Three-Dimensional Reconstruction from Three-Point Perspective Imagery

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ABSTRACT: In architectural drawing, there are graphical methods for constructing a perspective image of an object from orthographic views. Applied to the photogrammetric problem of reconstructing orthographic views from a perspective image, these techniques provide accurate estimates of object-space dimensions that are not easily found any other way. The methods use *a priori* information about geometry and scale. For manmade objects, such information includes known angles, parallel lines, circles, and planes. Whereas we previously described reconstruction from two-point perspective images, we now extend the discussion to three-point perspective. As in the previous paper, the imagery need not be formatted, and the camera need not be calibrated. The interior and exterior elements of the camera emerge from these methods, and the scaled camera coordinates may also be determined. The purpose of this paper is to give a fresh view of these rather classical techniques, whose photogrammetric applications are not widely described.

INTRODUCTION

THE STUDY of a technology can often benefit from a fresh look at classical procedures. In close-range photogrammetry, such a classical procedure is the graphical analysis of a single image using perspective. For example, in architectural drawing, there are graphical methods for constructing a perspective image of an object from orthographic views. Using these same techniques and some *a priori* information, it is often feasible to obtain three-dimensional measurements from a single photograph acquired by a non-metric camera (a camera that has not been calibrated). Such measurements are important in close-range forensic photogrammetry, for which camera orientation information and stereo views are not usually available. For manmade objects, the *a priori* information includes known angles, parallel lines, circles, and planes.

In a previous article (Williamson and Brill, 1987), graphical techniques were discussed for mensuration of two-point perspective images. Whereas previous treatments (e.g., Gracie *et al.*, 1967; Busby, 1981; Novak, 1986) required prior knowledge of the principal point of the image, our approach (similar to that of Kelley(1976–1983)) required no such knowledge. However, it was necessary to know at least one diagonal angle of a rectangle to reconstruct the principal point and camera station of the perspective image. These, in turn, were used to reconstruct orthographic views of an imaged object.

The present article extends these methods to the problems of three-point perspective geometry (in which none of the edges of the imaged solid are parallel to the image plane). In particular, we consider a rectangular box, and also solids with rectangular walls and two nonrectangular horizontal surfaces. The methods prove to be somewhat simpler than in two-point perspective, and knowledge of a diagonal angle is no longer required. As in the previous paper, the scaled camera position and attitude are determined together with the orthographic views of the solids.

Our methods are adapted from classical procedures in architectural drawing for constructing a perspective image from orthographic views (McCartney, 1963; Walters and Bromham, 1970). In the photogrammetric application, the methods are used to determine the orthographic views from the perspective image, rather than the reverse as in architectural drawing (Fry, 1969). These graphical techniques are a useful adjunct to the analytical techniques recently proposed to accomplish the same objective (Ethrog, 1984).

RECONSTRUCTING A RECTANGULAR BOX

Reconstruction of orthographic views of a rectangular box involves first performing a graphical resection that locates the attitude and camera-station coordinates (scaled) with respect to an arbitrary object-space coordinate system defined by the forward-most corner of the imaged box. Consider the three-point perspective image in Figure 1. The top of the box (**ABCD**) can be reconstructed in six steps.

Step 1. Construct the vanishing points by extending the three sets of parallel edges of the box so that they intersect. These constructed intersection points are called vanishing points, and are labeled **VPX, VPY,** and **VPZ,** according to the respective axes producing them.

