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A Combined Photogrammetric and Doppler Adjustment

Because of the massive dimensions of a combined adjustment, a sequential adjustment of Doppler and photogrammetric data is a more practical procedure.

Abstract: *The feasibility of a combined, simultaneous adjustment of aerial photogrammetric data and Doppler satellite observations at ground stations is studied. Photogrammetric and Doppler condition equations are developed and formed into one system for which a solution by the method of least squares is discussed. The resulting system of equations is of massive proportions so that a simultaneous adjustment is not practical. A sequential least-squares adjustment is possible and merits further study as a potential solution to the system.*

Introduction

During the past decade, analytical photogrammetric triangulation using aerial photography (the bundle adjustment) has been fully developed and is now accepted as a legitimate procedure for the determination of the positions for ground points. Concurrently with these advances in photogrammetry, a great deal of work has also occurred in the use of Doppler satellite signals for calculating basic geodetic positions. Doppler satellite positioning has particular relevance in the more remote and inaccessible parts of the world but can also be used for densification of existing networks in the more highly developed regions.

With the current interest in densifying control networks throughout the world and because a primary requisite for the aerial bundle adjustment is the presence of control points of known geodetic positions, it is natural to consider a combination of the photogrammetric and Doppler adjustments.

Combined adjustments of this type are not without precedent. Hartwell *et al.* (1973) developed a procedure in 1973 for a combined photogrammetric tracking and network analysis which includes photogrammetric observations of a satellite from ground stations, ranging measurements from ground stations to a satellite, and photogrammetric data from a sensor in the satellite. Brown and Trotter (1969) have a program for simultaneous reduction of Doppler satellite data and photogrammetric measurements from fixed ground stations. Hartwell's approach was aimed at improving orbital parameters using all available data. Brown's and Trotter's method was designed to obtain positions for ground points using Doppler observations and photogrammetric data from terrestrial cameras. Both of these procedures allow orbital parameters to be included as unknowns in the adjustment.

Along more conventional lines, the manufacturers of doppler receivers (JMR, Magnavox, etc.) usually provide a software package which permits calculating the positions of occupied points. Most users prefer to develop their own more general programs which frequently include some of the orbital parameters as unknowns. Kouba (1974) and Hittel and Kouba (1971) have developed programs which include three orbital parameters as unknowns in the adjustment. Wells (1974) also has a Doppler satellite reduction program in which the main objective is to determine ground point positions. Wells permits some relaxation of these orbital parameters but does not include them as unknowns in the solution.

The objective of this investigation is to study the feasibility of simultaneously treated conventional aerial photogrammetric data and Doppler satellite observations in a single combined adjustment. Emphasis is placed on a brief explanation of the photogrammetric adjustment, a review of Doppler reduction techniques, and the formation and solution of the resulting combined system of equations.

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Photogrammetric Bundle Adjustment

The basic relationship is the collinearity equation, which may be expressed for object point j from exposure station i as

$$
\begin{aligned}
x_{ij} &= -c\,\frac{(X_j - X_{0i})r_{11} + (Y_j - Y_{0i})r_{21} + (Z_j - Z_{0i})r_{31}}{(X_j - X_{0i})r_{13} + (Y_j - Y_{0i})r_{23} + (Z_j - Z_{0i})r_{33}} = -c\,\frac{T_x}{N} \\
y_{ij} &= -c\,\frac{(X_j - X_{0i})r_{12} + (Y_j - Y_{0i})r_{22} + (Z_j - Z_{0i})r_{32}}{(X_j - X_{0i})r_{13} + (Y_j - Y_{0i})r_{23} + (Z_j - Z_{0i})r_{33}} = -c\,\frac{T_y}{N}
\end{aligned}
\tag{1}
$$

in which

$(x, y)_{ij}$ = refined photographic coordinates of point j on photograph i,
$[X, Y, Z]_j^{\mathrm{T}}$ = terrain coordinates of point j,
$[X, Y, Z]_{0i}^{\mathrm{T}}$ = coordinate of exposure station i,
$r_{11}, r_{12}, \ldots, r_{33}$ = elements of orientation matrix **R** and are functions of ω, ϕ, κ rotations which define camera orientation, and
c = camera focal length (calibrated).

Equations 1 in functional form are

$$
\begin{aligned}
Fx_{ij} &= x_{ij} + c\,\frac{T_x}{N} \\
Fy_{ij} &= y_{ij} + c\,\frac{T_y}{N}
\end{aligned}
\tag{2}
$$

Linearization of Equations 2 using a Taylor series expansion (neglecting second and higher order terms) at initial approximations for $[X^0\ Y^0\ Z^0]_j^{\mathrm{T}}$, $[X^0\ Y^0\ Z^0]_{0i}^{\mathrm{T}}$, and $[\omega^0\ \phi^0\ \kappa^0]_{0i}^{\mathrm{T}}$ and at measured values for $(\mathrm{x}, \mathrm{y})_{\mathrm{ij}}$ yields

$$
\begin{bmatrix} v_x \\ v_y \end{bmatrix}_{ij} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \end{bmatrix}_{ij} \begin{bmatrix} \delta\omega \\ \delta\phi \\ \delta\kappa \\ \delta X \\ \delta Y \\ \delta Z \end{bmatrix}_{0i} + \begin{bmatrix} -b_{14} & -b_{15} & -b_{16} \\ -b_{24} & -b_{25} & -b_{26} \end{bmatrix}_j \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_j = \begin{bmatrix} fx \\ fy \end{bmatrix}_{ij}
\tag{3}
$$

in which

$$
b_{11} = \left.\frac{\partial Fx_{ij}}{\partial \omega_{0i}}\right|_{\omega_{0i}^0},\ b_{12} = \left.\frac{\partial Fx_{ij}}{\partial \phi_{0i}}\right|_{\phi_{0i}^0},\ \ldots,\ b_{26} = \left.\frac{\partial Fy_{ij}}{\partial Z_{0i}}\right|_{Z_{0i}^0}
$$

and fx_{ij}, fy_{ij} are calculated using ω_{0i}^0, ϕ_{0i}^0, . . . , Z_{0i}^0 in Equations 2.

Equations 3 can be expressed more compactly as

$$
\underset{2,1}{\mathbf{V}_{ij}} + \underset{2,6}{\boldsymbol{\alpha}_{ij}}\ \underset{6,1}{\boldsymbol{\delta}_i} + \underset{2,3}{\boldsymbol{\beta}_{ij}}\ \underset{3,1}{\boldsymbol{\Delta}_j} = \underset{2,1}{\mathbf{f}_{ij}}
\tag{4}
$$

that is of the general form

$$
\mathbf{A\,V} + \mathbf{B\,\Delta} = \mathbf{f}
\tag{5}
$$

where

$$
\mathbf{A} = \mathbf{I},\ \mathbf{B} = [\boldsymbol{\alpha}_{ij} | \boldsymbol{\beta}_{ij}],\ \text{and}\ \boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\delta}_i \\ \hline \boldsymbol{\Delta}_j \end{bmatrix}
$$

A collinearity equation (Equation 4) is formed for each image of all object points (known and unknown) and the resulting system is solved for corrections to estimates for unknown object points and exposure station parameters in a least-square adjustment. As a rule, weights are assigned to measured photographic coordinates as well as to exposure station and ground point parameters so that it is an adjustment with constraints on unknown parameters. Further details of such an adjustment can be found in Anderson *et al.* (1975) and Brown *et al.* (1964).

