Radarclinometry for the Venus Radar Mapper

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ABSTRACT: A mathematical theory and a corresponding algorithm have been developed to derive (in principle) topographic maps froom radar images as photometric arrays. Thus, as radargrammetry is to photogrammetry, so radarclinometry is to photoclinometry. Photoclinometry is endowed with a fundamental indeterminacy principle even for terrain homogeneous in normal albedo. This arises from the fact that the geometric locus of orientations of the local surface normal that is consistent with a given reflected specific intensity of radiation is more complicated than a fixed line in space. For a radar image, the locus is approximately a cone whose half-angle is the incidence angle and whose axis contains the radar. The indeterminacy is removed throughout a region if one possesses a control profile as a boundary condition. In the absence of such ground truth, a point-boundary condition will suffice only in conjunction with a heuristic assumption, such as that the strike-line runs perpendicularly to the line-of-sight. In the present study a more reasonable assumption, which I call "the hypothesis of local cylindricity," is implemented.

First, a general theory is derived, based solely on the implicit mathematical determinacy. The theory produces topography by an area integration of radar brightness, starting from a control profile, without need of additional idealistic assumptions. But we have also theorized separately a method of forming this control profile, which method does require an additional assumption about the terrain. That assumption is that the curvature properties of the terrain are locally those of a cylinder of inferable orientation, within a second-order mathematical neighborhood of every point of the terrain. While local strike-and-dip completely determine the radar brightness itself, they combine with the terrain curvature to determine the brightness gradient in the radar image. Therefore, the control profile is formed as a line integration of brightness and its local gradient starting from a single point of the terrain where the local orientation of the strike-line is estimated by eye.

Second, and independently, the calibration curve for pixel brightness versus incidence angle is produced. I assume that an applicable curve can be found from the literature or elsewhere so that our problem is condensed to that of properly scaling the brightness axis of the calibration curve. A first estimate is found by equating the average image brightness to the point on the brightness axis corresponding to the complement of the effective radar depression angle, an angle assumed given. A statistical analysis is then used to correct, on the one hand, for the fact that the average brightness is not the brightness that corresponds to the average incidence angle, as a result of the non-linearity of the calibration curve; and on the other hand, we correct for the fact that the average incidence angle is not the same for a rough surface as it is for a flat surface (and therefore not the complement of the depression angle).

INTRODUCTION

THE USE OF the photometric function of a surface for the purpose of translating image variegation into slope and relief information dates back at least to the work of van Diggelen (1951), who applied it to lunar maria when near the terminator; a condition for which a uniquely simple limiting form of photometric function was inferred through the Helmholtz reciprocity principle (Minnaert, 1941). Based on the constancy of surface brightness near the bright limb of the Moon, this inference has since fallen into disrepute (Wildey, 1978).

The development of a method for producing a topographic map from a photometric image begins with Rindfleisch (1966), and independently with Watson (1968), who devised a theory which was implemented by Lambiotte and Taylor (1967). This process is as elaborate compared to the former approach of Van Diggelen as photogrammetry is compared to basic trigonometric parallax. Accordingly, the term "photoclinometry" was invented by Jack McCauley in 1965, from the greek roots "photos" and "klinos," in reference to this process.

For a given illumination and normal albedo of a terrain point under consideration, the photometric function enables the prediction of a measurable surface brightness as seen from any direction through the specification of three angles: g, the phase angle,

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e, the emergence angle, and *i*, the incidence angle. While the first angle can be independently known, the latter two depend on the orientation of the local surface normal (equivalent to knowledge of local strike-and-dip). Thus, a knowledge of the local orientation of the terrain is essential to the prediction of surface brightness. In the inverse problem, where one has a measured surface brightness instead of known values of *e* and *i*, one degree of freedom remains in the form of an implied functional relation between *e* and *i*, rather than unique values thereof. That is, the measured surface brightness imposes a constraint on the local normal vector, without uniquely determining it. Its range of possibilities generates a surface. This is the fundamental indeterminacy of photoclinometry. In the case of Watson's theory, uniquely applicable to the lunar surface, the indeterminacy could be ignored by confining the process to a one-dimensional rather than two-dimensional topographic mapping (an ordinary topographic map, while inherently three dimensional in information content, is two dimensional in the sense that it represents height as a scalar function of two independent variables). The lunar photometric function, when particularly evaluated to correspond to the brightness of a single terrain point, generates a locus of surface normals that is a plane perpendicular to the phase plane. The measured surface brightness serves to specify the angular placement of the intersection of these two planes as a direction, as seen from the surface, algebraically between the direction to the illuminator and the direction to the observer. Watson showed that the intersection of the phase plane with the true surface, as a topographic profile, was uniquely determined by the variation in surface brightness along the corresponding line in the image. Unfortunately, no grounds exist for adjusting the relative range to two such profiles, which cannot intersect, taken from parallel data traverses in the same two-dimensional image.

In order to produce a two-dimensional topographic map purely by photoclinometry, the mathematical constraint imposed by an additional assumption is essential. There is no such thing as a completely reasonable assumption. But some assumptions are more absurd than others. One method of extending Watson's approach to two dimensions would be to adjust each parallel profile for a minimum of the integral of the square of height above an adopted lunar ellipsoid. While obviously not obeyed by the lunar surface, this assumption is nevertheless clearly superior to the assumption that the initial point on each profile is located precisely on the lunar ellipsoid. But only bodies of lunar-like reflective properties would qualify for this treatment. The photometric functions characterizing most of the various *Martian* terrain types are distinctly non-lunar in character. Thus, the first generalized two-dimensional photoclinometry was developed by Wildey (1974, 1975) in which the auxiliary assumption took the form of the Eulerian equations from the calculus-of-variations for the minimization of the total surface area. The analysis was mathematically interesting, the result plausible, and the algorithm eminently impractical.

