Deformation Analysis by Close-Range Photogrammetry

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ABSTRACT: Close-range-photogrammetry systems are extremely well suited for three-dimensional analysis of displacements and deformations of engineering objects and structures. The paper presents a new method of analysis where displacements and deformation of an object are identified with the velocities of a network of object-points. Following their least-squares estimation, the velocities are decomposed into global rates of displacement and deformation of the object and into residual components. Various systems of measurements are studied and their potential for estimating the above parameters is evaluated. Mathematical models of adjustment are discussed schematically, including linear constraints needed to define the reference system.

INTRODUCTION: KINEMATICS OF A NETWORK OF POINTS

ANALYSIS OF DISPLACEMENTS and deformations by close-range photogrammetry is not different in principle from that based on geodetic measurements. The identity of the parameters, the need for solving the datum problem (see Appendix B), and many other computational details are practically the same. Close-range photogrammetry, however, commands two typical and important advantages as shown by Fraser (1982):

- through a proper choice of the supplementary measurements, the need for a direct contact with the object which is being analyzed can be avoided; and
- the photogrammetric document (the photograph) remains available for "second thought" measurements of additional object points at later stages of the analysis.

At the outset we assume the existence and accessibility of a Cartesian coordinate system (XYZ) which is stable. It serves as a reference for describing the position, the orientation, and the motion of objects or structures which are being investigated. Through a process of discretization we actually investigate the behavior (with respect to XYZ) of a network of points which have been marked on that object.

Any point in the network is conceived as being in a state of motion with respect to the reference system. If we assume the velocities to be constant over a specified time interval, we can write the basic kinematic equation of the network as follows:

$$\mathbf{X}_j = \mathbf{X}_0 + \dot{\mathbf{X}} \cdot (T_j - T_0) \qquad (1)$$

where $\mathbf{X}_j^{\mathrm{T}} = (x_1 y_1 z_1 x_2 y_2 z_2 x_3 \ldots)_j$;

$\mathbf{X}_j, \mathbf{X}_0$ are vectors containing the Cartesian coordinates of the net points at the given moment T_j and at the standard epoch T_0, respectively; and

$\dot{\mathbf{X}}^{\mathrm{T}} = (\dot{x}_1 \dot{y}_1 \dot{z}_1 \dot{x}_2 \dot{y}_2 \ldots)$ is a vector of velocity components of the net points.

Equation 1 can be written differently as follows:

$$\dot{\mathbf{X}} = \frac{\mathbf{X}_j - \mathbf{X}_0}{T_j - T_0} = \frac{\Delta \mathbf{X}_{0j}}{\Delta T_{0j}}. \qquad (2)$$

The velocities are assumed to be constant. As a result, Equation 2 is valid for any two epochs of time; for example, T_m and T_n instead of T_0 and T_j. In this paper we adopt the linear velocities model as the only practical alternative in cases of small and variable accelerations. The validity of the linear model has to be tested, however, as an integral part of the computational process.

In displacement and deformation analysis we are interested in the variations in point coordinates which have taken place between two given epochs of time. Thus, according to Equation 2, only point velocities are needed for our analysis while the standard epoch coordinates can be regarded as nuisance (irrelevant) parameters (Papo and Perelmuter, 1983). The velocities of points in a network can be easily decomposed into a systematic (homogeneous) component (see Appendix A) and into residual (individual) velocities. The homogeneous component is parameterized by the rates of translation and rotation of the network with respect to the XYZ system and by its rates of deformation (strain and shear) which are independent of XYZ.

Assuming the availability of appropriate measurements (see next section), analysis of deformations according to the proposed method is performed in two steps:

(1) estimation of object-point velocities and respective covariance matrix from least-squares processing of the measurements, and

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(2) derivation of displacement and deformation parameters from the object-point velocity field as follows (Mase, 1970):

rates of translation		$\bar{x},\bar{y},\bar{z}$
rates of rotation		r_x,r_y,r_z
rates of deformation	(strain)	d_{xx},d_{yy},d_{zz}
	(shear)	d_{yz},d_{xz},d_{xy}

As the rates of translation and rotation are defined with respect to to *XYZ*, it is imperative to be more specific about its realization. It is common practice to select a subset of the network points (reference points) which are characterized by their stability or equivalently by having small (insignificant) velocities. As shown in Papo and Perelmuter (1983), through free-net constraints which are imposed on the velocities of those points, we define an *XYZ* system such that the rates of translation and rotation of the reference points are identically zero. The consistency in the behavior of the reference points is tested with the introduction of each new batch of measurements.

