Performance Evaluation of Two Bivariate Processes for DEM Using Transfer Functions

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ABSTRACT: The performance of linear least-squares and multiquadric surface interpolation techniques depends greatly on the expedient selection of the appropriate covariance or distance function. In processing digital elevation models data, these functions are not theoretically deterministic. They are either derived experimentally or simply assumed. The experimental determination of covariance and distance functions from observed data proved to be impractical, while an arbitrary chosen function may not be suitable for a given data set. This paper employs an analytical approach in which the interpolation process is applied to fictitious DEM data with known frequency content. Transfer functions are then determined to evaluate the fidelity achieved when using an *a priori* selected covariance or distance function. The frequencies of the generated transfer function surfaces are used as a measure of fidelity, and, hence, a relationship is established between the terrain type and the chosen function or its parameters. This relationship facilitates optimum choice of covariance or distance functions to be used with a given DEM data set. Moreover, it can be used in planning new DEMs by determining optimum sampling densities.

INTRODUCTION

THE LINEAR LEAST-SQUARES INTERPOLATION METHOD (LLS) has found its way as a powerful tool in filtering and processing digital elevation models data. This method is based on the general theory of linear prediction which is applicable to stationary random functions. In the bivariate mode, a random function is said to be homogeneous and isotropic if its autocorrelation function depends only on the distance between points (Schut, 1974). When the autocorrelation function is presented as a function of distance only, it is often called a covariance function (CV) (Hardy, 1977).

In linear prediction theory each observed value is assumed to consist of two components signal and noise. The problem of prediction is to separate signal from noise, which can only be achieved if the two components are uncorrelated with each having a known covariance function. However, in practice, covariance functions are usually unknown, and, therefore, the problem of estimating a covariance function most appropriate to a given set of data is extremely important.

Another bivariate interpolation process is that based on the multiquadric surfaces (MS) technique presented by Hardy (1977). Although this technique is not based on the theory of linear prediction, it is applied in practice mainly because it does not necessitate the stationarity of input data, which is sometimes considered a drawback in the LLS method. Multiquadric surfaces interpolation utilizes a distance function which does not exactly satisfy the conditions of a true covariance function.

The objective of this paper is to develop an analytical-rather than experimental-approach by which the performance of linear least-squares and multiquadric surfaces interpolation techniques can be evaluated using different covariance and distance functions for different types of fictitious double fourier surfaces. This analytical approach is based on the well known transfer functions (TF) concept. A transfer function is defined as the fourier transform of the impulse response function (IRF) of a given process (Chatfield, 1975). The impulse response functions and transfer functions are two ways of describing a linear process, in a somewhat similar way that the covariance function and spectral density function are two ways of describing a stationary random set of data, one function being the fourier transform of the other. Therefore, the impulse response function is applied in the spatial domain while the latter transfer function works in the frequency domain. The beauty of the transfer functions approach is that inspection of the diagonal profile curves of these transfer functions shows clearly which frequency components of the input will be emphasized or deemphasized by the bivariate process.

LINEAR LEAST SQUARES INTERPOLATION

DEM data can be considered, after trend removal,

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as a random field of reference points P_i, $i=1, 2, \ldots, n$, at each of which an observed elevation, Z_i, is given. According to the theory of linear prediction, Z_i is composed of two components, a signal or "true" elevation, S_i, and a noise or "error," r_i; i.e.,

$$Z_i = S_i + r_i. \quad (1)$$

The problem of interpolation is to find an estimate, $\hat{S}_j$, for the component, S_j, at a point, P_j, other than the reference points. This estimate must be determined using a linear combination of all the reference elevations, Z_i (Kraus and Mikhail, 1972); i.e.,

$$\hat{S}_j = \mathbf{C}^T \cdot \mathbf{Z} \quad (2)$$

where

$\mathbf{C}$ is a vector of coefficients $[C_1, C_2, \ldots, C_n]^T$ and
$\mathbf{Z}$ is the vector of reference elevations $[Z_1, Z_2, \ldots, Z_n]^T$.