Doppler Positioning Techniques

Two basic approaches are employed where

(1) Orbital data as derived from the broadcast ephemeris are assumed error free and locations are determined by

- Point positioning
- Translocation

(2) Orbital data are treated as observations and are permitted to adjust in the data reduction. Positions are determined by

- Short arc geodetic adjustment
- Short arc translocation

If the so called precise ephemeris is available, the need for adjustment of orbital parameters, although not eliminated, is reduced for many projects. The problem is that the precise ephemeris is difficult to acquire and is available for only certain satellites. Consequently, use of the precise ephemeris lowers the practicality of the method and will not be considered further in this study where all developments presuppose use of the broadcast ephemeris.

ORBITAL DATA CONSIDERED ERRORLESS

Point Positioning (using geodetic doppler receivers). In this case, one Doppler receiver is used and counts from a large number of passes (perhaps 100) are recorded. These data are then reduced independently for the occupied station holding the orbital data from the broadcast ephemeris fixed. The unknowns are the X, Y, Z coordinates of the observation station and a frequency shift for each satellite pass.

When data for all passes have been filtered and edited, then X, Y, Z coordinates in the geocentric system are computed for each single station using all acceptable counts in all passes observed by that station. The orbital parameters and frequency shift may be held fixed. Thus, in a multi pass adjustment, the unknowns are simply the X, Y, Z coordinates of the station. Alternatively, the frequency shift and three orbital biases (ΔE, correction to eccentric anomaly; Δa, correction to semi-major axis; and η, out of plane orbit component) may be relaxed and treated as unknowns with appropriate *a priori* constraints.

Using this method, it is estimated (Wells, 1974) that root mean square (RMS) errors of 2 to 3 m in ground point positions can be obtained when up to 100 passes per station are observed and the broadcast ephemeris is used. With a geodetic receiver such as the JMR 1, 100 to 200 counts per pass are obtainable. Thus, if one assumes 80 acceptable passes per station, 8,000 doppler equations are formed and reduced in a least-squares adjustment for three unknown coordinates. Acquiring 100 passes may take a week. However, the calculations are not too extensive since there are only three unknowns per station.

Translocation. Two receivers are used for this procedure with each receiver at different stations as at A and B in Figure 1.

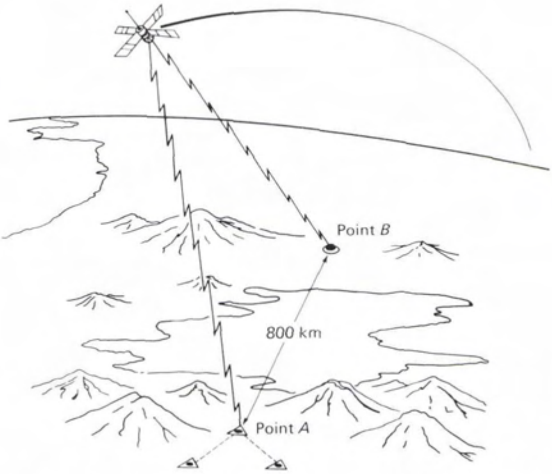

Fig. 1. Doppler positioning by translocation (JMR Instruments, Inc.).

A common set of observations is made simultaneously at each station. If the distance between the stations is approximately the same as the altitude of the satellite, then effects of orbital errors on the two stations are about the same. Thus, the relative position of the two stations is more accurate than their respective absolute locations. If more than two stations are to be located, one receiver is left at one station and the other is moved around to different points in the network. Data reduction is similar to that for point positioning, except positions are calculated using only the passes common to each pair. Then, employing all combinations of stations, ΔX_{ij}, ΔY_{ij}, ΔZ_{ij} are determined. In this way, groundpoint positions with RMS errors of 2 to 3 m can be obtained with only 25 passes of a satellite per station, a considerable reduction in time and effort compared to point positioning.

ORBITAL DATA TREATED AS OBSERVATIONS

Short arc adjustment. The short arc adjustment (Brown and Trotter, 1969, 1973) is a method in which all data including orbital parameters are given *a priori* weights and are permitted to adjust within the constraints imposed by these weights. This procedure is rigorous, permits use of broadcast ephemeris, and can provide RMS errors of less than one metre in ground point positions. However, the computational effort is substantially greater than for point positioning and translocation. The unknown parameters in the solution are

- X, Y, Z coordinates for all occupied stations in the network;
- Six orbital parameters for each pass; and
- Up to five error coefficients for each station for each pass (zero set, timing bias in Geoceiver clock, frequency offset, frequency drift, and coefficient of refraction).

For example, assume the following: two receivers are used on a network of six stations; a set of 250 good passes is acceptable for reduction; each pass is observed by the two receivers (not a necessary assumption); and 100 Doppler counts per pass are recorded. A system of (2)(100)(250) = 50,000 equations will be generated involving a total of

$$\begin{aligned} 6 \times 3 &= 18 \text{ unknown coordinates of stations,} \\ 6 \times 250 &= 1{,}500 \text{ unknown orbital parameters, and} \\ 2 \times 5 \times 250 &= 2{,}500 \text{ error coefficients.} \end{aligned}$$

Solution of such a system is possible only by use of the second order partitioning developed by Brown and Trotter (1969).

At this point, one begins to get an idea of the scope of the computations involved in the reduction of Doppler data.

Short Arc Translocation. In this procedure, the short arc geodetic adjustment is applied independently to pairs of ground stations occupied by Doppler receivers. When all combinations of pairs of unknown ground stations have been so treated, the resulting vectors between stations are used in a subsequent network adjustment. Consequently, this method is a special case of the general, geodetic adjustment by the short arc method. The procedure by Wells (1974) is a simplified version of this method where the orbit is adjusted parallel to itself for each pair of stations.

The rigorous short arc translocation is satisfactory when only two receivers are available. If three or more receivers are used, the general short arc adjustment is more effective.

ORBITAL PARAMETERS

From the preceding discussion, it is apparent that, in a combined adjustment of photogrammetric and Doppler data, one has the option of (a) holding the satellite orbital parameters fixed or (b) allowing satellite orbital parameters to be included as unknowns in the solution. If orbital data are assumed errorless, development of the Doppler condition equation is relatively straightforward and problems exist mainly in deciding what error coefficients to retain in the model. The disadvantage in this approach is that the most accessible orbital data are from the broadcast ephemeris which does contain errors, placing a definite limit on the attainable accuracy of the adjustment.

Thus, a procedure which allows improvement in orbital parameters and permits use of the broadcast ephemeris without compromising the accuracy of the results is appealing. In this solution, the parameters of the orbit (cartesian coordinates $[x, y, z]^T$ and velocities $[\dot{x}, \dot{y}, \dot{z}]^T$ in an Earth fixed system) are introduced as approximations with *a priori* constraints. Position and velocity vectors for the satellite can be calculated for a given time t referred to a certain epoch $t = 0$ on the time scale of the transmitted Doppler signal. To include these parameters in the solution, a functional relationship between $[x, y, z]^T$ and $[\dot{x}, \dot{y}, \dot{z}]^T$ of the satellite and the equations of motion is required.