We should note that in close-range photogrammetry there are three different types of points involved in deformation analysis as follows:

R, reference points	—define the reference *XYZ* system
P, object points	—represent the object which is being analyzed
O, projection centers	—are the principal points of the photogrammetric bundles.

ESTIMABLE PARAMETERS AND DATUM DEFECT IN A SYSTEM OF MEASUREMENTS

In this section we present the results of studying the datum defect of typical close-range photogrammetry systems and the identity of parameters of displacement and deformation which can be estimated (see Appendix B). The datum defect is associated with the inherent inability of a particular system of measurements to completely define the reference *XYZ* system.

Table 1 shows six types of elementary measurements between points in three-dimensional space made with respect to the *XYZ* system (Papo and Perelmuter, 1982; Papo and Shmutter, 1978). A directly measured spatial angle is a special case of the type-2 measurements where there are only two directions in the bundle. Measurements consisting of *x,y* photocoordinates and processed under the assumption that the photogrammetric system satisfies the collinearity conditions and the inner orientation parameters of the camera are perfectly known would result in a typical bundle of directions. Note that some of the measurement types are interrelated. For example, a combination of type-2 and type-6 measurements is equivalent to type-4 and type-5 measurements.

Eight different systems of elementary measurements were studied in terms of their datum defect and the identity of estimable parameters. The results are shown in Table 2. The datum defect is partitioned according to the need for defining the origin, the orientation, and the scale of the reference *XYZ* system. The types of points involved in the measurements are marked in the Table by "+". In those cases where the particular type of points is immaterial, "+/+" has been marked indicating that either type could be measured. It is assumed that the specified types of measurements have been made at two or more different epochs (Niemeier, 1981) and that there are no configuration defects.

TABLE 1. GEOMETRIC CLASSIFICATION OF MEASUREMENTS IN 3-D

Measurements		Number of Points Involved	Dependence on the Reference XYZ System
Type	Description		
1	spatial distance	2	none
2	bundle of directions	3 or more	none
3	elevation difference	2	Z axis
4	vertical angle	2	Z axis
5	azimuth	2	XYZ axes
6	bundle orientation	3 or more	XYZ axes

We summarize the contents of Table 2 by the following remarks:

- The photogrammetric bundles serve only as a flexible binder between the network points. They monitor variations in its form without introducing any scale bias (system 1).
- A reduction in datum defect for orientation can be obtained by the measurement of elevation differences or of vertical angles. However, the elevation differences, which are given conventionally in units of distance, do not have any effect on the scale defect (system 3).
- One or more directly measured spatial distances at two or more epochs eliminate the datum defect associated with scale-change and enable a complete solution of translation- and deformation-rates.
- In systems 1 and 4, where no measurements to the reference points were made, the datum defect is corrected by constraints imposed on the object-point velocities (instead of the reference-point velocities).
- Reduction of the datum defect and increase in the number of estimable parameters is achieved through direct measurements of types 3 through 5 where the particular group of points involved in those measurements is immaterial (systems 7, 8).

MODELS FOR LEAST-SQUARES ESTIMATION OF VELOCITIES

We turn our attention first to some technical aspects of the photogrammetric measurements and their combination with direct field measurements. The photogrammetric camera can be set on a tripod or can be carried on an airplane or on a helicopter.