Step 2. Locate the perspective principal point (**PP**) on the im-

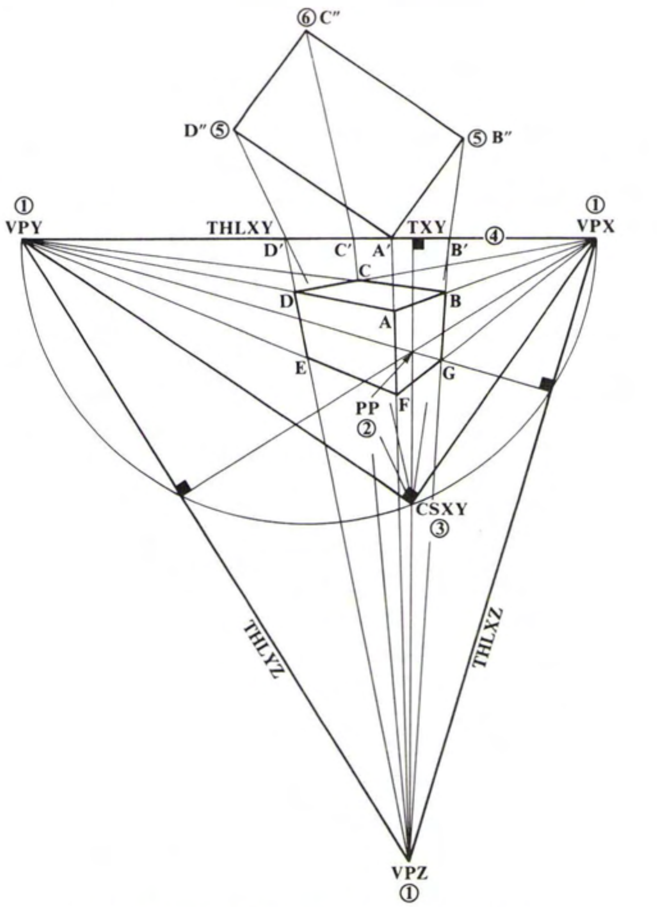

FIG. 1. Six Steps to **XY** plan view of rectangular box.

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age as the intersection of the three altitudes of the triangle formed by **VPX, VPY,** and **VPZ**.

Step 3. Construct a circle whose diameter is the segment **VPX-VPY**; label as the camera-station position **CSXY** the intersection of this circle with the altitude drawn through **VPZ**.

Step 4. Construct lines from **VPZ** through **A, B, C,** and **D** that intersect the line **VPX-VPY** (the true **XY** horizon, or **THLXY**). The intersection points are labeled **A′**, B′, **C′**, and **D′**.

Step 5. Construct a line through **A′** parallel to **CSXY-VPY**, and extend **CSXY-D′** to intersect this line at a point (**D″**). Similarly, construct a line through **A′** parallel to **CSXY-VPX**, and extend **CSXY-B′** to intersect this line at a point (**B″**).

Step 6. Construct **C″** by completing the rectangle **A′B″C″D″**. (**A** test of the drawing accuracy is the proximity of **C″** to the extended line **CSXY-C′**.) This completes the six-step construction.

The other two orthographic views of the solid can be obtained analogously. The orthographic view of face **ABGF** is constructed by projecting from **VPY** to **THLXZ** instead of from **VPZ** to **THLXY** (see Figure 1): face **AFED** is reconstructed by projecting from **VPX** to **THLYZ**. Of course, the scales of the three orthographic views obtained in this way will probably be different. To refer the views to a common scale, identify the same segment (e.g., representing edge **AB**) in two views, and then change the scale of one view so the two representations of **AB** have the same length. This can be done graphically by parallel displacement of the horizon line **THLYZ** or **THLXZ** prior to executing steps analogous to 5 and 6 above.

A GEOMETRIC JUSTIFICATION OF THE METHOD

The construction shown in Figure 1 can be understood geometrically in much the same way as the construction in the two-point perspective paper (Williamson and Brill, 1987).

First, we show the validity of the construction of **CSXY** and **PP** (the first three steps of the above method). The acquisition geometry of the image in three dimensions places the camera station (**CS**) one effective focal length away from the positive image plane (and object space) along the principal ray. Figure 2 illustrates this geometry, and the vanishing points of the perspective image. Revolving **CS** about the true horizon line **THLXY** (to the image plane) generates the point **CSXY**. The standard perspective projection model implies that a ray from the three-dimensional **CS** through any image point (of a positive image) will intercept the corresponding object point. These rays may be considered visual rays (see Figure 3). In this model, the constructed lines from the **CS** through the image vanishing points (**VPX, VPY, VPZ**) are parallel to edges of the object-space box. There is a tetrahedron formed by the image-space points **VPX, VPY, VPZ,** and **CS**, which may be thought of as a section of a corner of the box. In the image, this corner is viewed along the direction of tilt (see Figure 3). This tetrahedron has three right angles at its **CS** vertex. Because angle **VPY-CS-VPX** is a right angle (in the horizontal reference plane), then **CS** is on a circle (in the same reference plane) with **VPX-VPY** as a diameter. Similarly, **CSXY** is also on a circle with **VPX-VPY** as a diameter, and this circle lies in the image plane. The relation of **CS** to **CSXY** (by means of a revolution about **THLXY**) places **PP** on the line that is perpendicular to **THLXY** and contains the point **CSXY**.