An altogether different type of limitation on photoclinometry arises from the fact that planetary surfaces that are homogeneous in normal albedo are rare. Progress toward the alleviation of this difficulty has recently been made by Eliason *et al.* (1981), who used multi-color image sets of the same terrain in an operations-research approach to separate topography from albedo variation. Though representative of the greatest utility achieved to date to extract information in images due to topography, the auxiliary assumption used to render the photoclinometric portion of the study tractable was that the normal vector was confined to a plane containing the terrain point, the planetary center, and the sun. The mathematical independence of parallel down-sun lines of integration of the topography then produces, after filtering, a two-dimensional result with slopes in all directions, rather than merely toward the sun. A Lambertian photometric function was also assumed. Thus, at the level of fundamental assumptions, the metric integrity of true photogrammetry was not approached. The separation of insolation and albedo variations is the important contribution of that work.

The primary goal of the present study has been the adaptation of photoclinometry to radar imagery, whether it be the type in which the reflected pulse generates all of a down-range raster in the final image ("brute-force" radar imaging systems, hereinafter referred to as SLAR), or the type in which the reflected pulse, as complex electric amplitude, is but part of a contribution to a large synthesized aperture used to form the final image (SAR). The first application of the present technique has been published (Wildey, 1984). The present paper documents the underlying mathmatical theory. Its development occurs in response to a need for topographic information during the nominal mission of the Venus Radar Mapper, when radargrammetric results will be still unavailable.

The radar image as a candidate for photoclinometry offers both advantages and disadvantages when compared to ordinary optical images. The most obvious advantage is the simplicity of the photometric function. The radar becomes both the sun and the camera, so that $g = 0$ and $i = e$. Any tabulated function of the single resulting angle, *i*, can be readily accomodated. This, combined with the fact that radar imagery is essentially "strip-camera" in nature, causes the surface brightness to have a dependence on the azimuthal component of slope that is zero to first order, as will be shown. Thus, the consequences of ignoring this slope component in a down-range integration of topography are less severe. At the same time, however, the relative height adjustment of two such profiles adjacent to one another is made more uncertain by what is essentially poor photometric leverage. An additional advantage of radar stems from the subjective impression that terrain homogeneous in radar albedo (normal back-scattering cross-section density) is more common than at visual wavelengths. For an objective evaluation of this, see Schaber *et al*. (1976) and Birrer *et al*. (1982). A powerful advantage of radar arises from a canonical (independent of terrain type) factor in the dependence of image pixel signal on terrain orientation, discussed in the next section. Hence, the adjective, "quasi-photoclinometric."

Radar imagery presents one distinct complication over traditional optical imagery with respect to the physical nature of image coordinates. Photoclinometry is usually given serious consideration only when photogrammetry is impossible. That is, if the ratio of surface relief to camera range is so small that the relative parallax of features in all possible stereo-pairs is exceeded by image resolution, then photoclinometry offers the only possible source of topographic information. In the ensuing photoclinometric processing for the extraction of topography, no consideration need be given to a correction for mapping of a feature from image coordinates to coordinates on the mean datum (mean-datum plane) because of the feature's height. Such mapping simplicity can never exist in radar imagery because one image coordinate is essentially range itself. Probably the greatest disadvantage of radar in comparison to visual imagery lies in the area of the quality and uniformity of photometric (radiometric) calibration and the signal-to-noise ratio of a given digital element of surface brightness. At the very outset, relative photometric accuracy is limited by the repeatability of total transmitted pulse energy, an aspect of radar imagery not comparable to ordinary photographic imagery. Image photometry (radiometry) has not been a traditional consideration in the design of radar imaging systems, nor in their corresponding signal-processing algorithms, especially when of an analog nature. In this connection SLAR possesses great advantage over SAR. On the one hand, the power allocation to a final pixel is more generous in SLAR. But most especially, because SAR must preserve the electromagnetic wave's phase, it suffers from "speckle," a direct enhancement of the random error of pixel photometry, which can only be alleviated, from an engineering point of view, through the use of an unacceptably large number of "looks." Low-pass filtering of the image is the only alternative for dealing with this problem.

Throughout the sections which follow, it will be assumed that terrain homogeneous in "radar albedo" is being processed.

THEORY

Radiative Transfer

Consider the following argument, which assumes perfect geometrical optics. Let a telescope form an image in the usual way of a terrain viewed obliquely. Let the telescope be sufficiently distant that the rays coming from the object are nearly an orthographic projection. Let the image plane in the telescope occur at the receiving surface of a uniform array of radiation detectors of equal sensitive area. Then the total radiative power received by each sensor is the integral of the focal-plane flux over the intercepting area of each sensor. Next, consider the mapping of all the rays from the boundary of an arbitrarily chosen sensor, through the telescope, to their intersection with the terrain. In the immediate vicinity of the ground, the locus of these rays is a mathematical closed cylinder; and its intersection with a plane perpendicular to the rays forms a closed curve bounding an area of integration for contribution to the total radiative power received by the sensor. The quantity to be integrated will be the specific intensity, or surface brightness (watts/metre^2/steradian), emerging from the terrain into the direction of the telescope, multiplied by the solid-angle subtended at a point in the area of integration by the area of the telescope aperture. Given the constancy of the aforementioned solid angle, energy conservation dictates that the image-plane flux is directly proportional to, and a mapping of, the specific intensity distribution over a plane perpendicular to the line-of-sight near the ground. Given also that the sensors have equal receiving areas and are dynamically identical and linear, the DN (data number) value in the resulting digital image will be directly proportional to the specific intensity in the proper direction at the corresponding point of the viewed surface.