TABLE 2. ESTIMABLE PARAMETERS AND DATUM DEFECT

Measurements			Points Involved			Estimable Parameters			Datum Defect		
System	Type	Description	R	P	O	TRAN	ROT	DEFO	ORIG	ORNT	SCL
1	2	photo-bundles		+	+			5	3	3	1
2	2	photo-bundles	+	+	+	2	3	5	3	3	1
3	2	photo-bundles	+	+	+	2	3	5	3	1	1
	3	elevat-differ	+ /	+ /	+						
4	2	photo-bundles		+	+			6	3	3	
	1	spat-distance		+ /	+						
5	2	photo-bundles	+	+	+	3	3	6	3	3	
	1	spat-distance	+ /	+ /	+						
6	2	photo-bundles	+	+	+	3	3	6	3	1	
	1	spat-distance	+ /	+ /	+						
	3/	elevat-differ	+ /	+ /	+						
	/4	vertical-angl	+ /	+ /	+						
7	2	photo-bundles	+	+	+	3	3	6	3		
	1	spat-distance			+						
	6	bundle-orient	+		+						
8	2	photo-bundles	+	+	+	3	3	6	3		
	1	spat-distance	+ /	+ /	+						
	3/	elevat-differ	+ /	+ /	+						
	/4	vertical-angl	+ /	+ /	+						
	5	azimuth	+ /	+ /	+						

Without loss of generality, we limit our discussion to non-simultaneous photographs so that the index j of the epoch of measurements (T_j) identifies the particular bundle of directions.

In aerial photogrammetry the projection center O_j is in a state of motion which makes direct measurements involving O-type with P-type and R-type points difficult if not impossible. The bundle orientation angles are treated as free variables or as weight-constrained parameters (measurements) depending on the availability of auxilliary information.

In terrestrial photogrammetry it is convenient to perform measurements which involve the projection centers O_j. In many terrestrial cameras the measurement of bundle orientation angles is straightforward.

A typical close-range-photogrammetry campaign of measurements for displacement and deformation analysis is defined as a series of metric photographs taken together with other field measurements. Such a campaign can last anywhere from a few seconds up to a number of years.

We should note an interesting difference between the various point groups which are involved in the analysis:

- group P are by definition four-dimensional points. Their velocities are perceived as the primary objectives of deformation analysis.
- group R are also four-dimensional points. However, in order to qualify as reference points, their velocities have to be insignificant. Their significance is tested by an F-test of hypothesis (H_0: $\dot{\mathbf{X}}_R = 0$). Thus, in terms of the null-hypothesis, those points are defined actually in three dimensions.
- group O are by definition three-dimensional points as their principal function as projection centers of the photogrammetric bundles is instantaneous. They are defined only at the moment of exposure.

The complete list of parameters pertaining to a typical deformation analysis problem is as follows:

- $\mathbf{X}_P$ standard epoch (T_0) coordinates of P-type points
- $\mathbf{X}_R$ standard epoch (T_0) coordinates of R-type points
- $\mathbf{X}_{Oj}$ coordinates of O_j (projection center of the T_j bundle)
- $\mathbf{X}_{Oj}$ orientation angles of the T_j bundle
- $\dot{\mathbf{X}}_P$ velocities of the P-type points
- $\dot{\mathbf{X}}_{R1}$ velocities of the R-type points
- $\dot{\mathbf{X}}_{R2}$ velocities of the R-type points

The partitioning of $\dot{\mathbf{X}}_R$ into $\dot{\mathbf{X}}_{R1}$ and $\dot{\mathbf{X}}_{R2}$ is required by the particular method of imposing free-net constraints which in turn is inherent in the definition of the XYZ reference system (Perelmuter, 1979; Papo and Perelmuter, 1982). $\dot{\mathbf{X}}_{R2}$ is of size d equal to the datum defect of the particular system of measurements as shown in Table 3.

A detailed derivation of partial derivatives of the measurements in Table 1 with respect to the above parameters would be contra-productive in misplacing the emphasis from the main issues of this paper. Table 3 presents a schematic substitute of a partial derivatives matrix, including constraints