Obviously, there are infinite estimates of S_j depending on the choice of the elements of the vector $\mathbf{C}$. Of all these estimates it is required to determine the estimate which gives minimum variance. The deviation of $\hat{S}_j$ from the unknown "true" value, S_j, at point, P_j, is given by

$$V_j = S_j - \mathbf{C}^T \cdot \mathbf{Z}. \quad (3)$$

Applying the variance-covariance propagation technique, (Mikhail, 1976) the variance of V_j can be written as

$$\boldsymbol{\sigma}_v^2 = \mathbf{J} \cdot \mathbf{R} \cdot \mathbf{J}^T \quad (4)$$

where

$\boldsymbol{\sigma}_v^2$ is the variance of V_j;
$\mathbf{J}$ is the Jacobian $[1 \; -\mathbf{C}^T]$;

$$\mathbf{R} = \begin{bmatrix} \boldsymbol{\sigma}_s^2 \, \mathbf{B}_{zs} \\ \mathbf{B}_{zs} \mathbf{A} \end{bmatrix}$$

is the symmetric matrix of autocovariances and cross-covariances;
$\boldsymbol{\sigma}_s^2$ is the variance of S_j;
$\mathbf{B}_{sz} = \mathbf{B}_{zs}^T$ is the vector of crosscovariances between S_j and each element of $\mathbf{Z}$ (each value is obtained from the covariance function for each distance between P_j and P_i); and
$\mathbf{A}$ is the symmetrical autocovariance matrix of the data.

Substituting the values of $\mathbf{J}$ and $\mathbf{R}$ in Equation 4, we obtain

$$\boldsymbol{\sigma}_v^2 = \boldsymbol{\sigma}_s^2 - 2 \cdot \mathbf{B}_{sz} \cdot \mathbf{C} + \mathbf{C}^T \cdot \mathbf{A} \cdot \mathbf{C}. \quad (5)$$

Because it is required to minimize $\boldsymbol{\sigma}_v^2$, the vector of unknown coefficients $\mathbf{C}$ should be selected such that $\frac{\partial \boldsymbol{\sigma}_v^2}{\partial \mathbf{C}} = 0$. Therefore, we have

$$\hat{S}_j = \mathbf{B}_{sz} \cdot \mathbf{A}^{-1} \cdot \mathbf{Z}. \quad (6)$$

Using Equation 6, it is possible to either estimate the elevations of new points or to filter the elevations at the reference points. In this paper, Equation 6, based on linear least-squares interpolation, is used to densify DEM data in a bivariate mode by estimating elevations of new points. The values of the elements of the vector of cross-covariances and the autocovariances matrix are determined using a covariance function which is a function of distance, d, only and satisfies the following conditions (Kraus and Mikhail, 1972):

- Its value at the origin must be positive $\mathbf{CV}(o) > o$
- It must be an even function $\mathbf{CV}(-d) = \mathbf{CV}(+d)$
- Its value at any distance must be less or equal to its value at the origin $|\mathbf{CV}(d)| \leq \mathbf{CV}(o)$

The covariance functions tested will be discussed separately in more details.

MULTIQUADRIC SURFACES INTERPOLATION

This method employs an algebraic formulation which is almost identical to the one used in linear least squares. The only difference is that a conic or a hyperboloid function, which is not a covariance function, is used in this method.

The method of interpolation using multiquadric surfaces may be geometrically interpreted as a summation of surfaces of revolution such that each of them is centered at one of the reference points. In the case of conic summation, the elevation, z_j, of a new point, P_j, can be represented as follows:

$$Z_j = \sum_{i=1}^{n} f_i \, d_{ji} \quad (7)$$

where

Z_j is the interpolated elevation at point P_j;
f_i, $i=1,2,\ldots,n$ are coefficients which determine the properties of each surface; and
d_{ji} are the distances between reference points P_i, $i=1,2,\ldots,n$ and the interpolated point (P_j).

Now suppose that n equations in the form of Equation 7 are to be written for n reference points. These equations may be expressed in matrix form as follows:

$$\mathbf{Z} = \mathbf{A} \cdot \mathbf{F} \quad (8)$$

where

$\mathbf{Z}$ is the vector of elevations of reference points,
$\mathbf{A}$ is the symmetrical matrix of distances, and
$\mathbf{F}$ is the vector of coefficients of the conic surface.