In existing Doppler reduction programs such as those developed by Hartwell *et al.* (1973) and Brown and Trotter (1969) an *orbital integrator* employing a power series solution to the equations of motion provides the desired functional relationship. The orbital integrator is also used to furnish a power series solution to the variational equations so that errors in satellite position $[x, y, z]^T$ at time t can be related to errors in the six initial conditions, $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$, and the assumed center of mass $[X_{00}, Y_{00}, Z_{00}]^T$ in the Earth fixed system. Thus, coordinates for the center of mass can also be carried as unknowns so there is the possibility of including nine parameters in the Doppler equation in addition to the three coordinates of the occupied station and error model coefficients.

Further details concerning the orbital integrator and the cited functional relationships can be found in Brown and Trotter (1969, 1973) and Hartwell *et al.* (1973).

CONDITION EQUATION FOR DOPPLER SATELLITE MEASUREMENTS

The Transit system consists of five satellites in polar orbits about the Earth (Stansell, 1978). These satellites are spaced so that one of them is visible to any point on Earth at least once every two hours. Two frequencies, one at 400 MHz and one at 150 MHz, are transmitted. As the satellite approaches the ground station where the receiver is located, the transmitted frequency is received and compared to a reference frequency to yield a *beat frequency*. This reference frequency, generated by the receiver, is offset somewhat, so the beat frequency or Doppler count is always positive. Figure 2 shows several satellites in polar orbits about the Earth. Figure 3 shows the relationship of the satellite in orbit to the observer on the ground.

There are two basic forms for the Doppler condition equation. The first and perhaps more commonly seen equation, formulated using individual Doppler counts and range differences, is (Krakiwsky and Wells, 1971; Wells, 1974; Stansell, 1978)

$$r_k - r_{k-1} = \lambda \Delta N_k + \lambda (f_0 - f_0')(t_k - t_{k-1}) \tag{6}$$

in which

r_k = range at time t_k,
r_{k-1} = range at time t_{k-1},
λ = c/f_0 = wave length of transmitted frequency where c = the velocity of light,
f_0 = frequency of signal transmitted by satellite,
f_0' = reference frequency generated in receiver, and
ΔN_k = Doppler count from t_{k-1} to t_k.

An advantage in the use of Equation 6 is that effects of errors due to frequency drift and timing bias are suppressed and can be neglected. The effect of frequency offset is substantial, and it is included as an error term. Disadvantages of Equation 6 for geodetic receivers (in which cycle count is continuous and a short count is possible) are that successive range differences are correlated and the geometry is weak.

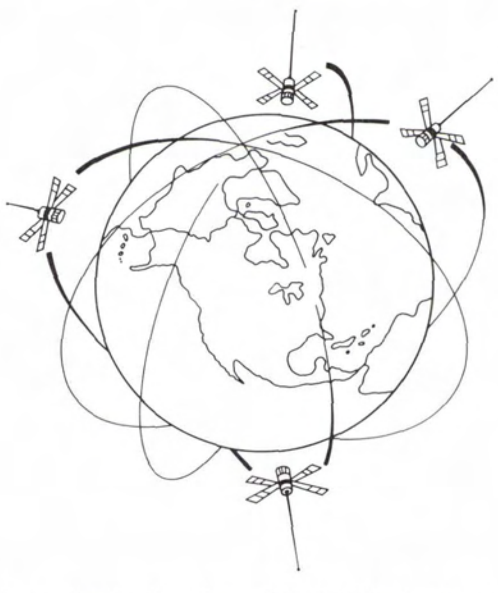

FIG. 2. Navigational satellite orbits (JMR Instruments, Inc.).

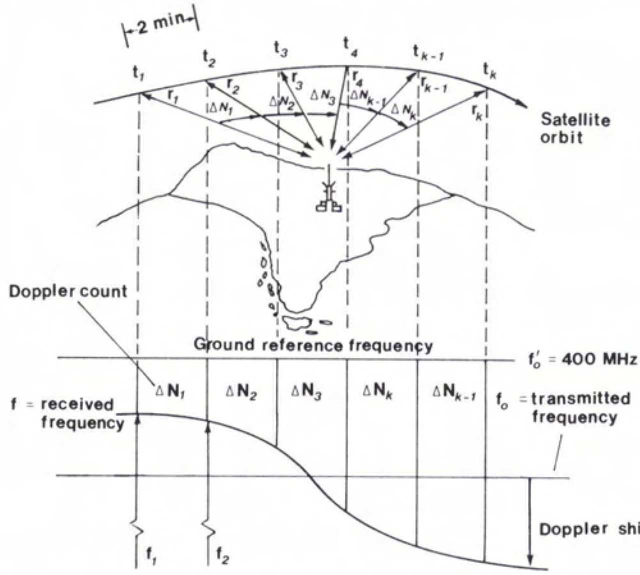

FIG. 3. Relationship of doppler count to observer (Magnavox Company).

To avoid the complication of correlation and to capitalize on the built-in features of geodetic receivers, the continuously integrated Doppler equation was proposed by Brown (1968) and Brown and Trotter (1969). This equation, formulated in terms of *range* and *cumulated Doppler count*, is

$$r_k = \lambda(N_k - \Delta f_0 t_k) + d_1 c_0 + d_2 c_1 + d_3 c_2 + d_4 c_3 + d_5 c_4 + d_6 c_5 \tag{7}$$

in which

r_k = range at time t_k,
λ = wave length transmitted from the satellite,
$\Delta f_0 = f_0 - f_0'$,
N_k = cumulated doppler counts from time $t = 0$, and
$d_1, d_2, \ldots, d_6$ = coefficients of error terms $c_0, c_1, \ldots, c_5$ designed to compensate for zero set timing, timing bias, frequency offset, frequency drift, frequency bias, and tropospheric refraction.

Substituting $r_k = [(X_k - X_j)^2 + (Y_k - Y_j)^2 + (Z_k - Z_j)^2]^{1/2}$ and $s^0 \doteq \lambda(N_k - \Delta f_0 t_k)$, Equation 7 can be written

$$\begin{aligned} s^0 + a_1 v_r + a_2 v_t + d_1 c_0 + d_2 c_1 + d_3 c_2 + d_4 c_3 + d_5 c_4 + d_6 c_5 \\ = [(X_k - X_j)^2 + (Y_k - Y_j)^2 + (Z_k - Z_j)^2]^{1/2} \end{aligned} \tag{8}$$

where v_r and v_t represent residuals in the timing associated with Doppler counts.