TABLE 3. SCHEMATIC PARTIAL DERIVATIVES MATRIX WITH CONSTRAINTS

Parameters		3	3	.	3	3	.	$3k$	$3m\text{-}d$	d	$3k$	$3m\text{-}d$	d	
Measurements	Type	x_{Q1}	x_{Q2}	.	x_{O1}	x_{O2}	.	x_P	x_{R1}	x_{R2}	x_P	x_{R1}	x_{R2}	
photo-bundle T_1	2	///		.	///		.	///	////	///	///	////	///	
photo-bundle T_2	2		///	.		///	.	///	////	///	///	////	///	
. . . .	. . . . .	. . .	. . .	/	. . .	. . .	/	. . .	. . . .	. . .	. . .	. . . .	. . .	
orien-bundle T_1	6	///		.			.							
. . . .	. . . . .	. . .	///	.	. . .	. . .	.	. . .	. . . .	. . .	. . .	. . . .	. . .	
spatial distn	1			.	///	///	.							
elevation diff	3			.	///	///	.							
vertical angle	4			.	///	///	.							
spatial distn	1			.			.	///	////	///	///	////	///	
elevation diff	3			.			.	///	////	///	///	////	///	
vertical angle	4			.			.	///	////	///	///	////	///	
azimuth	5			.			.	///	////	///	///	////	///	
Weight Constr.										///				d
Free Net Constr.												////////		d

which are imposed on some of the parameters. The datum defect of the various observational systems is eliminated by imposing two types of constraints as follows:

- weight constraints are applied on d of the $\mathbf{X}_R$ coordinates ($\mathbf{X}_{R2}$) for defining the XYZ system at the T_0 epoch, and
- d free-net constraints are applied on the $\dot{\mathbf{X}}_R$ velocities in order to define XYZ as a reference frame for rates of translation and rotation.

The free net constraints minimize the sum of squares of the reference-point velocities as shown in Perelmuter and Papo (1983).

$$\dot{\mathbf{X}}_R^{\mathrm{T}} \cdot \dot{\mathbf{X}}_R = \text{min.} \tag{4}$$

Equation 4 is transformed (through differentiation) into d independent linear conditions which are to be satisfied by the $\dot{\mathbf{X}}_R$ velocities

$$[\mathbf{C}_1\ \mathbf{C}_2] \cdot \begin{bmatrix} \dot{\mathbf{X}}_{R1} \\ \dot{\mathbf{X}}_{R2} \end{bmatrix} = 0 \tag{5}$$

where $[\mathbf{C}_1\mathbf{C}_2]$ is Helmert's transformation matrix.

$\mathbf{C}_2$ is inverted (it is a full rank matrix, as shown in Perelmuter (1979)) and used to form the linear relationship between $\dot{\mathbf{X}}_{R1}$ and $\dot{\mathbf{X}}_{R2}$: i.e.,

$$\dot{\mathbf{X}}_{R2} = -\mathbf{C}_2^{-1} \cdot \mathbf{C}_1 \cdot \dot{\mathbf{X}}_{R1}. \tag{6}$$

There are definite advantages in processing the measurements made at successive T_j epochs by a sequential mode of adjustment. At each T_j stage the normal equations formed for all of the above parameters are reduced by folding-in the nuisance ($\mathbf{X}_{Qj}$,$\mathbf{X}_{Oj}$) and the datum ($\mathbf{X}_{R2}$,$\dot{\mathbf{X}}_{R2}$) parameters. The nuisance parameters $\mathbf{X}_p$ and $\dot{\mathbf{X}}_{R1}$ are not folded-in as they are common to all the T_j batches. The $\dot{\mathbf{X}}_{R2}$ parameters are evaluated from $\dot{\mathbf{X}}_{R1}$ through Equation 6. Each new update (T_j) of $\dot{\mathbf{X}}_P$ and $\dot{\mathbf{X}}_R$ is subjected to the following testing and transformation procedures:

- The consistency of the R-points' motion is tested through an F-test null-hypothesis imposed on the $\dot{\mathbf{X}}_R$ velocities. A rejection of the null-hypothesis signifies that one or more of the R-points deviate significantly in their motion as compared to the other R-points. Those points are identified and are transfered temporarily from the R-group.
- The P-type points are partitioned as necessary into P-subgroups according to external (subjective) information. The velocities of each P-subgroup are transformed into rates of translation and rotation with respect to the R-group (the XYZ reference system) and into rates of deformation. The residual velocities of the P-subgroups are inspected visually or are tested statistically as a feasibility check of the partitioning.