The unknown coefficient vector $\mathbf{F}$ can then be determined according to Equation 9: i.e.,

$$\mathbf{F} = \mathbf{A}^{-1} \cdot \mathbf{Z}. \quad (9)$$

The interpolated elevation at any point P_j is determined using Equation 10: i.e.,

$$Z_j = \mathbf{B}_{ji} \cdot \mathbf{A}^{-1} \cdot \mathbf{Z} \quad (10)$$

where $\mathbf{B}_{ji}$ is the row vector of distances between point P_j and each reference point P_i, $i=1,2,\ldots,n$.

Although Equations 6 and 10 are similar in form, matrices $\mathbf{B}$ and $\mathbf{A}$ are determined differently for the two interpolation processes.

COVARIANCE AND DISTANCE FUNCTIONS

The covariance function of a stationary random set of data is defined as the fourier transform of the spectral density function of these data. The covariance function, $\mathbf{CV}(d)$, is a function of the separation (distance) between two values. The determination of $\mathbf{CV}(d)$ from observed values cannot be performed accurately if the number of observations is not too large. Therefore, in practice a covariance function, which hopefully agrees well with the data, is assumed. In this paper, the performance of the following covariance functions are tested using linear least-squares interpolation: i.e.,

$$\mathbf{CV}(d) = k\ e^{-ad^2} \quad \text{(Gaussian)} \qquad (11)$$

$$\mathbf{CV}(d) = k\ e^{-a|d|} \quad \text{(Exponential)} \qquad (12)$$

$$\mathbf{CV}(d) = k/(e^{d} + e^{-d}) \qquad (13)$$

$$\mathbf{CV}(d) = k/(2|d|) \qquad (14)$$

$$\mathbf{CV}(d) = k \cos(a\pi d) \qquad (15)$$

In the case of multiquadric surface interpolation, the conic and hyperboloid functions are used. However, because these surfaces do not qualify as covariance functions, they are called distance functions. The following equation is tested in this case (Hardy, 1977):

$$\mathbf{CV}(d) = k(d^2 + a)^{1/2}. \qquad (16)$$

In all these functions, d denotes the distance while a and k are the function parameters.

The most commonly used covariance functions are the Gaussian and the exponential. The Gaussian covariance function (Equation 11) describes a smooth surface with continuous first derivatives while the exponential covariance function (Equation 12) represents an irregular surface with sharp peaks at the location of reference points (Mikhail, 1976). Covariance functions represented by Equations 13 and 14 are modifications of the Gaussian and exponential forms, whereas Equation 15 represents a cosine covariance function. The distance function (Equation 16) represents a conic surface if $a=0$, or a hyperboloid of revolution if $a \neq 0$. Graphs of the six functions using different values of the parameters k and a are shown in Figure 1. The graph of the covariance function is sometimes called the autocovariogram. If the ordinates of points are divided by the value of the sample variance, the resulting graph is then called the autocorrelogram.

PERFORMANCE EVALUATION USING TRANSFER FUNCTIONS

Linear least-squares and multiquadric interpolation algorithms are used to densify DEM data by interpolating elevations at intermediate points. In order to evaluate the efficiency and fidelity of these algorithms, fictitious data are used. The frequency characteristics of these data must be known, so that it would be possible to draw conclusions about the behavior of each method with respect to different inputs. An efficient way of assessing the performance of these interpolation algorithms is the use of the well known concept of the frequency response functions, usually known as the transfer function (Laan, 1973). This function shows how a linear system responds to sinusoids at different frequencies. The frequency here can be used as an indication of terrain undulations. In order to calculate the values of the transfer function surface in the bivariate mode, a double fourier series is used as an input. The basic element of such a series is the term $\sin(\omega x + \psi y)$, which is analogous to $\sin(\omega x)$ in the univariate case. The terms ω and ψ are the frequencies in the x and y directions, respectively. Accordingly, the function $z(x,y)$ may be expressed as follows:

$$z(x,y) = \sin(\omega x + \psi y). \qquad (17)$$

This function has unit amplitude and zero phase. Equation 17 is used to generate input data for the computation of the transfer function surface. In order to sample this function, a set of points (x_q, y_ℓ); $q = 1,2,\ldots,n$; and $\ell = 1,2,\ldots,m$, is chosen and the corresponding elevations are computed. To minimize the computational effort, sampling is taken in a homogenious square grid with an interval of Δx in both directions. The square Δx by Δx is called a cell. A patch consists of four cells and has a local coordinate system as shown in Figure 2.

