

This Kalman mechanism is designed for a constant size state vector. A modification, which also accommodates a state vector of varying size, is given by Mikhail and Helmering (1973). In the same publication an expensive updating formula for the $\mathbf{Q}_{vv}$ matrix is suggested.

Triangular Factor Update (TFU). Assuming that Equation 4a represents the linearized estimation model at the stage $k - 1$ of the sequential process, we get the following system if one or more observation equations are added, including new parameters $\mathbf{x}_{(k)}$ and $\mathbf{t}_{(k)}$:

$$\begin{aligned} -\mathbf{e} &= \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{t} - \mathbf{l};\ \mathbf{P} \\ -\mathbf{e}_{(k)} &= \mathbf{A}_{1(k)}\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_{(k)}\end{bmatrix} + \mathbf{A}_{2(k)}\begin{bmatrix}\mathbf{t}\\ \mathbf{t}_{(k)}\end{bmatrix} \\ &\quad - \mathbf{l}_{(k)};\ \mathbf{P}_{(k)} \end{aligned} \tag{22}$$

The updated normal equations of the stage k are of the form

$$\dot{\mathbf{N}}\begin{bmatrix}\hat{\dot{\mathbf{x}}}\\ \hat{\dot{\mathbf{t}}}\end{bmatrix} = \begin{bmatrix}\dot{\mathbf{l}}_x\\ \dot{\mathbf{l}}_t\end{bmatrix}, \tag{23}$$

with the recursions

$$\hat{\dot{\mathbf{x}}} \triangleq \begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{x}}_{(k)}\end{bmatrix},\ \hat{\dot{\mathbf{t}}} \triangleq \begin{bmatrix}\hat{\mathbf{t}}\\ \hat{\mathbf{t}}_{(k)}\end{bmatrix}$$

and

$$\dot{\mathbf{N}} = \begin{bmatrix}\dot{\mathbf{N}}_{xx} & \dot{\mathbf{N}}_{xt}\\ \dot{\mathbf{N}}^{\mathrm{T}}_{xt} & \dot{\mathbf{N}}_{tt}\end{bmatrix},\ \begin{aligned}\dot{\mathbf{l}}_x &= \mathbf{l}_x^{(0)} + \mathbf{A}^{\mathrm{T}}_{1(k)}\mathbf{P}_{(k)}\mathbf{l}_{(k)}\\ \dot{\mathbf{l}}_t &= \mathbf{l}_t^{(0)} + \mathbf{A}^{\mathrm{T}}_{2(k)}\mathbf{P}_{(k)}\mathbf{l}_{(k)}\end{aligned},$$

$$\begin{aligned}\dot{\mathbf{N}}_{xx} &= \mathbf{N}_{xx}^{(0)} + \mathbf{A}^{\mathrm{T}}_{1(k)}\mathbf{P}_{(k)}\mathbf{A}_{1(k)},\\ \dot{\mathbf{N}}_{xt} &= \mathbf{N}_{xt}^{(0)} + \mathbf{A}^{\mathrm{T}}_{1(k)}\mathbf{P}_{(k)}\mathbf{A}_{2(k)}, \text{ and}\\ \dot{\mathbf{N}}_{tt} &= \mathbf{N}_{tt}^{(0)} + \mathbf{A}^{\mathrm{T}}_{2(k)}\mathbf{P}_{(k)}\mathbf{A}_{2(k)}.\end{aligned}$$

The superscripts (0) indicate that, if new parameters $\mathbf{x}_{(k)}$ and $\mathbf{t}_{(k)}$ are added, the column/row spaces of the original $\mathbf{N}_{xx}$, $\mathbf{N}_{xt}$, and $\mathbf{N}_{tt}$ matrices have to be extended by zero vectors and the row spaces of the original vectors $\mathbf{l}_x$ and $\mathbf{l}_t$ by zero elements accordingly.