SYSTEM SENSITIVITY TESTING

We have seen above that estimability of the parameters depends on proper selection of the type of measurements which constitute the system. It is equally important to design the accuracies of the various measurements so that the parameters would be estimated at a desired level of significance.

Fraser (1982) describes experiments which were simulated to test the sensitivity of a number of photogrammetric measurement systems. Applied to a list of marginal velocities of object-points, the same method can be used to test the capacity of the system to detect those velocities at a certain significance level.

A variable U which has an F-distribution is computed as follows:

$$U = \frac{\dot{\mathbf{X}}_P^{\mathrm{T}} \cdot \dot{\mathbf{Q}}_P^{-1} \cdot \dot{\mathbf{X}}_P}{3k \cdot \sigma_0^2} \quad (7)$$

where $\dot{\mathbf{Q}}_P$ is the covariance matrix of the velocities of k object points (which form a P-subgroup) (the $\dot{\mathbf{Q}}_P$ matrix is obtained by a simulation of the particular measurements system which is being tested);

σ_0^2 is the variance of unit weight which would be estimated from the weighted sum of squares of measurement corrections divided by f, the simulated degrees of freedom (in Equation 7 its value can be set equal to 1.0); and

$\dot{\mathbf{X}}_P$ are values of the marginal velocities of the k object points. Those values are specified by the party interested in the analysis.

If the null-hypothesis ($\dot{\mathbf{X}}_P = 0$) is not rejected (in case $U < F_{3k,f,\alpha}$) it would mean that the geometry and the quality of the simulated measurements ($\dot{\mathbf{Q}}_P$) are inadequate for determining $\dot{\mathbf{X}}_P$ at the α level of significance.

Through proper modification (improvement) of the simulated measurements, an eventual situation should be reached where the null-hypothesis is finally rejected. Then we can be confident (at the α level) that the system of measurements when performed in practice will be sensitive enough to detect velocities of the magnitude of $\dot{\mathbf{X}}_P$.

SUMMARY AND CONCLUSIONS

We have shown in some detail how the proposed method for analysis of deformations can be applied in close-range photogrammetry. Compared with other methods employed in geodesy and photogrammetry (Perelmuter and Papo, 1983), it stands out by its general and straightforward mathematical model. The validity of the few assumptions which are made is tested as an integral part of the computational procedure.

The main purpose of the paper was to draw attention to two problem areas in deformation analysis, namely, the need for datum definition in a dynamic environment and the limitations in type and identity of parameters which can be determined from the measurements. An attempt was also made to present the principal goals of deformation analysis in simple geometric (or rather kinematic) terms.

The computational procedure was presented schematically emphasizing the sequence of elementary steps rather than going into the details of partial differentiation of the various measurements. So far we have had no opportunity to apply our method to concrete practical cases in close-range photogrammetry. Judging, however, from many simulations as well as from the sound theoretical foundation, we feel confident in the merits and feasibility of the proposed method of analysis.

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APPENDIX A

DECOMPOSITION OF A VELOCITY FIELD INTO HOMOGENEOUS AND RESIDUAL COMPONENTS

We apply principles of continuum mechanics to the analysis of a point-velocity field. The degree of cohesion of solid objects and structures has a direct bearing on their motion and deformation. The structures are usually represented by a network of marked points which have strongly correlated velocities. Those correlations are directly proportional to the cohesion of the structure.

Let us have such a network defined at a given moment with respect to a stable reference system by the vectors of position **X** and velocity $\dot{\mathbf{X}}$ of its k points. First we evaluate an average velocity for the network:

$$[\bar{\dot{x}}\ \bar{\dot{y}}\ \bar{\dot{z}}] = \frac{1}{k}\sum_{i=1}^{k}[\dot{x}_i\ \dot{y}_i\ \dot{z}_i] \tag{A1}$$

The vector $(\bar{\dot{x}},\bar{\dot{y}},\bar{\dot{z}})$ identifies the velocity of the network's center of mass with respect to the reference system.