The sampling density (S) in the X and Y directions is defined as the number of sampling intervals Δx per period: i.e.,

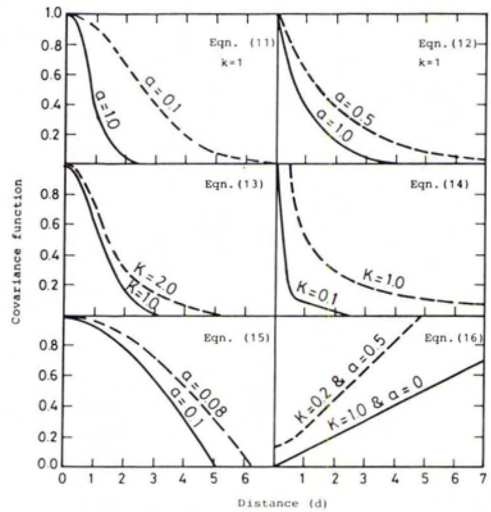

FIG. 1. Graphical representation of the six covariance and distance functions (autocovariogram) used with the bivariate processes.

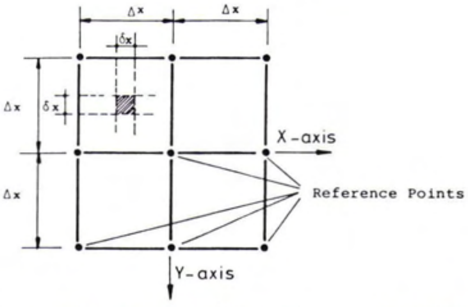

FIG. 2. A patch composed of four cells and nine reference points.

$$S = 2\pi/\omega.\Delta x. \quad (18)$$

This sampling density depends on the value of the product $\omega.\Delta x$. Therefore, when Δx is kept constant for all surfaces, the sampling density becomes inversely proportional to the frequency. The transfer ratio τ at frequencies ω and ψ are determined at intervals of $1/S$. Fictitious elevations at reference points as well as points within each patch are computed based on Equation 17. Meanwhile, interpolated elevations at these points are computed using the methods of linear least-squares and multiquadric surfaces techniques (Equations 6 and 10, respectively). The difference betwéen the interpolated and fictitious elevations is known as the error of interpolation. The root-mean-square error of all errors over the area is used to determine the transfer ratio, τ, of the used algorithm. Generally, the transfer ratio, τ, represents a relationship between input and output of a dynamic system (Schwarzenbach and Gill, 1979), indicating the fidelity of interpolation. This transfer ratio is calculated for all frequencies up to the Nyquist frequency $\pi/\Delta x$ as follows:

$$\tau(\omega,\psi) = \frac{a - c \cdot e(\omega,\psi)}{a} \quad (19)$$

where

- $\tau(\omega,\psi)$ is the transfer ratio at frequencies ω and ψ in the x and y directions, respectively;
- a is the amplitude of the fourier function, assumed to be unity;
- $e(\omega,\psi)$ is the root-mean-square error of interpolation over the whole area; and
- c is a numerical coefficient.

The numerical value of the coefficient c is determined by taking the case in which $\Delta x = 2\pi$ and $\omega = \psi = 1$, where the reference points will all have zero elevations. Therefore, all interpolated elevations will have zero values and the error of interpolation will thus be equal to the value of the function $Z(x,y)$ at the desired points. The root-mean-square error of interpolation will be equal to the integration of the sine function (Equation 17), which equals $1/\sqrt{2}$. Because the value of $\tau(\omega,\psi)$ is this case must be zero, it then follows that the numerical value of c should be equal to $\sqrt{2}$. Therefore,

$$\tau(\omega,\psi) = 1 - \sqrt{2} \cdot e(\omega,\psi) \quad (20)$$

It is possible, using Equation 20, to determine the values of $\tau(\omega,\psi)$ at each combination of ω and ψ, which results in a surface transfer function. In order to visually represent these transfer functions, diagonal profiles are plotted resulting from different covariance and distance functions. These curves would give adequate information about the performance of the algorithm under consideration at different frequencies. Figures 3 to 9 show the diagonal profiles of the transfer functions corresponding to the six covariance and distance functions expressed by Equations 11 to 16. These curves allow a quick comparison between the fidelity resulting from different functions for a certain parameter value or for comparing the effect of different parameter values on the fidelity of a certain function. This will be discussed in more details next. In this research, transfer functions were obtained by changing the frequency while maintaining the sampling interval constant. Needless to say, the same results would have been obtained by maintaining a fixed frequency and varying the sampling interval. The physical interpretation of this argument is that the first case corresponds to having different fictitious terrain types with one sampling density, while the later case represents one terrain type with different sampling densities.