With formal notations the updating of the k − 1 stage normals can be described as

$$(\mathbf{N} + \Delta\mathbf{N})\begin{bmatrix}\hat{\dot{\mathbf{x}}}\\ \hat{\dot{\mathbf{t}}}\end{bmatrix} = \begin{bmatrix}\mathbf{l}_x + \Delta\mathbf{l}_x\\ \mathbf{l}_t + \Delta\mathbf{l}_t\end{bmatrix}. \tag{24}$$

The addition of the term $\Delta\mathbf{N}$ to the $k - 1$ normals will result in alterations of the matrix factors $\mathbf{L}$ and $\mathbf{U}$; i.e.,

$$(\mathbf{U} + \Delta\mathbf{U})\begin{bmatrix}\hat{\dot{\mathbf{x}}}\\ \hat{\dot{\mathbf{t}}}\end{bmatrix} = (\mathbf{L} + \Delta\mathbf{L})^{-1}\begin{bmatrix}\mathbf{l}_x + \Delta\mathbf{l}_x\\ \mathbf{l}_t + \Delta\mathbf{l}_t\end{bmatrix}. \tag{25}$$

If the prereduction concept is applied, we obtain

$$(\mathbf{N}_R + \Delta\mathbf{N}_R)\,\hat{\dot{\mathbf{t}}} = \mathbf{l}_R + \Delta\mathbf{l}_R, \tag{26}$$

and finally

$$(\mathbf{N}_{RR} + \Delta\mathbf{N}_{RR})\,\hat{\dot{\mathbf{t}}} = \mathbf{l}_{RR} + \Delta\mathbf{l}_{RR}. \tag{27}$$

For the deletion of observations and/or parameters, the same approach can be used; just the signs of the correction terms have to be reversed and the matrix/vector spaces have to be adjusted. These formal presentations do not indicate the computational amount involved to compute the corrections $\Delta\mathbf{N}_R$ and $\Delta\mathbf{l}_R$ or $\Delta\mathbf{N}_{RR}$ and $\Delta\mathbf{l}_{RR}$. The specific sparsity structures of the normal matrices, especially the one of the $\mathbf{N}_{xx}$ matrix, allow for a very efficient updating procedure (for details, see Gruen (1982)). First, the $\mathbf{N}_{xx}$ hyperdiagonal structure yields significant storage savings, because the normals of only one object point have to be kept in core memory during the factorization. Second, if observations belonging to a specific object point i have to be manipulated, only those submatrices $\mathbf{N}_{xx}^i$ and $\mathbf{N}_{xt}^i$ which refer to this point have to be altered and refactorized in sequential estimation. This allows, even on minicomputers, for fast "core memory only" operations.

Individual elements/rows of the $\mathbf{Q}_{vv}$ matrix can be efficiently computed by utilizing the upper triangle $\dot{\mathbf{U}} = \mathbf{U} + \Delta\mathbf{U}$ of the latest stage and applying the method of "unit observation vector", as explained by Gruen (1982). Another option utilizes the standard formula

$$\mathbf{Q}_{vv} = \overline{\mathbf{P}}^{-1} - \mathbf{A}\dot{\mathbf{N}}^{-1}\mathbf{A}^{\mathrm{T}}, \qquad (28)$$

where $\mathbf{A}$ is the total design matrix of Equation 22 and $\overline{\mathbf{P}}$ the total weight matrix.

$\dot{\mathbf{N}}^{-1}$ can be replaced by the Gauss decomposition

$$\dot{\mathbf{N}}^{-1} = (\dot{\mathbf{U}}^{\mathrm{T}}\dot{\mathbf{D}}\dot{\mathbf{U}})^{-1} = \dot{\mathbf{U}}^{-1}\dot{\mathbf{D}}^{-1}(\dot{\mathbf{U}}^{-1})^{\mathrm{T}}, \qquad (29)$$

resulting in

$$\mathbf{Q}_{vv} = \overline{\mathbf{P}}^{-1} - \mathbf{A}\dot{\mathbf{U}}^{-1}\dot{\mathbf{D}}^{-1}(\mathbf{A}\dot{\mathbf{U}}^{-1})^{\mathrm{T}}. \qquad (30)$$