Substitution of $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ from the orbital generator and including (X_{00}, Y_{00}, Z_{00}) in Equation 8 yields

$$s = f(X_j, Y_j, Z_j, X_{00}, Y_{00}, Z_{00}, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, t) \tag{9}$$

where s represents the left side of Equation 8. Linearization of Equation 9 in a Taylor series expansion gives

$$\underset{1,2}{\ddot{\mathbf{A}}}\ \underset{2,1}{\mathbf{V}_d} + \underset{1,3}{\ddot{\boldsymbol{\gamma}}}\ \underset{3,1}{\boldsymbol{\Delta}_j} + \underset{1,3}{\ddot{\mathbf{C}}}\ \underset{3,1}{\boldsymbol{\Delta}_{00}} + \underset{1,12}{\ddot{\mathbf{D}}}\ \underset{12,1}{\boldsymbol{\delta}_d} = \underset{1,1}{\ddot{\mathbf{f}}} \tag{10}$$

the equation for a single cumulated Doppler count N_k from station j at time t_k. In Equation 10

$$\ddot{\mathbf{A}} = [a_1\ a_2],\ \mathbf{V}_d = \begin{bmatrix} V_r \\ V_t \end{bmatrix},\ \ddot{\boldsymbol{\gamma}} = \begin{bmatrix} \frac{\partial s}{\partial X_j} & \frac{\partial s}{\partial Y_j} & \frac{\partial s}{\partial Z_j} \end{bmatrix},$$

$$\mathbf{\Delta}_j^{\mathrm{T}} = [\Delta X\ \Delta Y\ \Delta Z]_j^{\mathrm{T}},\ \ddot{\mathbf{C}} = \left[\frac{\partial s}{\partial X_{00}}\ \frac{\partial s}{\partial Y_{00}}\ \frac{\partial s}{\partial Z_{00}}\right],\ \mathbf{\Delta}_{00}^{\mathrm{T}} = [\Delta X_{00}\ \Delta Y_{00}\ \Delta Z_{00}]^{\mathrm{T}},$$

$$\ddot{\mathbf{D}} = \left[\ddot{\mathbf{D}}\middle|\ddot{\mathbf{E}}_k\right] = \left[\frac{\partial s}{\partial x_0}\ \frac{\partial s}{\partial y_0}\ \ldots\ \frac{\partial s}{\partial \dot{z}_0}\middle|\ d_1\ d_2\ \ldots\ d_6\right],$$

$$\mathbf{\delta}_d^{\mathrm{T}} = [\Delta x_0\ \Delta y_0\ \Delta z_0\ \Delta\dot{x}_0\ \Delta\dot{y}_0\ \Delta\dot{z}_0|\Delta d_1\ \Delta d_2\ \ldots\ \Delta d_6] = \left[\mathbf{\delta}_p^{\mathrm{T}}\middle|\mathbf{\delta}_e^{\mathrm{T}}\right],\ \text{and}$$

$$\ddot{\mathbf{f}}^{\mathrm{T}} = f(X_j^0,\ Y_j^0,\ \ldots,\ \dot{z}_0,\ t) - \hat{r}_k,$$

where the superscript ^{0} indicates an approximate value and $\hat{r}_k$ is the measured range from station j to the satellite at time t_k. Equation 10 is used in this study as the Doppler equation for the combined adjustment.

System of Equations

Consider the system of photogrammetric and Doppler condition equations generated for a block of $i = 1, 2, \ldots, m$ photographs containing images of $j = 1, 2, \ldots, n$ object points of which n_1 are pass points and n_2 are occupied by Doppler receivers ($n_1 + n_2 = n$). Assume that the n_2 points register signals from $p = 1, 2, \ldots, l$ satellite passes with $q = 1, 2, \ldots, k$ counts per pass.

PHOTOGRAMMETRIC EQUATIONS

Each object point j in photograph i generates a pair of equations (Equations 4). If j is visible on $i = 1, 2, \ldots, m$ photos, we have

$$\underset{2m,1}{\begin{bmatrix} \underset{2,1}{\mathbf{V}_{1j}} \\ \mathbf{V}_{2j} \\ \vdots \\ \mathbf{V}_{mj} \end{bmatrix}} + \underset{2m,6m}{\begin{bmatrix} \underset{2,6}{\mathbf{\alpha}_{1j}} & & 0 \\ & \mathbf{\alpha}_{2j} & \\ & & \ddots \\ 0 & & \mathbf{\alpha}_{mj} \end{bmatrix}} \underset{6m,1}{\begin{bmatrix} \underset{6,1}{\mathbf{\delta}_1} \\ \mathbf{\delta}_2 \\ \vdots \\ \mathbf{\delta}_m \end{bmatrix}} + \underset{2m,3}{\begin{bmatrix} \underset{2,3}{\mathbf{\beta}_{1j}} \\ \mathbf{\beta}_{2j} \\ \vdots \\ \mathbf{\beta}_{mj} \end{bmatrix}} \underset{3,1}{\mathbf{\Delta}_j} = \underset{2m,1}{\begin{bmatrix} \underset{2,1}{\mathbf{f}_{1j}} \\ \mathbf{f}_{2j} \\ \\ \mathbf{f}_{mj} \end{bmatrix}} \tag{11}$$

or

$$\mathbf{V}_j + \mathbf{\alpha}_j\,\mathbf{\delta} + \mathbf{\beta}_j\,\mathbf{\Delta}_j = \mathbf{f}_j. \tag{12}$$

Next, assume all points $j = 1, 2, \ldots, n$ are visible on $i = 1, 2, \ldots, m$ photos

$$\underset{2mn,1}{\begin{bmatrix} \underset{2m,1}{\mathbf{V}_1} \\ \mathbf{V}_2 \\ \vdots \\ \mathbf{V}_n \end{bmatrix}} + \underset{2mn,6m}{\begin{bmatrix} \underset{2m,6m}{\mathbf{\alpha}_1} \\ \mathbf{\alpha}_2 \\ \vdots \\ \mathbf{\alpha}_n \end{bmatrix}} \underset{6m,1}{\begin{bmatrix} \mathbf{\delta}_1 \\ \mathbf{\delta}_2 \\ \vdots \\ \mathbf{\delta}_m \end{bmatrix}} + \underset{2mn,3n}{\begin{bmatrix} \underset{2m,3}{\hat{\mathbf{\beta}}_1} & & 0 \\ & \hat{\mathbf{\beta}}_2 & \\ & & \ddots \\ 0 & & \hat{\mathbf{\beta}}_n \end{bmatrix}} \underset{3n,1}{\begin{bmatrix} \underset{3,1}{\mathbf{\Delta}_1} \\ \mathbf{\Delta}_2 \\ \vdots \\ \mathbf{\Delta}_n \end{bmatrix}} = \underset{2mn,1}{\begin{bmatrix} \underset{2m,1}{\mathbf{f}_1} \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix}} \tag{13}$$

or

$$\mathbf{V}_p + \mathbf{\alpha}\,\mathbf{\delta} + \hat{\mathbf{\beta}}\,\hat{\mathbf{\Delta}} = \mathbf{f}. \tag{14}$$