DISCUSSION AND CONCLUSIONS

The fidelity of any bivariate process is based on the fact that the total loss of information due to prediction can be represented by the area above the transfer function diagonal profile. The smaller the area above the profile, the better would be the algorithm used. Furthermore, the range of frequencies in these diagonal curves may be subdivided into two or more regions. The smaller the value of 1/S, the lower would be the frequencies corresponding to flatter terrain type. The larger the value of 1/S, the higher would be the frequencies corresponding to more undulating and rougher terrain types. Accordingly, the diagonal profile curves of the transfer functions (Figures 3 to 9) resulting from using the different covariance and distance functions for the two bivariate processes considered offer an objective way of assessing their performances.

Linear least-squares interpolation renders optimum results if the input data have the characteristics of a stationary random function, provided that the proper covariance function is used. The fidelity of this method depends strongly on the values of the parameters as well as the chosen covariance function. Using a Gaussian covariance function (Figures 3 and 4), it was found that the loss of in-

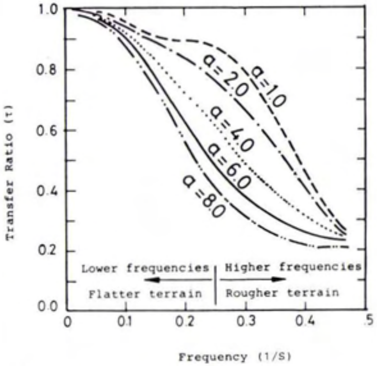

FIG. 3. Diagonal profiles of the transfer function using the Gaussian covariance function **CV**(d) = $k.e^{-ad^2}$, with linear least-squares interpolation (k = 1).

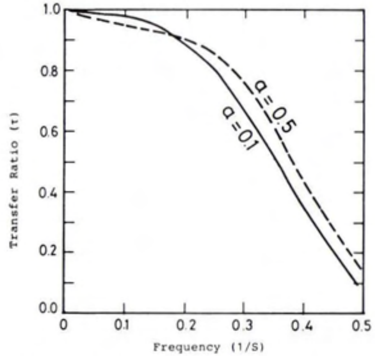

FIG. 4. Diagonal profiles of the transfer function using the Gaussian covariance function **CV**(d) = $k.e^{-ad^2}$, with linear least-squares interpolation (k = 1).

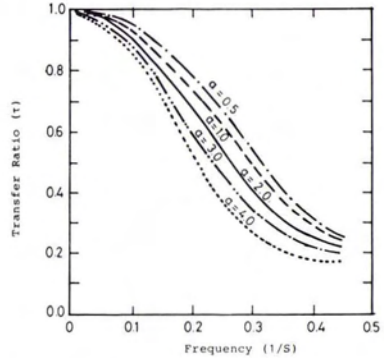

FIG. 5. Diagonal profiles of the transfer functions using the exponential covariance function **CV**(d) = $K.e^{-a|d|}$, with linear least-squares interpolation (k = 1).

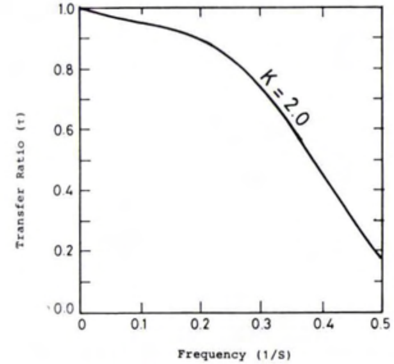

FIG 6. Diagonal profile of the transfer functions using the covariance function **CV**(d) = $k/(e^{d} + e^{-d})$, with linear least-squares interpolation (k = 2).