Equation 30 is computationally more efficient than Equation 28 because it uses the inverse $\dot{\mathbf{U}}^{-1}$, which is an upper triangle, instead of the full inverse $\dot{\mathbf{N}}^{-1}$. Hence, $\dot{\mathbf{N}}^{-1}$ (equivalent to $\mathbf{Q}_{k/k}$ of the Kalman update Equation 21c) is not necessary for $\mathbf{Q}_{vv}$ computations, and need not to be computed unless it is required for certain statistical analysis purposes. The basic TFU operations of deletion/addition of image points, object points, and photographs are described in detail with the help of examples in Gruen (1982) and supported by more examples in Wyatt (1982).

For statistical analysis, e.g., for blunder detection, the variance factor σ_0^2 is needed. $\hat{\sigma}_0^2$ can be computed with the TFU technique without computing first the solution vectors $\mathbf{x}$ and $\mathbf{t}$. Expand the right hand side in Equation 23 by the element $\mathbf{l}_s = \mathbf{l}^{\mathrm{T}}\mathbf{P}\mathbf{l} + \mathbf{l}^{\mathrm{T}}_{(k)}\mathbf{P}_{(k)}\mathbf{l}_{(k)}$. Reduce the extended normal equation system $\dot{\mathbf{N}}_E$ to the upper triangular form

$$\dot{\mathbf{N}}_E = \left[\begin{array}{c|c} \dot{\mathbf{N}} & \begin{matrix}\dot{\mathbf{l}}_x \\ \dot{\mathbf{l}}_t\end{matrix} \\ \hline \dot{\mathbf{l}}_x\ \dot{\mathbf{l}}_t & \mathbf{l}_s \end{array}\right] \rightarrow \left[\begin{array}{c|c} \dot{\mathbf{U}} & \bar{\mathbf{l}} \\ \hline \mathbf{0} & \Omega \end{array}\right]. \qquad (31)$$

It can be shown that Ω is

$$\Omega = (\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v})^{1/2}, \qquad (32)$$

so we get $\hat{\sigma}_0^2$ to

$$\hat{\sigma}_0^2 = \frac{\Omega^2}{r} \qquad (33)$$

in which r is the system redundancy.

Computationally, this extended reduction is easy to handle. The reduction which generates $\mathbf{l}_{RR} + \Delta\mathbf{l}_{RR}$ simply has to be carried one step further, thus including the element $\mathbf{l}_s$. This approach can be readily included in the sequential TFU mechanism.

Orthogonalization with Givens Transformations. Sequential estimation with orthogonal transformations using QR decompositions is described by Lawson and Hanson (1974). Both additions/deletions of column and row vectors of the $\mathbf{A}$ matrix are discussed there. Householder transformations as well as Givens rotations are used. Blais (1983) recommended the application of Givens rotations for the sequential treatment of surveying and photogrammetry networks. This approach uses the estimation model (Equation 4a). Instead of obtaining the updated upper triangular matrix $\dot{\mathbf{U}} = \mathbf{U} + \Delta\mathbf{U}$ (Equation 25) by means of Gauss factorization of the normal equations, it applies Givens transformations directly to the upper triangular matrix $\mathbf{U}$ of the previous stage.

At stage $k-1$ the reduced system (Equation 10) takes the form

$$\mathbf{U}\begin{bmatrix}\hat{\mathbf{x}} \\ \hat{\mathbf{t}}\end{bmatrix} = \mathbf{L}^{-1}\begin{bmatrix}\mathbf{l}_x \\ \mathbf{l}_t\end{bmatrix} = \mathbf{d}. \qquad (10)$$