The velocities of the individual points, in addition to the above global translational velocity, are due also to rotation as well as to network deformation, as shown in Mase (1970). The residual velocities (after subtracting the above homogeneous components) can be regarded as virtually random quantities. If, however, the network is moving as a system of several rigid subnets, we have to identify respective rates of translation, rotation, and deformation for every subnet. As a reference we can use the same or an alternative system, whichever is more convenient and useful.

Partitioning of the $\dot{\mathbf{X}}$ velocities vector into homogeneous and residual components is accomplished through **L**, the velocities' tensor. **L** is basically a transformation matrix analogous to the incremental tensor as shown by Brunner (1979). Its 3 × 3 = 9 elements contain the information required to define the rotation and the deformation of the network. **L** is defined implicitly through Equation A2 which expresses the relationship between position and velocity of a point in the network: i.e.,

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_i - \begin{bmatrix}\bar{\dot{x}}\\ \bar{\dot{y}}\\ \bar{\dot{z}}\end{bmatrix} = (\mathbf{L}-\mathbf{I})\cdot\left[\begin{bmatrix}x\\ y\\ z\end{bmatrix}_i - \begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix}\right] + \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{il} \tag{A2}$$

where **L** is the network velocity tensor,
I is the 3 × 3 unit matrix,
$(\dot{x},\dot{y},\dot{z})_{il}$ are the components of residual velocity of point P_i, and
$(\bar{x},\bar{y},\bar{z})$ are the instantaneous coordinates of the network's mass-center.

Matrix **L** is partitioned into a unique pair of a symmetric and a skew-symmetric matrices as follows:

$$\mathbf{L} = \mathbf{R} + \mathbf{D} = \begin{bmatrix}0 & -r_z & r_y\\ r_z & 0 & -r_x\\ -r_y & r_x & 0\end{bmatrix} + \begin{bmatrix}d_{xx} & d_{xy} & d_{xz}\\ d_{yx} & d_{yy} & d_{yz}\\ d_{zx} & d_{zy} & d_{zz}\end{bmatrix} \tag{A3}$$

where **R** is the vorticity tensor of the network, while matrix **D** is identified as the deformation-rate tensor (Mase, 1970).

Now Equation A2 is written in a more meaningful form: i.e.,

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_i = \begin{bmatrix}\bar{\dot{x}}\\ \bar{\dot{y}}\\ \bar{\dot{z}}\end{bmatrix} + \mathbf{R}\cdot\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix}_i + (\mathbf{D}-\mathbf{I})\cdot\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix}_i + \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{il} \tag{A2'}$$

where $(\bar{x},\bar{y},\bar{z})_i$ are now the coordinates of point P_i with respect to a Cartesian system which is parallel to the reference system and has its origin at the network's center of mass.

The velocity components in Equation A2′ due to translation and to rotation can be rewritten as follows:

$$\begin{bmatrix}\bar{\dot{x}}\\ \bar{\dot{y}}\\ \bar{\dot{z}}\end{bmatrix} + \mathbf{R}\cdot\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix}_i = \begin{bmatrix}1&0&0&0&\bar{z}_i&-\bar{y}_i\\ 0&1&0&-\bar{z}_i&0&\bar{x}_i\\ 0&0&1&\bar{y}_i&-\bar{x}_i&0\end{bmatrix}\cdot\begin{bmatrix}\bar{\dot{x}}\\ \bar{\dot{y}}\\ \bar{\dot{z}}\\ r_x\\ r_y\\ r_z\end{bmatrix} = \mathbf{C}_i^{\mathrm{T}}\cdot\mathbf{p} \tag{A4}$$

where **p** is a 6 × 1 vector of rates of displacements (translation and rotation) of the network with respect to the reference system, and
$\mathbf{C}_i$ is part of the well known Helmert's transformation matrix **C**.