If the foregoing picture is replaced by one in which a microwave CW (continuous wave) transmitter and receiver pair, of very narrow and symmetric main lobe in the antenna pattern, operates by serially positioning itself at points in a two-dimensional raster of equally spaced *angular* directions whose spacing is at least as large as the antenna pattern, then a digital image results about which the same conclusion can be made. Here it depends on the constancy of transmitter power, the ability of the receiver to measure power accurately, and, of course, the constancy of effective antenna area.

The situation is rather more complicated when one examines radar images. The "doubly brute-force" equivalent of any radar imaging system, neglecting speckle, will be considered to be one in which the azimuthal resolution is not only directly the width of the antenna pattern, but the range resolution is also directly the length of the transmitted radar pulse, and the matched dwell-time of the receiver detecting the returned waveform. In this view, the power, or more precisely the energy, that is allocated to the DN number (datum corresponding to signal strength) of a given image pixel all comes from a range interval, approximately also the range resolution, that is a constant of the image frame. The width in azimuth of this resolution-cell is also a constant of the frame. The difficulty in identifying the DN value of a pixel in a radar frame as proportional to the specific intensity of reflected radar radiation back-scattered from the corresponding point on the ground is thus seen to be a result of the fact that an image resolution cell does not map into a constant area in the plane near the ground that is perpendicular to the line-of-sight. Thus, the emergent specific intensity, though always multiplied by a constant solid angle, is multiplied by a variable area in forming the energy that is transduced into the DN value of a pixel. In fact, the geometry of the situation reveals that the boundaries of the integrating area for the specific intensity that correspond to the range-resolution-interval, Δr, are separated by the distance $\Delta r \cot i$, where i is the local incidence angle of the terrain. These boundary lines are perpendicular to the plane containing the local normal to the terrain and the line-of-sight from the radar to the terrain point. If the constant azimuthal interval, Δy, corresponding to azimuthal resolution, is bounded by two loci of constant azimuth, they will cut the two range-related lines so as to form a small parallelogram as the area for specific-intensity integration. The azimuth-related lines will have separation, Δy. But this makes the area of the parallelogram $\Delta r\ \Delta y \cot i \sec \Phi$, where Φ is the position angle of the local normal with respect to the local vertical as seen from the radar. While $\Delta r\ \Delta y$ is a constant, $\cot i \sec \Phi$ is not. Insofar as $\sec \Phi$ is never far from 1, it can be neglected. In that case, a backscatter-intensity versus incidence-angle curve, obtained from CW active microwave measurements as proposed to provide a photometric function for radar photoclinometry, need only be multiplied by $\cot i$ in order to provide an effective curve that may be applied *as though* the DN values in the radar image were proportional to "specific intensity." This approximation is adopted for the present, and noted that it allows the image to be converted directly to an array of values of the cosine of the incidence angle.

Differential Geometry

Photoclinometry is quite generally described by an inhomogeneous nonlinear first-order partial differential equation in the topography, considered in the form of height-above-mean-datum, z, as a function of Cartesian coordinates x and y on the mean datum. Consider any photometric function, $\Phi(g,i,e)$. Let the topography be given by $z = z(x,y)$. In three dimensions, the topography can be viewed in terms of an equipotential surface

$$V(x,y,z) = z - z(x,y) = 0, \quad (1)$$

where z appears in the dual context of independent variable and name of a function: The gradient of this potential function will point perpendicularly to the equipotential surface, i.e., the topography.

$$\nabla V = -\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}. \quad (2)$$

The unit normal vector that will always have a positive z component is then

$$\hat{\mathbf{n}} = \frac{-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \quad (3)$$

Let unit vectors $\hat{\mathbf{e}}$ and $\hat{\mathbf{s}}$ point toward the observer and the illuminator, respectively. They may or may not be nearly constant vectors in an image, but they are in any event known functions of no more than x and y. The same may be said of the phase angle, g. Now we will have

$$\cos e = \hat{\mathbf{e}} \cdot \hat{\mathbf{n}} \quad (4)$$
$$\cos i = \hat{\mathbf{s}} \cdot \hat{\mathbf{n}} \quad (5)$$

If we now let the given specific intensity of the image, transformed if necessary, be $b(x,y)$, then photoclinometry consists in solving the equation

$$\Phi(g,i,e) = C\, b(x,y) \quad (6)$$

for the function $z(x,y)$. Herein, C is a calibration constant dependent on the illuminating flux and the normal albedo. Because g, i, and e depend on x, y, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\left(\frac{\partial z}{\partial x}\right)^2$, and $\left(\frac{\partial z}{\partial y}\right)^2$ through Equations 3, 4, and 5, Equation 6 is a non-linear first-order partial differential equation with a driving function, $Cb(x,y)$. In the corresponding numerical analysis, an equation involving both first derivatives is a triangular three-point condition on a corresponding integration mesh of discrete values of x and y. If a single profile in z is given for all x at a constant value of y, profiles in x for the adjacent values of y may be generated, assuming an axial orientation does not correspond to a singularity. Thus, the fundamental indeterminacy of photoclinometry reduces to the requirement of a one-dimensional boundary condition.