Adding one observation equation, including a set of new parameters $\mathbf{y}$, to this system results in stage k and gives (with $\mathbf{P}_{(k)} = \mathbf{I}$)

$$\left[\begin{array}{c|c} \mathbf{U} & \mathbf{0} \\ & \\ \mathbf{a}^{\mathrm{T}}_{(k)} & \end{array}\right]\left[\begin{array}{c} \mathbf{x} \\ \mathbf{t} \\ \hline \mathbf{y}\end{array}\right] = \left[\begin{array}{c} \mathbf{d} \\ \hline \mathbf{l}_{(k)}\end{array}\right] \qquad (34)$$

in which

- $\mathbf{y}$ is the new parameter vector of length p,
- $\mathbf{a}^{\mathrm{T}}_{(k)}$ is the row vector with coefficients of new observation equation, and
- $\mathbf{l}_{(k)}$ is the right hand side of new observation equation.

Applying a series of orthogonal Givens transformations

$$\mathbf{G} = \mathbf{G}_n\mathbf{G}_{n-1}\ \ldots\ \mathbf{G}_1 \qquad (35)$$

(n is the total number of system parameters) to Equation 34 results in

$$\mathbf{G}\left[\begin{array}{cc} \mathbf{U} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline \mathbf{a}^{\mathrm{T}}_{(k)} & \end{array}\right]\begin{array}{l}\}\ n-p \\ \}\ p \\ \}\ 1\end{array} = \left[\begin{array}{c} \dot{\mathbf{U}} \\ \hline 0 \end{array}\right]\begin{array}{l}\Big\}\ \mathbf{n} \\ \}\ 1\end{array}, \qquad (36a)$$

$$\mathbf{G}\left[\begin{array}{c} \mathbf{d} \\ \hline \mathbf{0} \\ \hline \mathbf{l}_{(k)} \end{array}\right]\begin{array}{l}\}\ n-p \\ \}\ p \\ \}\ 1\end{array} = \left[\begin{array}{c} \dot{\mathbf{d}} \\ \hline \bar{\mathbf{l}}_{(k)} \end{array}\right]\begin{array}{l}\Big\}\ n \\ \}\ 1\end{array}. \qquad (36b)$$

The updated solution vector can be found by back-substitution into

$$\dot{\mathbf{U}}\begin{bmatrix}\hat{\mathbf{x}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{y}}\end{bmatrix} = \dot{\mathbf{d}}. \qquad (37)$$

The sparsity patterns of both $\mathbf{U}$ and $\mathbf{a}^{\mathrm{T}}_{(k)}$ can be exploited advantageously. The updated covariance matrix of the parameters can be generated using essentially the same approach as for the updated parameters. Another option is to derive it from the upper triangle **U** using the equivalent Equation 14. Methods for the deletion of observations and the addition and deletion of parameters are described in Golub (1969) and Lawson and Hanson (1974). Some of these methods fit nicely into the mechanization of the Givens approach. Deletion of observations can be handled by introducing these observation equations with negative weights into the standard format (Equation 34). Complex arithmetic is avoided in computations.

For the deletion of parameters one simply cuts out the corresponding columns of the upper triangle **U** and transforms the remaining matrix to upper triangular form with Givens matrices. The transformation of vector **d** is also necessary.

Several options are available for the computation of the $\mathbf{Q}_{vv}$ matrix. If the orthogonal transformation matrix $\mathbf{G} = \mathbf{G}_n\mathbf{G}_{n-1} \ldots \mathbf{G}_1$ was stored, which is not very likely if storage is at a premium on small, dedicated computers, $\mathbf{Q}_{vv}$ as a whole or individual elements/columns can be computed according to Equation 41 which is derived in the following. With a weight matrix **P**, the factorization of $\mathbf{P}^{1/2}\mathbf{A}$ with Givens transformations results in

$$\mathbf{P}^{1/2}\mathbf{A} = \overline{\mathbf{A}} = \mathbf{G}^{\mathrm{T}} \begin{bmatrix} \mathbf{U} \\ \hline \mathbf{0} \end{bmatrix} \begin{matrix} \} \; n \\ \} \; m-n \end{matrix} . \tag{38}$$