The components in Equation A2′ due to deformation can be also rewritten as follows:

$$(\mathbf{D}-\mathbf{I})\cdot\begin{bmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{bmatrix}+\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}_{il}=\begin{bmatrix}\bar{x}_i & 0 & 0 & 0 & \bar{z}_i & \bar{y}_i\\ 0 & \bar{y}_i & 0 & \bar{z}_i & 0 & \bar{x}_i\\ 0 & 0 & \bar{z}_i & \bar{y}_i & \bar{x}_i & 0\end{bmatrix}\cdot\begin{bmatrix}d_{xx}{}^{-1}\\ d_{yy}{}^{-1}\\ d_{zz}{}^{-1}\\ d_{yz}\\ d_{xz}\\ d_{xy}\end{bmatrix}=\mathbf{F}_i^{\mathrm{T}}\cdot\mathbf{q} \tag{A5}$$

where **q** and $\mathbf{F}_i$ are analogous to **p** and $\mathbf{C}_i$. They pertain to homogeneous deformation.

We extend now Equation A2′ to include all the points in the network while substituting Equations A4 and A5: i.e.,

$$\dot{\mathbf{X}}=(\mathbf{C}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}})\cdot\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix}+\dot{\mathbf{X}}_l=\mathbf{H}^{\mathrm{T}}\cdot\mathbf{g}+\dot{\mathbf{X}}_l \tag{A6}$$

where $\dot{\mathbf{X}}_l$ is the vector of residual velocities.

An important property of the $\dot{\mathbf{X}}_l$ velocities is their compliance with the following conditions similar to the well known inner adjustment constraints as formulated by Meissl (1969):

$$\mathbf{H}\cdot\dot{\mathbf{X}}_l=0 \tag{A7}$$

Equation A7 holds for an arbitrary reference system, or in other words, another choice of a reference system would result in exactly the same residual velocities (Meissl, 1969; Perelmuter and Papo, 1983).

Let us have a system of measurements which when processed according to least-squares principles results in estimates of $\dot{\mathbf{X}}$ and its covariance matrix $\dot{\mathbf{Q}}$. Provided the network is composed of at least four noncoplanar points, the **H** matrix is of full (12) rank and Equation A6 can serve as a model for unbiased least-squares estimation of **g**. The normal equations are formed on the basis of the following minimum condition (least squares):

$$\dot{\mathbf{X}}_l^{\mathrm{T}}\cdot\dot{\mathbf{Q}}^{-1}\cdot\dot{\mathbf{X}}_l=\min. \tag{A8}$$

The solutions for **g** and its covariance matrix are derived in the usual way:

$$\mathbf{g}=(\mathbf{H}^{\mathrm{T}}\cdot\dot{\mathbf{Q}}^{-1}\cdot\mathbf{H})^{-1}\cdot(\mathbf{H}^{\mathrm{T}}\cdot\dot{\mathbf{Q}}^{-1}\cdot\dot{\mathbf{X}})$$
$$\mathbf{\Sigma}_g=(\mathbf{H}^{\mathrm{T}}\cdot\dot{\mathbf{Q}}^{-1}\cdot\mathbf{H})^{-1} \tag{A9}$$

The above is not the only rigorous solution of the problem and not necessarily the best one. The residual velocities $\dot{\mathbf{X}}_l$ can be treated as stochastic variables with an autocovariance matrix which is virtually independent of the measurement errors. Such an approach would entail the use of methods similar to those proposed by Hein and Kisterman (1984).

APPENDIX B

CONCEPTS IN ESTIMATION THEORY

Certain terms referred to frequently in the paper are defined in this Appendix in order to preclude ambiguity and misinterpretation of their meaning.

Parameters in the mathematical model of a set of measurements (Papo, 1973) which can be determined (estimated) uniquely from processing those measurements without the need for any additional information are *estimable parameters* (Grafarend and Schafrin, 1976). The above concept can be extended to include a case where the set of observation equations is appended by linear constraints. In such a case the parameters are estimable subject to those constraints.

The number of non-estimable parameters in a system define the size of its *defect*. One part of the defect which can be corrected by additional measurements of types already present in the existing set is known as *configuration* defect (Welsch, 1979; Wolf, 1983). The remaining defect which can be corrected either by introducing new types of measurements or by imposing linear *constraints* on the parameters is defined as the *datum* defect of the existing system of observation equations. It is customary to associate the datum defect with the need to define a reference coordinate system. In a system of observation equations whose defects have been corrected by additional measurements and constraints, all the parameters are estimable subject to those additions.