Let us now direct ourselves specifically toward radarclinometry. Because $\Phi = \Phi(e)$ for a given C, the image may be readily conceived directly as cosine of incidence angle, rather than specific intensity. Let $\mu = \cos i$. Then equation (6) becomes

$$\hat{\mathbf{e}} \cdot \hat{\mathbf{n}} = \mu(x,y), \tag{7}$$

or

$$\frac{e_z - e_x \frac{\partial z}{\partial x} - e_y \frac{\partial z}{\partial y}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} = \mu\,(x,y),$$

which is more readily useable in the standard quadratic form

$$\begin{aligned}&\{[\mu(x,y)]^2 - e_y{}^2\} \left(\frac{\partial z}{\partial y}\right)^2 \\ &- 2\, e_y \left(e_x \frac{\partial z}{\partial x} - e_z\right) \frac{\partial z}{\partial y} \\ &+ \{[\mu(x,y)]^2\, [1 + \left(\frac{\partial z}{\partial x}\right)^2] \\ &- \left(e_x \frac{\partial z}{\partial x} - e_z\right)^2\} = 0.\end{aligned} \tag{8}$$

A further simplification is possible, with a proper choice of coordinates, due to the inherently "strip-camera" nature of radar imagery. Let us choose the x-axis positive directly down-ground-range from the sub-radar position in the zero-Doppler direction. Let the positive y-axis point co-parallel to the direction of motion of the radar platform. It will then always turn out for all x and y that $e_y = 0$.

Equation 8 quickly solves to yield

$$\frac{\partial z}{\partial y} = \pm \sqrt{\left(\frac{e_x \frac{\partial z}{\partial x} - e_z}{\mu}\right)^2 - [1 + \left(\frac{\partial z}{\partial x}\right)^2]} \tag{9}$$

Let σ and ν be integers identifying the numerical lattice of integration. If one is at point (x_σ, y_ν) on a boundary profile or the profile of an immediately preceding integration, $\frac{\partial z}{\partial x}$ is formed from $(z_{\sigma+1,\nu} - z_{\sigma,\nu})/(x_{\sigma+1} - x_\sigma)$, while μ, e_x, e_z are known. Formula 9 then yields $\frac{\partial z}{\partial y}$; and the value of z at the mesh point $(x_\sigma, y_{\nu+1})$ is then increased over the value at (x_σ, y_ν) by $\triangle y$ times this amount.

As mentioned earlier, x and y are the true coordinates of features on the mean datum. They are the ultimately preferred coordinates. In terms of image coordinates, x' and y', we must use the transformation

$$\begin{aligned} y' &= y \\ x' &= x - z(x,y) \tan \alpha \end{aligned} \tag{10}$$

where α is the depression angle of the radar, and $\tan \alpha = -e_z/e_x$. It is implicit in Equation 10 that the radar wavefront is approximately planar over the relief range expected. The elements of the Jacobeans for the direct and inverse transformations are

$$\begin{aligned}
\frac{\partial x'}{\partial x} &= 1 + e_z/e_x \frac{\partial z}{\partial x} \\
\frac{\partial x'}{\partial y} &= e_z/e_x \frac{\partial z}{\partial y} \\
\frac{\partial y'}{\partial x} &= 0 \\
\frac{\partial y'}{\partial y} &= 1 \\
\frac{\partial x}{\partial x'} &= \frac{1}{1 + e_z/e_x \frac{\partial z}{\partial x}} \\
\frac{\partial x}{\partial y'} &= -\frac{e_z/e_x \frac{\partial z}{\partial y}}{1 + e_z/e_x \frac{\partial z}{\partial x}} \\
\frac{\partial y}{\partial x'} &= 0 \\
\frac{\partial y}{\partial y'} &= 1
\end{aligned} \tag{11}$$

One integrates in the domain of the image, accumulating the necessary transformation to ensure that each new value of z is put in proper correspondence with x and y. The sign ambiquity in Equation 9 can be resolved by making the rather reasonable assumption that the local isophote describes a surface line-element along which the normal vector is locally unchanging. If the isophotic direction is compared with the sign of $\frac{\partial z}{\partial x}$ as found from the current profile, a reasonable sign for $\frac{\partial z}{\partial y}$ can be found. For example, if the isophote trends from $(-,-)$ to $(+,+)$ in the x', y' plane, and $\frac{\partial z}{\partial x}$ is locally positive, then $\frac{\partial z}{\partial y}$ should be negative.

The foregoing analysis represents an operationally complete scheme, in principle, and emphasizes the requirement for one-dimensional ground truth. It seems reasonable to suppose that, if one can supply such a boundary profile, one has sufficient technology at one's disposal as to make reliance on photoclinometry unnecessary in any event. We therefore need to consider auxiliary mathematical assumptions that will enable the generation of such a boundary profile from initial point conditions photoclinometrically. Assuming that all slopes along a line of constant y are either directly toward or away from the radar seems unacceptable. One does not obtain the correct down-radar slope to the exclusion of the cross-radar slope when one does this. $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are not mathematically separable in the equations of photoclinometry. One simply obtains the wrong value of $\frac{\partial z}{\partial x}$. Mental reflection regarding radarclinometry performed on a hemispherical convexity, employing such a simplistic assumption, reveals that the profile which bisects the structure will be correct. For all parallel profiles, the slopes down-radar will be underestimated (effectively rotating $\hat{\mathbf{n}}$ about $\hat{\mathbf{e}}$ into the vertical plane) so that the derived structure will have a ridgeline oriented down range with depressed sides. A bilaterally symmetric ridge, or convex mathematical cylinder, trending obliquely, would be scaled down non-linearly in the vertical dimension, and the flat terrain on the opposite sides would separate in elevation.