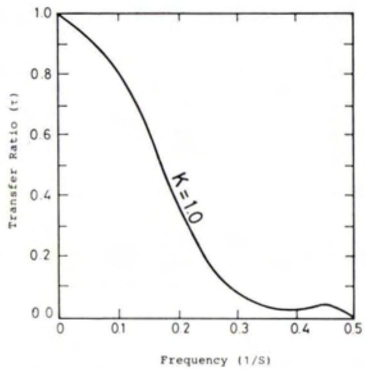

FIG. 7. Diagonal profile of the transfer functions using the covariance function **CV**(d) = $k/(2|d|)$, with linear least-squares interpolation (k = 1).

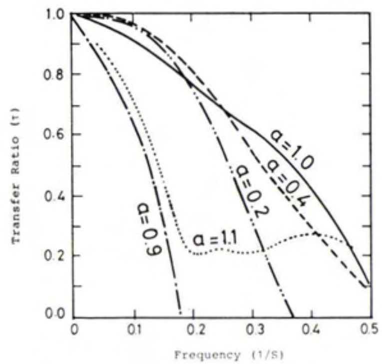

FIG. 8. Diagonal profiles of the transfer functions using the covariance function **CV**(d) = $k.\cos(a \cdot \phi \cdot d)$, with linear least-squares interpolation (k = 1).

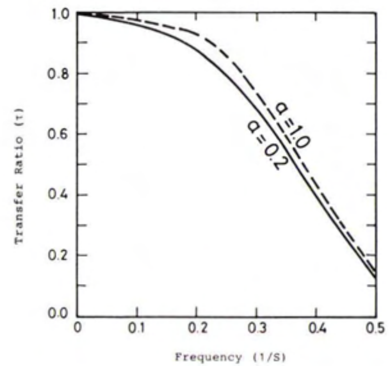

FIG. 9. Diagonal profiles of the transfer function using the distance function **CV**(d) = $k.(d^2 + a)^{1/2}$, with multiquadric surfaces interpolation (k = 1).

formation due to least-squares interpolation increased rapidly as the numerical value of parameter a was greater than 1 or less than 0.5. However, at very low frequencies (i.e., flat terrain), the transfer ratio improved with $a < 0.5$. A value of $a = 0.1$ is thus recommended for use with flat surface digital elevation models (Figure 4).

The exponential covariance function (Equation 12) gave results which are generally inferior to those obtained from the Gaussian function (Figure 5). This observation is obtained by comparing the diagonal profile curves of the transfer functions of Figures 3 and 4 with Figure 5. This behavior agrees with theoretical expectations, because the exponential function leads to surfaces that exhibit discontinuous slope changes and peaks at reference points (Mikhail, 1976). However, using a value for a in the vicinity of 0.5, the process gives improved results comparable to those obtained with the Gaussian function. With lower values of a, the fidelity of the process slightly decreased, but the stability of the model was still maintained.

The covariance function represented by Equation 13 gave excellent results with $K = 2$ (Figure 6). The fidelity using this function approaches and sometimes exceeds the fidelity resulting from the Gaussian function, particularly at medium frequencies. This similarity is attributed to the resemblance between the graphical representation of the covariance functions of Equations 11 and 13 shown in Figure 1.

The covariance function represented by Equation 14 showed a clear deterioration in the fidelity of the process by increasing the frequencies (Figure 7). This function may thus be recommended for low frequencies only, corresponding to flat or slightly undulating terrain.

The use of the cosine covariance function (Equation 15) should be considered with some care. Good results were obtained only when the numerical value of the parameter a ranged between 0.3 and 0.5. Beyond these limits the behavior of the function seems to be unstable (Figure 8). In some cases, the interpolation algorithm failed completely when the value of a exceeded 0.6, and thus this function should not be recommended for use.

Finally, the hyperbolic distance function used with the multiquadric surfaces interpolation algorithm resulted in the diagonal profile transfer function shown in Figure 9. With values of a equal to 1 and 2, these curves behaved nearly similar to the Gaussian covariance function shown in Figure 3. With frequencies less than 0.2, the fidelity using the hyperbolic function was even slightly superior.

In all cases, the use of the concept of transfer functions proved to be readily applicable in the bivariate mode. It provides an objective tool for assessing the performance of different interpolation algorithms with different covariance and distance functions. The use of this approach with real DEM data could lead to the development of several diagonal profile transfer function curves using different covariance function with different terrain types. Such curves could facilitate the choice of the most appropriate covariance function for a given DEM data set. They could also serve as a planning tool, where the sampling density could be determined to meet the requirement of an *a priori* specified transfer ratio.

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