For subsequent developments, it will be useful to include the vector

$$\underset{3,1}{\mathbf{\Delta}_{00}},$$

corrections to the coordinates of the center of mass from Equation 10, as the last three corrections in the vector $\hat{\mathbf{\Delta}}$ from Equation 14. This is accomplished by augmenting $\mathbf{\beta}$ with a $2m,3$ matrix $\mathbf{\beta}_{00}$. The final array of collinearity equations is then

$$\mathbf{V}_p + \mathbf{\alpha}\,\mathbf{\delta} + \mathbf{\beta}\,\mathbf{\Delta} = \mathbf{f} \tag{15}$$

where

$$\mathbf{\beta} = \underset{2mn,3n+3}{\left[\underset{2mn,3n}{\hat{\mathbf{\beta}}}\ \middle|\ \underset{2mn,3}{\mathbf{\beta}_{00}}\right]},\ \mathbf{\Delta} = \underset{3n+3,1}{\begin{bmatrix} \underset{3n,1}{\hat{\mathbf{\Delta}}} \\ \underset{3,1}{\mathbf{\Delta}_{00}} \end{bmatrix}},\ \text{and } \mathbf{\beta}_{00} = 0$$

DOPPLER EQUATIONS

Given one station j receiving $q = 1$ count from $p = 1$ pass, write one equation (10). With $q = 1$, 2, . . . , k counts at station j from one pass,

$$\begin{bmatrix} \underset{1,2}{\dddot{\mathbf{A}}_1} & & & 0 \\ & \dddot{\mathbf{A}}_2 & & \\ & & \ddots & \\ 0 & & & \dddot{\mathbf{A}}_k \end{bmatrix}_{k,2k} \begin{bmatrix} \underset{2,1}{\dddot{\mathbf{V}}_1} \\ \dddot{\mathbf{V}}_2 \\ \vdots \\ \dddot{\mathbf{V}}_k \end{bmatrix}_{2k,1} + \begin{bmatrix} \underset{1,3}{\dddot{\boldsymbol{\gamma}}_1} & \underset{1,3}{\dddot{\mathbf{C}}_1} \\ \dddot{\boldsymbol{\gamma}}_2 & \dddot{\mathbf{C}}_2 \\ \vdots & \vdots \\ \dddot{\boldsymbol{\gamma}}_k & \dddot{\mathbf{C}}_k \end{bmatrix}_{k,6} \begin{bmatrix} \underset{3,1}{\boldsymbol{\Delta}_j} \\ \underset{3,1}{\boldsymbol{\Delta}_{00}} \end{bmatrix} + \begin{bmatrix} \underset{1,6}{\dddot{\mathbf{D}}_1} & \underset{1,6}{\dddot{\mathbf{E}}_1} \\ \dddot{\mathbf{D}}_2 & \dddot{\mathbf{E}}_2 \\ \vdots & \vdots \\ \underset{k,6}{\dddot{\mathbf{D}}_k} & \underset{k,6}{\dddot{\mathbf{E}}_k} \end{bmatrix} \begin{bmatrix} \underset{6,1}{\boldsymbol{\delta}_p} \\ \hline \underset{6,1}{\boldsymbol{\delta}_e} \end{bmatrix}_{12,1} = \begin{bmatrix} \dddot{\mathbf{f}}_1 \\ \dddot{\mathbf{f}}_2 \\ \vdots \\ \dddot{\mathbf{f}}_k \end{bmatrix}_{k,1} \tag{16}$$

or

$$\bar{\mathbf{A}}\,\bar{\mathbf{V}} + [\bar{\boldsymbol{\gamma}}|\bar{\mathbf{C}}] \left[\frac{\boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_{00}}\right] + [\bar{\mathbf{D}}|\bar{\mathbf{E}}] \left[\frac{\boldsymbol{\delta}_p}{\boldsymbol{\delta}_e}\right] = \bar{\mathbf{f}}. \tag{17}$$

Next, assume all points ($j = n_1 + 1, n_1 + 2, \ldots, n$ or $j = 1, 2, \ldots, n_2$) receive $q = 1, 2, \ldots, k$ counts from pass 1; then,

$$\begin{aligned} &\begin{bmatrix} \underset{k,2k}{\bar{\mathbf{A}}_1} & & 0 \\ & \bar{\mathbf{A}}_2 \ddots & \\ 0 & & \bar{\mathbf{A}}_{n2} \end{bmatrix}_{kn_2,2kn_2} \begin{bmatrix} \underset{2k,1}{\bar{\mathbf{V}}_1} \\ \bar{\mathbf{V}}_2 \\ \vdots \\ \bar{\mathbf{V}}_{n2} \end{bmatrix}_{2kn_2,1} + \left[\begin{array}{ccc|c} \underset{k,3}{\bar{\boldsymbol{\gamma}}_1} & & 0 & \underset{k,3}{\bar{\mathbf{C}}_1} \\ & \bar{\boldsymbol{\gamma}}_2 \ddots & & \bar{\mathbf{C}}_2 \\ & & & \vdots \\ 0 & & \bar{\boldsymbol{\gamma}}_{n2} & \bar{\mathbf{C}}_{n2} \end{array}\right]_{kn_2,3n_2 \quad kn_2,3} \begin{bmatrix} \underset{3,1}{\boldsymbol{\Delta}_1} \\ \boldsymbol{\Delta}_2 \\ \vdots \\ \boldsymbol{\Delta}_{n_2} \\ \boldsymbol{\Delta}_{00} \end{bmatrix}_{(3n_2+3),1} \\ &+ \left[\begin{array}{c|ccc} \underset{k,6}{\bar{\mathbf{D}}_1} & \underset{k,6}{\bar{\mathbf{E}}_1} & & \\ \bar{\mathbf{D}}_2 & & \bar{\mathbf{E}}_2 \ddots & \\ \vdots & & & \\ \mathbf{D}_{n2} & & & \bar{\mathbf{E}}_{n2} \end{array}\right]_{kn_2,6 \quad kn_2,6n_2} \begin{bmatrix} \boldsymbol{\delta}_{p1} \\ \boldsymbol{\delta}_{e1} \\ \boldsymbol{\delta}_{e2} \\ \vdots \\ \boldsymbol{\delta}_{en2} \end{bmatrix}_{(6n_2+6)} = \begin{bmatrix} \underset{k,1}{\bar{\mathbf{f}}_1} \\ \bar{\mathbf{f}}_2 \\ \\ \bar{\mathbf{f}}_{n2} \end{bmatrix}_{kn_2,1} \end{aligned} \tag{18}$$

or

$$\underset{kn_2,2kn_2}{\dot{\mathbf{A}}} \; \underset{2kn_2,1}{\dot{\mathbf{V}}} + [\, \underset{kn_2,3n_2}{\dot{\boldsymbol{\gamma}}} \mid \underset{kn_2,3}{\dot{\mathbf{C}}} \,] \begin{bmatrix} \underset{3n_2,1}{\boldsymbol{\Delta}} \\ \hline \underset{3,1}{\boldsymbol{\Delta}_{00}} \end{bmatrix} + \underset{kn_2,(6n_2+6)}{\left[\begin{array}{c|c} \underset{kn_2,6}{\dot{\mathbf{D}}} & \underset{kn_2,6n_2}{\dot{\mathbf{E}}} \end{array}\right]} \underset{(6n_2+6),1}{\begin{bmatrix} \dot{\boldsymbol{\delta}}_{\mathrm{p}} \\ \dot{\boldsymbol{\delta}}_{\mathrm{e}} \end{bmatrix}} = \underset{kn_2,1}{\dot{\mathbf{f}}}\,.$$