The search for a reasonable auxiliary assumption begins by examining the significance of information expansion in the form of not only the value of μ at each x' and y' in the image but of the two-dimensional gradient of μ as well. The components expressed in terms of the coordinates on the mean-datum will be needed. By the chain-rule,

$$\frac{\partial \mu}{\partial x} = \frac{\partial \mu}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \mu}{\partial y'}\frac{\partial y'}{\partial x}$$
$$\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial \mu}{\partial y'}\frac{\partial y'}{\partial y} \qquad (12)$$

In Equations 12, the derivatives with respect to x' and y' are directly the pixel differences in the image, and other derivatives are from Equations 11. Equations 12 represent the transformed measurements which apply. On the other hand, the theoretical equivalents of the left-hand-sides of Equations 12 are obtained by differentiating Equation 7 with e_y set to zero: i.e.,

$$\frac{\partial \mu}{\partial x} = [1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2]^{-3/2}$$
$$\left\{\left(e_x \frac{\partial z}{\partial x} - e_z\right)\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y}\right.$$
$$\left. - e_x [\left(\frac{\partial z}{\partial y}\right)^2 + 1]\frac{\partial^2 z}{\partial x^2} - e_z \frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x^2}\right\}$$

$$\frac{\partial \mu}{\partial y} = [1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2]^{-3/2}$$
$$\left\{\left(e_x \frac{\partial z}{\partial x} - e_z\right)\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial y^2}\right.$$
$$\left. - e_x [\left(\frac{\partial z}{\partial y}\right)^2 + 1]\frac{\partial^2 z}{\partial x \partial y} - e_z \frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial y}\right\} \qquad (13)$$

We do not expect that merely looking at the gradient introduces determinacy, and that is surely the case, for by adding the *two* gradient equations we have introduced *three* unknown second derivatives not present in our original Equation 7. Thus, instead of promoting determinacy, we have now, on the contrary, a theory with two remaining degrees of freedom instead of one. The *"hypothesis-of-local-cylindricity"* will now be invoked. The second derivatives now involved in the argument describe fully the local curvature properties of the topography. We will assume the curvature is *locally* cylindrical in nature, of arbitrary orientation. That is, there exists a direction in space in which the curvature is maximum and another direction perpendicular to the first in which it is zero. This defines a local tangent plane to the topography. The local curvature in all other directions in the plane is a projection of the maximum. Thus is defined a local tangent cylinder of equivalent local curvature as well. I emphasize that this is a *local* and not a *global* assumption about the topography. To be reasonable, it is only necessary that local curvature possess a fairly dominant direction. Aside from such features as granite domes and the summits of mountain peaks, local cylindricity (LC) seems to me to be fairly consistent with topography on the Earth, probably due to the dominance of down-slope movement in the erosion-transport process. If the hypothesis tends to fail the more as curvature of any kind becomes slight, there is compensation in the fact that it means local slope is changing negligibly from an already established value.

What are the consequences of LC for the equations at hand? Consider an alternative set of locally definable x and y axes, say x'' and y'', that are rotated about the z axis with respect to x and y. Assuming that LC prevails, let the axis of the local tangent-and-equivalently-curved cylinder be parallel to the $y''z$ plane. It may have an arbitrary axial elevation angle. The following facts then follow from LC:

$$\left.\begin{array}{l} \frac{\partial z}{\partial y''} \neq 0 \\ \frac{\partial z}{\partial x''} \neq 0 \\ \frac{\partial^2 z}{\partial x''^2} \neq 0 \end{array}\right\} \text{ but } \left\{\begin{array}{l} \frac{\partial^2 z}{\partial y''^2} = 0 \\ \frac{\partial^2 z}{\partial x'' \partial y''} = 0 \end{array}\right. \qquad (14)$$

Let the angle of rotation into the (x,y,z) system from the x'', y'', z) system be θ. The transformation matrix for two-dimensional vectors defined in the (x,y) plane, is, thus,

$$\gamma_{\sigma\nu} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{15}$$

Now the topography certainly has a physical meaning independent of the coordinate system used to represent it. Its functional representation is a true scalar in the group-theoretic sense. Its second derivatives, therefore, have a second-rank tensor character, and the appropriate transformation laws may be used to re-represent the set of values shown in Equations 14 in terms of x and y: i.e.,

$$\frac{\partial^2 z}{\partial x_\xi \partial x_\eta} = \sum_{\substack{\sigma=1\\ \nu=1}}^{2,2} \gamma_{\xi\sigma}\,\gamma_{\eta\nu}\,\frac{\partial^2 z}{\partial x_\sigma \partial x_\nu} \tag{16}$$

If we let σ and ν each refer to the (x'', y'') system and ξ and η each refer to the (x,y) system, applying Equation 16 using the values shown in Equations 15 and 14, we have

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \cos^2\theta\,\frac{\partial^2 z}{\partial x''^2} \\ \frac{\partial^2 z}{\partial x \partial y} &= -\sin\theta\cos\theta\,\frac{\partial^2 z}{\partial x''^2} \\ \frac{\partial^2 z}{\partial y^2} &= \sin^2\theta\,\frac{\partial^2 z}{\partial x''^2} \end{aligned} \tag{17}$$

the essential importance of which is that it leads to the conclusion that there exists a parameter λ, which we may identify as tan θ, such that

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= -\lambda\,\frac{\partial^2 z}{\partial x^2} \\ \text{and} \quad \frac{\partial^2 z}{\partial y^2} &= \lambda^2\,\frac{\partial^2 z}{\partial x^2} \end{aligned} \tag{18}$$

In Equations 18 one quickly sees that the three unknown second derivatives have been traded for two unknown values of λ and $\frac{\partial^2 z}{\partial x^2}$. Equations 18 can be used to substitute into Equations 13. Thus, we have reduced two remaining degrees of freedom back to one.