With

$$(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1} = (\overline{\mathbf{A}}^{\mathrm{T}}\overline{\mathbf{A}})^{-1} = \mathbf{U}^{-1}\mathbf{U}^{-\mathrm{T}},$$

$$\overline{\mathbf{v}} = \mathbf{P}^{1/2}\mathbf{v}, \; \mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \hline \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{matrix} \} \; n \\ \} \; m-n \end{matrix} \tag{39}$$

where

m = number of observation equations,
n = number of system parameters, and
U is the upper triangle of the latest stage,

we obtain

$$\mathbf{Q}_{\overline{v}\overline{v}} = \mathbf{I} - \overline{\mathbf{A}}(\overline{\mathbf{A}}^{\mathrm{T}}\overline{\mathbf{A}})^{-1}\overline{\mathbf{A}}^{\mathrm{T}}$$

$$= \mathbf{I} - \mathbf{G}^{\mathrm{T}} \begin{bmatrix} \mathbf{U} \\ \mathbf{0} \end{bmatrix} \mathbf{U}^{-1}(\mathbf{U}^{-1})^{\mathrm{T}} \, [\mathbf{U}^{\mathrm{T}} \, \mathbf{0}] \, \mathbf{G} \tag{40}$$

$$= \mathbf{I} - \mathbf{G}^{\mathrm{T}} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} [\mathbf{I} \; \mathbf{0}] \, \mathbf{G} = \mathbf{I} - \begin{bmatrix} \mathbf{G}_{11}^{\mathrm{T}} \\ \mathbf{G}_{12}^{\mathrm{T}} \end{bmatrix} [\mathbf{G}_{11} \; \mathbf{G}_{12}],$$

or finally

$$\mathbf{Q}_{vv} = \mathbf{P}^{-1} - \mathbf{P}^{-1/2} \begin{bmatrix} \mathbf{G}_{11}^{\mathrm{T}} \\ \mathbf{G}_{12}^{\mathrm{T}} \end{bmatrix} [\mathbf{G}_{11} \; \mathbf{G}_{12}] \, \mathbf{P}^{-1/2}. \tag{41}$$

Equation 41 utilizes only the submatrices $\mathbf{G}_{11}$ and $\mathbf{G}_{12}$ of the Givens matrix **G**. The equation is particularly inexpensive if **P** is diagonal.

If **G** has not been stored, one can use for the $\mathbf{Q}_{vv}$ computation the same techniques that have been suggested for the TFU Update. Either the method of "unit observation vector" or Equation 30 might be used, both exploiting the latest stage upper triangular matrix **U**. $\hat{\sigma}_0^2$ can be computed similarly as with the TFU technique.

An orthogonal factorization of the observation Equation 4a yields

$$\mathbf{G} \, \mathbf{P}^{1/2} \, [\mathbf{A}_1 \; \mathbf{A}_2] = \begin{bmatrix} \mathbf{U} \\ \mathbf{0} \end{bmatrix}, \; \mathbf{G}\mathbf{P}^{1/2}\mathbf{l} = \begin{bmatrix} \mathbf{d} \\ \mathbf{d}_2 \end{bmatrix}, \tag{42}$$

and the least-squares solution is obtained from

$$\mathbf{U} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{t}} \end{bmatrix} = \mathbf{d}. \tag{43}$$

It can be shown (Lawson and Hanson, 1974, p. 6) that

$$\Omega = (\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v})^{1/2} = \mathbf{d}_2^{\mathrm{T}}\mathbf{d}_2. \tag{44}$$

Hence $\hat{\sigma}_0^2 = \Omega^2/r$ can easily be derived from the lower portion $\mathbf{d}_2$ of the transformed right hand side.