Finally, consider all stations with p = 1, 2, . . . , l passes

$$\begin{bmatrix} \underset{kn_2,2kn_2}{\dot{\mathbf{A}}_1} & & 0 \\ \dot{\mathbf{A}}_2 & \ddots & \\ 0 & & \dot{\mathbf{A}}_l \end{bmatrix}_{kln_2,2kln_2} \begin{bmatrix} \underset{2kn_2,1}{\dot{\mathbf{V}}_1} \\ \dot{\mathbf{V}}_2 \\ \vdots \\ \dot{\mathbf{V}}_l \end{bmatrix}_{2kln_2,1} + \left[\begin{array}{c|c} \underset{kn_2,3n_2}{\dot{\boldsymbol{\gamma}}_1} & \underset{kn_2,3}{\dot{\mathbf{C}}_1} \\ \dot{\boldsymbol{\gamma}}_2 & \dot{\mathbf{C}}_2 \\ \vdots & \vdots \\ \dot{\boldsymbol{\gamma}}_l & \dot{\mathbf{C}}_l \end{array}\right]_{kln_2,3n_2 \quad kln_2,3} \begin{bmatrix} \underset{3,1}{\boldsymbol{\Delta}_1} \\ \boldsymbol{\Delta}_2 \\ \vdots \\ \boldsymbol{\Delta}_{n2} \\ \boldsymbol{\Delta}_{00} \end{bmatrix}_{(3n_2+3),1} + \begin{bmatrix} \underset{kn_2,6+6n_2}{[\dot{\mathbf{D}}|\dot{\mathbf{E}}]_1} & & 0 \\ & [\dot{\mathbf{D}}|\dot{\mathbf{E}}]_2 \ddots & \\ 0 & & [\dot{\mathbf{D}}|\dot{\mathbf{E}}]_l \end{bmatrix}_{kln_2,(6l+6ln_2)} \begin{bmatrix} \underset{6,1}{\dot{\boldsymbol{\delta}}_{p1}} \\ \dot{\boldsymbol{\delta}}_{e1} \\ \dot{\boldsymbol{\delta}}_{p2} \\ \dot{\boldsymbol{\delta}}_{e2} \\ \vdots \\ \dot{\boldsymbol{\delta}}_{pl} \\ \dot{\boldsymbol{\delta}}_{el} \end{bmatrix}_{(6l+6ln_2),1} = \begin{bmatrix} \underset{kn_2,1}{\mathbf{f}_1} \\ \mathbf{f}_2 \\ \vdots \\ \dot{\mathbf{f}}_l \end{bmatrix}_{kln_2,1} \tag{20}$$

to give

$$\mathbf{A}_d\,\mathbf{V}_d + [\ddot{\boldsymbol{\gamma}}|\ddot{\mathbf{C}}] \begin{bmatrix} \underset{3n_2,1}{\boldsymbol{\Delta}} \\ \underset{3,1}{\boldsymbol{\Delta}_{00}} \end{bmatrix} + \mathbf{D}\,\dot{\boldsymbol{\delta}} = \mathbf{f}_d \tag{21}$$

The matrix $[\ddot{\boldsymbol{\gamma}}|\ddot{\mathbf{C}}]$ is augmented by a zero matrix so the entire vector of corrections to all object points $j = 1, 2, \ldots, n$ appears in the equations

$$\mathbf{A}_d\mathbf{V}_d + \underset{kln_2,3n+3}{\left[\underset{kln_2,3n_1}{\mathbf{0}} \,\middle|\, \underset{kln_2,3n_2}{\ddot{\boldsymbol{\gamma}}} \,\middle|\, \underset{kln_2,3}{\ddot{\mathbf{C}}}\right]} \underset{3n+3,1}{\begin{bmatrix} \underset{3n_1,1}{\boldsymbol{\Delta}_p} \\ \underset{3n_2,1}{\boldsymbol{\Delta}_d} \\ \underset{3,1}{\boldsymbol{\Delta}_{00}} \end{bmatrix}} + \underset{kln_2,t}{\mathbf{D}}\ \underset{t,1}{\dot{\boldsymbol{\delta}}} = \underset{kln_2,1}{\mathbf{f}_d} \tag{22}$$

where $t = 6l + 6ln_2$ or

$$\mathbf{A}_d\ \mathbf{V}_d + \boldsymbol{\gamma}\ \boldsymbol{\Delta} + \mathbf{D}\ \dot{\boldsymbol{\delta}} = \mathbf{f}_d, \tag{23}$$

which represents the total system of doppler equations.

COMBINED SYSTEM OF EQUATIONS

Combining Equations 15 with 23 and including constraints on $\boldsymbol{\delta}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}_{00}$, and $\dot{\boldsymbol{\delta}}$, we have the total system (note: $t = (6l + 6ln_2)$ where n_2 = number of Doppler stations),

$$\begin{array}{cccccccccc}
 & \underset{2mn,1}{\mathbf{V}_p} & + & \underset{2mn,6m}{\boldsymbol{\alpha}}\ \underset{6m,1}{\boldsymbol{\delta}} & + & \underset{2mn,3n+3}{\boldsymbol{\beta}}\ \underset{3n+3,1}{\boldsymbol{\Delta}} & & & = & \underset{2mn,1}{\mathbf{f}} \\
\underset{kln_2,2kln_2}{\mathbf{A}} & \underset{2kln_2,1}{\mathbf{V}_{\mathrm{d}}} & & & + & \underset{kln_2,3n+3}{\boldsymbol{\gamma}}\ \underset{3n+3,1}{\boldsymbol{\Delta}} & + & \underset{kln_2,t}{\mathbf{D}}\ \underset{t,1}{\dot{\boldsymbol{\delta}}} & = & \underset{kln_2,1}{\mathbf{f}_{\mathrm{d}}} \\
 & \underset{6m,1}{\tilde{\mathbf{V}}} & - & \underset{6m,1}{\boldsymbol{\delta}} & & & & & = & \underset{6m,1}{\tilde{\mathbf{f}}} \\
 & \underset{3n+3,1}{\hat{\mathbf{V}}} & & & - & \underset{3n+3,1}{\boldsymbol{\Delta}} & & & = & \underset{3n+3,1}{\hat{\mathbf{f}}} \\
 & \underset{t,1}{\dot{\mathbf{V}}} & & & & & - & \underset{t,1}{\dot{\boldsymbol{\delta}}} & = & \underset{t,1}{\dot{\mathbf{f}}}
\end{array} \tag{24}$$

that can be expressed more compactly as

$$\mathbf{A}\ \mathbf{V} + \mathbf{B}\ \mathbf{X} = \mathbf{F}, \tag{25}$$

for which the normal equations are

$$[\mathbf{B}^{\mathrm{T}}(\mathbf{A}\ \mathbf{Q}\ \mathbf{A}^{\mathrm{T}})^{-1}\ \mathbf{B}]\ \mathbf{X} = \mathbf{B}^{\mathrm{T}}\ (\mathbf{A}\ \mathbf{Q}\ \mathbf{A}^{\mathrm{T}})^{-1}\ \mathbf{F} \tag{26}$$

where $\mathbf{Q} = \mathbf{W}^{-1}$ is the cofactor matrix of measurements and parameters further defined as follows:

$$\mathbf{Q} = \text{diag.}\left\{ \underset{2mn,2mn}{\mathbf{Q}_e}\ \ \underset{2kln_2,2kln_2}{\mathbf{Q}_d}\ \ \underset{6m,6m}{\tilde{\mathbf{Q}}}\ \ \underset{\substack{(3n+3),\\(3n+3)}}{\hat{\mathbf{Q}}}\ \ \underset{t,t}{\dot{\mathbf{Q}}} \right\}. \tag{27}$$

Individual cofactor matrices are

$\mathbf{Q}_e$, photographic coordinates,
$\mathbf{Q}_d$, Doppler measurements,
$\tilde{\mathbf{Q}}$, exposure station parameters,
$\hat{\mathbf{Q}}$, object point coordinates and center of mass, and
$\dot{\mathbf{Q}}$, orbital parameters and error terms.