The parameter λ deserves deeper inquiry. It is the tangent of the angle, as we have seen, between the y axis and the projection of the local cylindrical axis onto the mean datum. The line in the surface that is in the cylindrical-axial direction must be a locus of constant $\hat{\mathbf{n}}$. As such, its image must be an isophote. In terms of the image mapping in x and y, we therefore conclude

$$\lambda = -\left(\frac{\partial \mu}{\partial y}\right)\Big/\left(\frac{\partial \mu}{\partial x}\right) \tag{19}$$

This follows from the requirement that $d\mu = 0$ along the line-element for which $\frac{dy}{dx} = \frac{1}{\lambda}$, the isophote direction. In terms of the directly measured gradient in the image, using the chain rule and Equations 11,

$$\lambda = -\left[\left(\frac{\partial \mu}{\partial y'}\right)\Big/\left(\frac{\partial \mu}{\partial x'}\right) + e_z/e_x\,\frac{\partial z}{\partial y}\right]\Big/\left(1 + e_z/e_x\,\frac{\partial z}{\partial x}\right) \tag{20}$$

With the appearance of Equation 20, the illusion is created that we have finally arrived at complete determinacy of all first and second derivatives. One extra equation has been found to account for the remaining degree of freedom. Such is not the case. If we combine Equations 11, 12, and 13, and substitute for the mixed derivative and the second derivative with respect to y from Equations 18, thereafter eliminating the parameter λ thus introduced by substitution from Equation 20, and finally eliminating the radical by substitution for it from Equation 7, we extract the following two equations for the two components of the image gradient:

$$\begin{aligned} &\mu^{-3}\left(\frac{\partial \mu}{\partial x'}\right)^2\left(e_z - e_x\frac{\partial z}{\partial x}\right)^3\left(1 + e_z e_x^{-1}\frac{\partial z}{\partial x}\right)^2 \\ &+ \frac{\partial \mu}{\partial y'}\left(e_z - e_x\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x^2} + e_x^{-1}\frac{\partial \mu}{\partial x'}\left[e_z^2\left(\frac{\partial z}{\partial x}\right)^2 \right. \\ &\left. + \left(\frac{\partial z}{\partial y}\right)^2\right]\frac{\partial^2 z}{\partial x^2} + \frac{\partial \mu}{\partial x'}\left(e_x + 2\,e_z\frac{\partial z}{\partial x}\right)\frac{\partial^2 z}{\partial x^2} = 0 \end{aligned} \tag{21}$$

and a second equation which turns out to be none other than Equation 21 multiplied on the left-hand-side by the factor:

$$\frac{\partial \mu}{\partial y'} + e_z e_x^{-1}\,\frac{\partial \mu}{\partial x'}\,\frac{\partial z}{\partial y}.$$

Hence, Equation 21 does not have a companion that is linearly independent in the three second derivatives. Thus, the LC hypothesis does not result in a set of equations which are algebraically determinant. It *does*, however, result in a set of equations which are *one-dimensionally* integrable starting from a *point* boundary condition. They therefore complement the earlier set of non-LC equations which are two-dimensionally integrable starting from a line boundary condition.

In principle, then, the integration of an entire topography can now proceed as follows. We may define the basic quantity which is stepped in the integration producing the boundary profile, aside from z, as the ratio of the slope components

$$\eta = \left(\frac{\partial z}{\partial y}\right) / \left(\frac{\partial z}{\partial x}\right),$$

which must be specified initially by eyeball estimate. Equation 7 may be rewritten in terms of $\frac{\partial z}{\partial x}$ and η as

$$[\mu^2 (1 + \eta^2) - e_x^2] \left(\frac{\partial z}{\partial x}\right)^2 + 2\, e_z e_x \frac{\partial z}{\partial x} + (\mu^2 - e_z^2) = 0 \tag{22}$$

For a given value of η that is either the starting value or the current value from the last integration step, one solves Equation 22, using the immediate image datum, μ, for $\frac{\partial z}{\partial x}$. Discrimination between the two roots of the quadratic is on the basis that no slope steeper than the incident wave plane of the radar has been admitted into the processing (terrain inobedient to this condition must be avoided in both radarclinometry and radargrammetry). With η known, $\frac{\partial z}{\partial y}$ immediately follows. These two slope components together with the immediate pixel differences, $\frac{\partial \mu}{\partial x'}$ and $\frac{\partial \mu}{\partial y'}$, can be inserted into Equation 21 which is then easily solved for $\frac{\partial^2 z}{\partial x^2}$. One then uses Equation 20 to find λ, and one of Equations 18 to find $\frac{\partial^2 z}{\partial x \partial y}$. We now step to the next integration point as follows:

$$\frac{\partial z}{\partial x} \rightarrow \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} \triangle x$$

$$\frac{\partial z}{\partial y} \rightarrow \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x \partial y} \triangle x \tag{23}$$

$$\eta \rightarrow \frac{\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x \partial y} \triangle x}{\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} \triangle x}$$

$$z \rightarrow z + \frac{\partial z}{\partial x} \triangle x + \frac{1}{2} \frac{\partial^2 z}{\partial x^2} (\triangle x)^2$$

The process can now be repeated. An important acknowledgment is necessary in order that a local strike-line may evolve successfully through the plane containing the local vertical and the radar line-of-sight. It will be noted that η is not updated through a differential formula, but from new values of the slope components that are thus linearized individually and updated. It would not otherwise be possible for η to pass through the perfectly meaningful discontinuity ($+ \infty \leftarrow\rightarrow - \infty$). While $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous variables, η is simply a discrete set.