The sequential updating of Ω can be achieved by simply adding Ω to **d** and updating Ω with Givens transformations (Gentleman, 1973).

$$\mathbf{G} \begin{bmatrix} \mathbf{U} \\ 0 \\ \mathbf{a}^{\mathrm{T}}_{(k)} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{U}} \\ 0 \\ 0 \end{bmatrix}, \; \mathbf{G} \begin{bmatrix} \mathbf{d} \\ \Omega \\ \mathrm{l}_{(k)} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{d}} \\ \dot{\Omega} \\ 0 \end{bmatrix}. \tag{45}$$

Operational considerations

Truly sequential on-line triangulation algorithms can be evaluated in two ways. Theoretical operation counts and storage requirement counts can be combined with conceptual considerations and the experience gained in other application areas. A better basis for evaluation is provided by the actual programming of the algorithms, and the testing and comparison in a real world environment, which includes all the overhead for sorting routines, bookkeeping, and I/0 operations as well.

The major requirements concerning an efficient algorithm can be summarized as follows:

- low computer storage;
- operations in central memory;
- fast response to a variety of operations, such as the addition/deletion of individual and groups of image point observations, point parameters, and camera parameters, the test and modification of a set of additional parameters, and the computation of individual elements or submatrices of the $\mathbf{Q}_{vv}$ matrix; and
- accommodation of an updating procedure for initial values.

In recent years some general objections to the Kalman Covariance Update have been formulated, such as the problem of selecting proper *a priori* statistics, the effect of unmodeled parameters, the divergence due to the presence of non-linearities, and the effect of computer round-off arising from the generally low numerical stability of this technique (Bierman, 1977). Especially the non-linearity problem can be very critical in on-line triangulation applications (Gruen, 1983b). The major drawbacks of this method as applied to on-line triangulation problems, however, are its poor adaptability to sparse matrix systems and its computational and storage requirements associated with the updating of a state vector of varying size.

In Wyatt (1982) and Gruen and Wyatt (1983) a thorough investigation of the Kalman Covariance Update and the Triangular Factor Gauss/Cholesky Update was performed, based on comparisons of computing times and storage requirements. The following on-line triangulation operations were considered:

- additions/deletions of individual and groups of observations,
- additions/deletions of groups of parameters, and
- computations of individual elements of the $\mathbf{Q}_{vv}$ matrix.

The results demonstrated clearly the superiority of the TFU technique. Critical computing times were better by factors of 4 to 31, depending on the type of operation and the size of the system. Storage savings factors of 3 to 5 were reported.

Although the TFU technique is a fairly fast algorithm, without the use of special computer hardware such as array processors the truly sequential estimation in medium and large size blocks still would result in computing times that could not satisfy real-time requirements. Hence, in Gruen (1982) an on-line triangulation concept was outlined which acknowledges the facts that the internal reliability of photogrammetric blocks has a very local structure and that $\mathbf{v}$ and $\mathbf{Q}_{vv}$ are invariant with respect to a non-redundant datum choice. It was suggested that only blocksubsystems be used in a truly sequential mode. A 3 by 3 = 9 photograph subsystem was recommended as a maximum configuration in case of 60 percent sidelap. This subsystem is used in a window mode, and shifted systematically over the entire block.

Major problems have been encountered with sequential estimation in a non-linear system such as the bundle method. Sequential estimation requires a unique set of initial values to be used throughout the entire process. A sequential updating of the parameters is not allowed, unless the whole system with all its coefficients in observation and normal equations, etc., is updated. Obviously, if the initial values are rather crude, the sequential solution will constantly be evaluating larger and larger corrections for parameters, until the parameter vector virtually "drifts off," i.e., the solution will no longer be correct. Furthermore, blunders can seriously deteriorate the system if not discovered and removed before the corresponding observations are finally introduced into the system. This might happen if the local reliability for erroneous observations is still weak at a certain stage because not all available rays have been included yet, thus not allowing the detection of the blunder at this stage. There are several options to tackle these problems. Some of the difficulties can be avoided if a rather strict method for the computation of initial values is used, such as a dependent relative orientation plus scale transfer for the newly incoming photographs prior to the truly sequential updating. The most drastic and safest measure is the evaluation of a simultaneous solution at various stages of the process. This is always appropriate when the operator is busy with measurements or other tasks that do not require the sequential routine to be activated. An optimal moment would be after the introduction of a new model or photograph, because the greatest errors in initial values are to be expected by then.