Measurements and parameters are assumed uncorrelated so that $\mathbf{Q}$ is a diagonal matrix of diagonal submatrices.

A detailed examination of $(\mathbf{A}\ \mathbf{Q}\ \mathbf{A}^{\mathrm{T}})^{-1}$ in Equation 26 reveals that

$$(\mathbf{A}_d\ \mathbf{Q}_d\ \mathbf{A}_d^{\mathrm{T}})^{-1} = \mathbf{I}$$

when the cofactors for doppler measurements are absorbed into the appropriate elements of $\ddot{\boldsymbol{\gamma}}$ so that

$$(\mathbf{A}\ \mathbf{Q}\ \mathbf{A}^{\mathrm{T}})^{-1} = \text{diag. }\{\mathbf{W}_e\ \mathbf{I}\ \tilde{\mathbf{W}}\ \hat{\mathbf{W}}\ \dot{\mathbf{W}}\} \tag{28}$$

and the general normal equations are

$$\begin{bmatrix} \dot{\mathbf{N}} + \tilde{\mathbf{W}} & \overline{\mathbf{N}} & 0 \\ \overline{\mathbf{N}}^{\mathrm{T}} & \ddot{\mathbf{N}} + \ddot{\mathbf{U}} + \hat{\mathbf{W}} & \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{D} \\ 0 & \mathbf{D}^{\mathrm{T}}\boldsymbol{\gamma} & \hat{\mathbf{D}} + \dot{\mathbf{W}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\Delta} \\ \dot{\boldsymbol{\delta}} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{c}} - \tilde{\mathbf{W}}\ \tilde{\mathbf{F}} \\ \tilde{\bar{\mathbf{c}}} - \hat{\mathbf{W}}\ \hat{\mathbf{F}} \\ \bar{\mathbf{c}} - \dot{\mathbf{W}}\ \dot{\mathbf{f}} \end{bmatrix} \tag{29}$$

in which

$$\begin{aligned} \dot{\mathbf{N}} &= \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}_e\boldsymbol{\alpha},\ \overline{\mathbf{N}} = \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}_e\boldsymbol{\beta},\ \ddot{\mathbf{N}} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{W}_e\boldsymbol{\beta},\ \ddot{\mathbf{U}} = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma} \\ \hat{\mathbf{D}} &= \mathbf{D}^{\mathrm{T}}\mathbf{D},\quad \dot{\mathbf{c}} = \boldsymbol{\alpha}^{\mathrm{T}}\tilde{\mathbf{W}}_e\mathbf{f},\ \ \ddot{\mathbf{c}} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{W}_e\mathbf{f},\ \ \bar{\mathbf{c}} = \mathbf{D}^{\mathrm{T}}\mathbf{f}_d. \end{aligned} \tag{29a}$$

The system represented by Equations 29 is not amenable to direct solution, but does exhibit some of the characteristics of a *banded bordered* system (Mikhail, 1976, pp. 306-309) or a *second order partitioned* system (Brown and Trotter, 1969). In order to more nearly approach the system of Brown, $\boldsymbol{\delta}$ and $\boldsymbol{\Delta}_p$ for pass points must be eliminated. Solving Equation 29 for $\boldsymbol{\delta}$ gives

$$\boldsymbol{\delta} = (\dot{\mathbf{N}} + \tilde{\mathbf{W}})^{-1}\,(\dot{\mathbf{c}} - \tilde{\mathbf{W}}\,\tilde{\mathbf{f}} - \overline{\mathbf{N}}\,\boldsymbol{\Delta}) \tag{30}$$

which is then substituted into the second of Equations 29 to yield

$$\begin{bmatrix} \ddot{\mathbf{N}} + \ddot{\mathbf{U}} + \hat{\mathbf{W}} - \overline{\mathbf{N}}^{\mathrm{T}}\,(\dot{\mathbf{N}} + \tilde{\mathbf{W}})^{-1}\,\overline{\mathbf{N}} & \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{D} \\ \mathbf{D}^{\mathrm{T}}\boldsymbol{\gamma} & \hat{\mathbf{D}} + \dot{\mathbf{W}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta} \\ \dot{\boldsymbol{\delta}} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{N}}^{\mathrm{T}}\,(\dot{\mathbf{N}} + \tilde{\mathbf{W}})^{-1}\,(\tilde{\mathbf{W}}\,\tilde{\mathbf{f}} - \dot{\mathbf{c}}) + (\ddot{\mathbf{c}} - \hat{\mathbf{W}}\,\hat{\mathbf{f}}) \\ \bar{\mathbf{c}} - \dot{\mathbf{w}}\,\dot{\mathbf{f}} \end{bmatrix} \tag{31}$$

that are the partially reduced normal equations. Examination of these equations reveals that if $\boldsymbol{\Delta}_p$ for the pass points were to be eliminated from $\boldsymbol{\Delta}$ (see Equations 22 and 23) the remaining equations correspond in all essential aspects to the normal equations for Brown's short arc adjustment which is solvable by first or second order partitioned regression (Brown and Trotter, 1969). Unfortunately, elimination of $\boldsymbol{\Delta}_p$ involves the inversion of a $3n_1 \times 3n_1$ matrix.

In a combined adjustment of photogrammetric and Doppler data the number of pass points n_1 might well be relatively large. Consequently, inversion of a $3n_1 \times 3n_1$ matrix was considered impractical and the stated approach was discarded.