When the boundary profile is complete, one proceeds according to the non-LC approach described in the first part of this section. Presumably the integration mesh is constant in x' rather than x, so that one operates pixel-to-pixel in the image. The value of $\triangle x$ in the foregoing is therefore given by the Taylor expansion,

$$\triangle x = \frac{\partial x}{\partial x'} \triangle x' + \frac{1}{2} \frac{\partial^2 x}{\partial x'^2} (\triangle x')^2 \tag{24}$$

where the second order term is available because topographic second derivatives have been determined. Noting that $\frac{\partial x}{\partial x'}$ is a function of $\frac{\partial z}{\partial x}$ in Equations 11, one writes by way of the chain-rule

$$\frac{\partial^2 x}{\partial x'^2} = - \frac{1}{\left(1 + e_z/e_x \frac{\partial z}{\partial x}\right)^2} e_z/e_x \left(\frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial x'} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial x'}\right) \tag{25}$$

whose parameters are known every step of the way.

CALIBRATION

The combined radar transfer characteristic and applicable reflectance function provides all the information corresponding to Equation 6 except C. We first evaluate C under the assumption that the average value of μ equals e_z, according to the convergent iterative equation for corrections to C, $\triangle C$, based on Newton's method: i.e.,

$$\triangle C = [e_z - \langle U(Cb) \rangle] / \langle \frac{dU}{dC} \rangle.$$

The overall calibration may be read for signal, $Cb = \rho(\mu) \equiv \varrho\,(i) \equiv \cot i\ \Phi(o,i,i)$, or for cosine of incidence angle, $\mu \equiv U(Cb)$, which defines the function, U. The starting value of C is $\rho(e_z)/\langle b \rangle$. The averages are frame averages.

A small correction to the value of C determined above is now made, based on the departure of $\langle \mu \rangle$ from e_z for a rough surface. Let A be any locally defineable property of the terrain. Thus, A may be considered a function of x and y (or x' and y' for that matter). We shall subsequently want A to represent μ and μ^2. Let A also be functionally determined by the local components of slope, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, which we shall temporarily refer to as z_x and z_y, in order to avoid some unweildy expressions. We will assume that the probability distribution for slopes is isotropic and gaussian as related to sampling that is uniform on the mean-datum.

$$P(z_x,z_y) = \frac{1}{2\pi\sigma} e^{-(z_x^2 + z_y^2)/2\sigma} \quad (26)$$

Accordingly, we will have alternative prescriptions for the average value of A.

$$\langle A\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(z_x, z_y)\, A(z_x, z_y) dz_x\, dz_y \quad (27)$$

$$\langle A\rangle = \int_{y_1}^{y_2}\int_{x_1}^{x_2} A(x,y)dxdy/[(y_2-y_1)(x_2-x_1)] \quad (28)$$

If we take an average over a radar picture-frame we do not have the kind of average indicated in Equations 27 and 28, because we are not sampling uniformly on the mean-datum. We know that slopes toward the radar are rendered in diminished duration compared to slopes away from the radar. If we let $P'(z_x, z_y)$ be a corresponding probability distribution for uniform sampling over the radar image-frame, then the average value we will measure, for example, of μ, by converting brightnesses, pixel-by-pixel, to μ and summing line-by-line over the frame, would be represented by substituting P' for P in Equation 27 and x' for x and y' for y in Equation 28. Our immediate problem is to find P'. In order to do this, we must rethink Equation 27 as a condensed integral over the actual slopes of the surface, with P representing the normalized multiplicity of distinct pairs of z_x and z_y. If Equation 28 is mapped from the space of x and y to the space of z_x and z_y, the result can be called the equivalent of Equation 27 and interidentification of terms can be achieved. In order to do this properly, the integral of Equation 28 should be divided up into however may integrals over separate domains are required in order that a one-to-one mapping between each individual domain in x and y and the single domain in z_x and z_y is achieved. Let n label such a domain and let there be a total of N of them. Inasmuch as the domains are contiguous, then, the leading reciprocal of integration range in Equation 28 is unaffected and the equation becomes

$$\langle A\rangle = (y_2-y_1)^{-1}(x_2-x_1)^{-1} \sum_{n=1}^{N}\int_{y_{1n}}^{y_{2n}}\int_{x_{1n}}^{x_{2n}} A(x,y)dxdy, \quad (29)$$

the mapped form of which is

$$\langle A\rangle = (y_2-y_1)^{-1}(x_2-x_1)^{-1} \sum_{n=1}^{N}\int_{z_{y1n}}^{z_{y2n}}\int_{z_{x1n}}^{z_{x2n}} A(z_x, z_y)\left[\frac{\partial(x,y)}{\partial(z_x,z_y)}\right]_n dz_x dz_y$$