The choice of Givens transformations over other orthogonalization methods such as Householder or Gram-Schmidt is based on the inherent advantages of Givens transformations for sequential updating. The sparsity of matrices is better preserved and exploited with Givens transformations. The Householder and Gram-Schmidt algorithms produce many intermediate fill-ins, which might finally be reduced to zero, but the minimum storage requirements for the upper triangle $\mathbf{U}$ are exceeded considerably and also the computing times are increased. While Householder and Gram-Schmidt transformations usually require access to all columns of the unreduced part of $\mathbf{A}$ during computation, Givens transformations allow the processing of rows of $\mathbf{A}$ one by one, so $\mathbf{A}$ can be accessed in a natural way, according to the acquisition sequence of observations. Permutation of matrices for the purposes of better exploiting sparsity patterns or numerical stabilization is not appropriate for on-line triangulation. The total sparsity pattern is available anyway only after the last row of $\mathbf{A}$ is accumulated. One should rather acquire the measurements in a sequence that leads naturally to near-optimal matrix patterns. Any possible gain achieved by permutations is widely offset by the additional sorting load involved in these operations.

Some relevant aspects related to the solution of sparse systems using Givens transformations are discussed by George and Heath (1980). The often cited advantage of orthogonal transformations with respect to their superior numerical stability is not a critical issue in on-line triangulation applications.

When comparing algorithms with each other, one should consider realistic on-line triangulation operations. To allow for the variation in the size of the solution vector is one of those indispensible demands. Another fact to be considered is the rhythm for the acquisition and checking of image coordinate observations. An operator surely would not like to interrupt his measurement procedure after each single image point in order to check its agreement with the adjustment model. By acquiring sets of observations and by sequentially inserting them as such into the adjustment model, the reliability and thus the error detection properties of the system improve faster, and the operator is not permanently disturbed and distracted.

Conclusions

The development of data processing equipment and triangulation techniques has reached a stage where further significant improvements in efficiency and reliability of the overall triangulation procedure can only be achieved through the utilization of sophisticated on-line triangulation techniques. As emphasized in this paper, the computational algorithm is a core element in any on-line triangulation procedure. Powerful sequential estimation algorithms are available. If they are to be used in on-line triangulation, some of them need still to be tuned and adjusted to the particular situation with regard to matrix structures, acquisition rhythm of measurements, expansion or reduction of solution vectors, and variables required for statistical analysis. Those algorithms which have already been suggested for on-line triangulation need to be compared to each other in a practical environment. Recommendations for the professional community, in particular for equipment manufacturers, must be formulated as to the use of these algorithms and to modifications and expansions of measurement procedures. For instance, in order to detect blunders at a most early stage, measurements and computations across the strip direction on analytical plotters should be done as early as possible. This allows for a rapid improvement of the system's internal reliability. The operator should utilize the ease and speed with which remeasurements can be performed on analytical plotters. Whenever possible (small format photographs, large stage analytical plotters), more than one model should be placed on the stages. Access to different models is then feasible without significant time delay. With analytical across strip positioning, the marking and transfer of artificial tie points is no longer necessary. What all this amounts to is that one should get rid of conventional measurement and computational concepts that are related to the capabilities of analog instruments and comparators and that do not consider the existing possibilities that are offered by modern analytical plotters.

Future developments will see the use of array processors for further speed-up of computations. Thus, the sequentially manageable block size will increase. Sequential algorithms will be developed and tailored to the specific matrix structures arising with the use of non-conventional sensor geometries. The on-line triangulation procedure with its quasi-real-time responses will not only serve for data processing, but will also support the design of a triangulation network. Softcopy analytical plotters will be used that allow one to perform measurements and remeasurements even faster than is currently possible. With the advent of more powerful and reliable correlation methods, the measurement and transfer of tie points will be performed automatically in the digital domain. Semi-automatic or fully automatic digital real-time triangulation systems can thus be envisioned for a future not too far ahead.

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