A re-examination of the general normal Equations 29 in detail (beyond the scope of this paper, see Anderson (1982)) indicates that, if the vector of corrections to parameters is reordered keeping orbital terms ($\boldsymbol{\delta}_{p_i}$) and Doppler error terms ($\dot{\boldsymbol{\delta}}_{e_1}$) together then the general normals can be reduced to the form

$$\begin{bmatrix} \underset{6m,6m}{\dot{\mathbf{N}}} & \underset{6m,3n_1}{\overline{\mathbf{N}}_{n1}} & \underset{6m,3n_2+3}{\overline{\mathbf{N}}_{n2}} & 0 & 0 \\ \underset{3n_1,6m}{\overline{\mathbf{N}}^{\mathrm{T}}_{n1}} & \underset{3n_1,3n_1}{\mathbf{U}} & 0 & 0 & 0 \\ \underset{3n_2+3,6m}{\overline{\mathbf{N}}^{\mathrm{T}}_{n2}} & 0 & \underset{s,s}{\dot{\mathbf{U}}} & \underset{s,6l}{\mathbf{M}} & \underset{s,6ln_2}{\overline{\mathbf{M}}} \\ 0 & 0 & \underset{6l,s}{\overline{\mathbf{M}}^{\mathrm{T}}} & \underset{6l,6l}{\hat{\mathbf{U}}} & \underset{6l,6ln_2}{\tilde{\mathbf{U}}} \\ 0 & 0 & \underset{6ln_2,s}{\overline{\mathbf{M}}^{\mathrm{T}}} & \underset{6ln_2,6l}{\tilde{\mathbf{U}}} & \underset{6ln_2,6ln_2}{\hat{\mathbf{E}}} \end{bmatrix} \begin{bmatrix} \underset{6m,1}{\boldsymbol{\delta}} \\ \underset{3n_1,1}{\boldsymbol{\Delta}_p} \\ \underset{s,1}{\boldsymbol{\Delta}_d} \\ \underset{6l,1}{\dot{\boldsymbol{\delta}}_p} \\ \underset{6ln_2,1}{\dot{\boldsymbol{\delta}}_e} \end{bmatrix} = \begin{bmatrix} \underset{6m,1}{\bar{\mathbf{c}}} \\ \underset{3n_1,1}{\hat{\mathbf{c}}} \\ \underset{s,1}{\hat{\mathbf{c}}_d} \\ \underset{6l,1}{\dddot{\mathbf{c}}} \\ \underset{6ln_2,1}{\mathbf{c}_d} \end{bmatrix} \tag{32}$$

where $s = 3n_2 + 3$ and in which the submatrices $\dot{\mathbf{N}}$, $\overline{\mathbf{N}}$, $\mathbf{U}$, $\dot{\mathbf{U}}$, $\mathbf{M}$, $\overline{\mathbf{M}}$, $\tilde{\mathbf{U}}$, and $\hat{\mathbf{E}}$ can be expressed in terms of the appropriate submatrices of photogrammetric Equations 11 to 14, Doppler Equations 16 to 18, and Equation 27. Successive elimination of $\dot{\boldsymbol{\delta}}_e$, $\dot{\boldsymbol{\delta}}_p$, $\boldsymbol{\Delta}_d$, and $\boldsymbol{\Delta}_p$ from Equation 32 yields a solution for $\boldsymbol{\delta}$; i.e.,

$$\underset{6m,1}{\boldsymbol{\delta}} = \begin{bmatrix} \underset{6m,6m}{\hat{\mathbf{N}}} - \underset{6m,3n_1}{\overline{\mathbf{N}}_{n1}}\ \underset{3n_1,3n_1}{\mathbf{U}^{-1}}\ \underset{3n_1,6m}{\overline{\mathbf{N}}^{\mathrm{T}}_{n1}} \end{bmatrix}^{-1} \begin{bmatrix} \underset{6m,1}{\bar{\mathbf{c}}_p} - \underset{6m,3n_1}{\overline{\mathbf{N}}_{n1}}\ \underset{3n_1,3n_1}{\mathbf{U}^{-1}}\ \underset{3n_1,1}{\hat{\mathbf{c}}} \end{bmatrix}$$

or

$$\underset{6m,1}{\boldsymbol{\delta}} = \underset{6m,6m}{\mathbf{N}^{-1}}\ \underset{6m,1}{\mathbf{c}} \tag{33}$$

In a block of aerial photographs which has the proper ordering of point and photograph numbers, the matrices $\hat{\mathbf{N}}$ and $\mathbf{N}$ are banded, with $\mathbf{N}$ having a band width of $B = 6\,(2s' + 2)$ for cross-strip numbering (s' = number of strips) and $B = 6\,(r + 3)$ for down strip numbering (r = number of photographs per strip). Thus, it is possible to compute $\boldsymbol{\delta}$ in Equation 33 by using a method of recursive partitioning such as the one developed by Gyer (1967). When $\boldsymbol{\delta}$ has been found, then $\boldsymbol{\Delta}_p$, $\boldsymbol{\Delta}_d$, $\dot{\boldsymbol{\delta}}_p$, and $\dot{\boldsymbol{\delta}}_e$ can be determined in a back solution.

Even though a certain degree of efficiency can be built into this process, the computations involved are still of a very large magnitude so that the practicality of the approach is questionable. In this respect, a re-examination of the normal equations shows that the link, provided by ground stations common to the Doppler and photogrammetric systems of equations, is relatively weak. For example, in a typical problem, doppler stations might comprise 10 to 15 percent of total number of ground control stations.

Because the two systems are not likely to be strongly coupled, another option is to adjust the Doppler network separately, propagate the covariance matrix for the Doppler stations, and use this propagated covariance matrix in a subsequent photogrammetric bundle adjustment. In other words, a sequential adjustment can be performed. If the full variance-covariance matrix for Doppler stations is propagated, the results will be the same as would be obtained by a simultaneous adjustment. If propagation of the full covariance matrix is not practical, then a reasonable approximation could be achieved by using the diagonal 3×3 submatrices of the covariance matrix and neglecting the covariance terms.

Conclusions

Photogrammetric and Doppler condition equations have been combined into one system and the detailed structure of the resulting normal equations has been examined for the general and partially reduced cases.

So far as the photogrammetric equations are concerned, they are of a well known, standard format representing the state of the art at the present time. The only improvement possible would be to include additional parameters to compensate for residual uncorrected perturbations in the photogrammetric system.

Two forms of the Doppler equation were studied. The form chosen was the range equation which includes, as unknowns, the orbital parameters of the satellite and parameters to compensate for errors in the signal frequency and timing. Considerable simplification would result by using the equation for range difference. Specifically, the number of unknowns per satellite pass could be reduced from twelve to five or, at the very least, one. Inclusion of the orbital parameters and error model lends to the rigor of the solution and promises higher accuracy. Judicious programming could be such as to allow only a subset or none of these terms to be enforced, thus simplifying the solution. This latter course appears to be the best approach and is recommended.

The system of equations which results from combining these two adjustments is of massive dimensions. A detailed analysis of the normal equations which result from a single simultaneous least-squares adjustment of this system reveals that, although a solution is theoretically possible, it is not a practical approach. A sequential adjustment of Doppler and photogrammetric data is a more practical procedure (Mikhail, 1976). In such an adjustment, the Doppler network would be adjusted separately using second-order partitioned regression as suggested by Brown and Trotter (1969). If it then develops that the full covariance matrix for adjusted Doppler stations can be propagated in a practical way from this adjustment, then this covariance matrix can be employed in a subsequent separate photogrammetric adjustment of the block. In this case, the final result would be the same as a simultaneous adjustment of both systems, assuming that a rigorous method of treating weighted parameters is used. Should propagation of the entire covariance matrix for adjusted Doppler stations not be practical, because use of a full covariance matrix is generally not compatible with the banded structure of the photogrammetric structure, then a reasonable approximation can be achieved by using the three by three covariance matrices propagated for each point and neglecting the minimal correlation between the widely separated Doppler stations.

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