The Jacobean of the foregoing equation is the reciprocal of the determinant whose elements are the second derivatives of the surface function, z; and its behavior in any one domain is independent of its behavior in all the others, so that it can properly be given a subscript, n. Equation 29 interidentifies with Equation 27 and would constitute the empirical basis for the probability distribution, P, whose form we have assumed. The original integration relating to P is defined on the mean datum. We may find P' by combining this result with the result of the same procedure applied to mapping the integration over the radar image frame into the space of z_x and z_y. Remember that z_x and z_y still represent $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$; not $\frac{\partial z}{\partial x'}$ nor $\frac{\partial z}{\partial y'}$. The repeat performance yields

$$\langle A\rangle' = (y_2' - y_1')^{-1}(x_2' - x_1')^{-1}\sum_{n=1}^{N}\int_{z_{y'1n}}^{z_{y'2n}}\int_{z_{x'1n}}^{z_{x'2n}} A(z_x, z_y)\left[\frac{\partial(y',y')}{\partial(z_x,z_y)}\right]_n dx_x dz_y, \quad (30)$$

which interidentifies with Equation 27 written with P' substituted for P. If the Jacobeans for the transformations between (x,y) and $(x'y')$, both direct and inverse, vanish nowehere in the regions of interest (there are no terrain slopes steeper than the radar wavefront), then the topological relations between the slope domain and the mean datum are the same as between the slope domain and the image frame. Therefore $N' = N$, and the individual domains in Equation 29 can be interidentified with those of Equation 30 on a one-to-one basis.

Now the probability, P, under discussion, can be interpreted as a sum of individual probabilities, P_n, each one of which is a joint probability of (1) being in the domain n, and (2) the slope probability distribution applicable to the particular domain. This enables the detailed interidentification between Equations 27 and 29: i.e.,

$$P_n = (y_2-y_1)^{-1}(x_2-x_1)^{-1}\left[\frac{\partial(x,y)}{\partial(z_x,z_y)}\right]_n$$

and, similarly,

$$P_n' = (y_2'-y_1')^{-1}(x_2'-x_1')^{-1}\left[\frac{\partial(x',y')}{\partial(z_x,z_y)}\right]_n.$$

We assume that the frame is sufficiently large that there is negligible difference in overall integration ranges: i.e., $y_2'-y_1' = y_2-y_1$ and $x_2'-x_1' = x_2-x_1$. And, therefore,

$$P_n' = P_n\left[\frac{\partial(x',y')}{\partial(z_x,z_y)}\right]_n \Big/ \left[\frac{\partial(x,y)}{\partial(z_x,z_y)}\right]_n$$

But by Jacobi's theorem, the ratio of the two Jacobeans is

$$[\frac{\partial(x',y')}{\partial(x,y)}]_n.$$

A quick glance at Equation 11, which correspond to this Jacobean, reveals that it is uniquely defined by the values of z_x and z_y at the point of the contribution to the integral over z_x and z_y, so that the subscript n is no longer necessary or significant. Factorization is therefore possible in the following result:

$$P' = \sum_{n=1}^{N} P_n' = \frac{\partial(x',y')}{\partial(x,y)} \sum_{n=1}^{N} P_n,$$

or

$$P'\ (z_x,z_y) = P\ (z_x,z_y)\,\frac{\partial(x',y')}{\partial(x,y)}; \qquad (33)$$

$$\text{i.e., } P'(z_x,z_y) = \left(\frac{1 + e_z z_x/e_x}{2\,\pi\,\sigma}\right) e^{-(z_x{}^2 + z_y{}^2)/2\sigma} \qquad (34)$$

With this expression for P' substituted for P in Equation 27, we may compute $\langle\mu\rangle$ and $\langle\mu^2\rangle$ in terms of σ, using Equation 7. The results are

$$\langle\mu^2\rangle = e_z^2 - (4e_z^2 - e_x^2)\sigma$$

and $\langle\mu\rangle = e_z(1-2\sigma)$, from which, (35)

to 2nd order, $\sigma = \langle(\triangle\mu)^2\rangle/e_x{}^2$. (36)

We compute $\langle\mu^2\rangle$ and $\langle\mu\rangle = e_z$ from the calibration as it exists up to this point, averaging over the frame. We therefore calculate σ in Equation 34 from these frame averages, $\langle(\triangle\mu)^2\rangle = \langle\mu^2\rangle - \langle\mu\rangle^2$, using Equation 36, and then use it to revise $\langle\mu\rangle$ through Equation 35. We finally revise our calibration constant by multiplying it by the ratio $\rho(e_z)/\rho(\langle\mu\rangle)$.

DISCUSSION

As indicated in the experimental counterpart (Wildey, 1984) of this theoretical paper, a test of metric precision will not be possible until radiometrically calibrated radar systems are developed. The Venus Radar Mapper is projected to fly such a system and the present theory, to whatever precision it has been extended at that time, will be used to extract topographic information during the nominal mission. This forecast is possible because the experimental paper has already demonstrated a facility for extracting enough topographic information from uncalibrated images to enhance their geologic interpretation (Soderblom, 1983). The greatest limitation of the present theory lies in the fact that azimuthal integration is unsatisfactory, so that topographic maps are obtained by producing a set of independent line integrations in ground-range, involving LC; which set is then adjusted in relative zero-point of height and mean slope so as to promote an isotropic autocorrelation of the topography. Nevertheless, the theory is intended to possess no *fundamental* limitations that prevent it from approaching the metric integrity of radargrammetry to the degree permitted by *practical* limitations. Efforts in this direction will continue.

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