Next, we show that **PP** is the intersection of the altitudes of triangle **VPX-VPY-VPZ**. By symmetry, if **PP** is on one of the altitudes, it is on all of them, so it will suffice to show that **PP** is on the altitude from **VPZ**. This property becomes readily evident by visualizing the situation in three dimensions, and representing as vectors the differences between the points. The vector **CS-VPZ** is perpendicular to the **XY** plane generated by the vectors **CS-VPX** and **CS-VPY**. Thus, vector **CS-VPZ** is perpendicular to the line **THLXY**. Also, the vector **CS-PP** is perpendicular to the image plane, and hence is perpendicular to **THLXY**. Because vectors **CS-VPZ** and **CS-PP** are both perpendicular to **THLXY**, their difference, **VPZ-PP**, is also perpendicular to **THLXY**. Hence, **PP** is on the altitude of triangle **VPX-VPY-VPZ** from **VPZ**.

Note that the preceding paragraph is an alternative proof to the plane-geometry theorem (Perfect, 1986) that the altitudes of a triangle are concurrent. The proof achieves its simplicity by making the triangle (represented above as **VPX-VPY-VPZ**) the base of a tetrahedron whose apex forms a rectangular corner (represented above as **CS**).

Having justified geometrically the first three steps of our reconstruction—involving the placement of the vanishing points, **CSXY**, and **PP**—we now justify the steps that actually produce the orthographic views. Imagine that the box is moved in the +**Z** direction (in object space) until the top of the box lies in the horizontal reference plane (**XY** horizon plane) containing **CS**. (Here, the positive **Z** direction is represented by **FA, ED,** and **GB** in the image, and raising the box is a construction

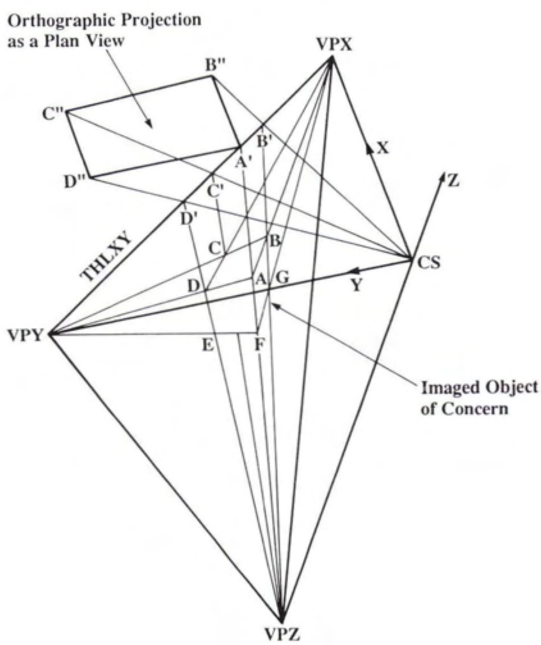

FIG. 2. Illustration of plan-view construction, showing **VPX-VPY-VPZ-CS** tetrahedron.

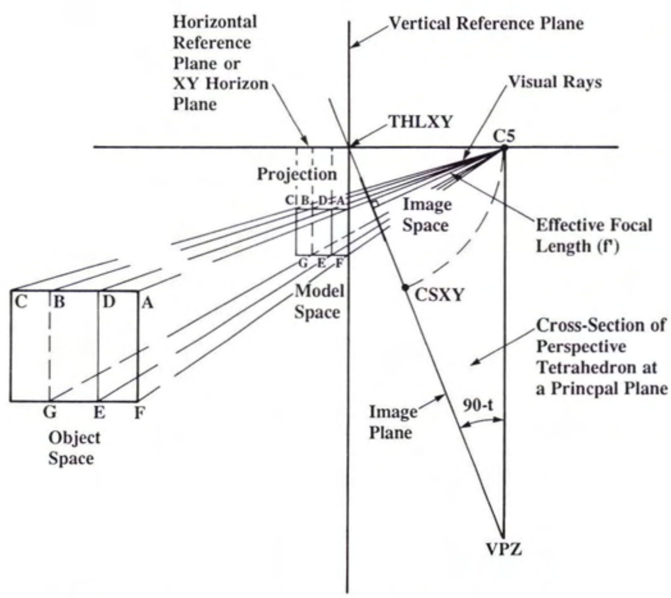

FIG. 3. Object space, model space, and image space.

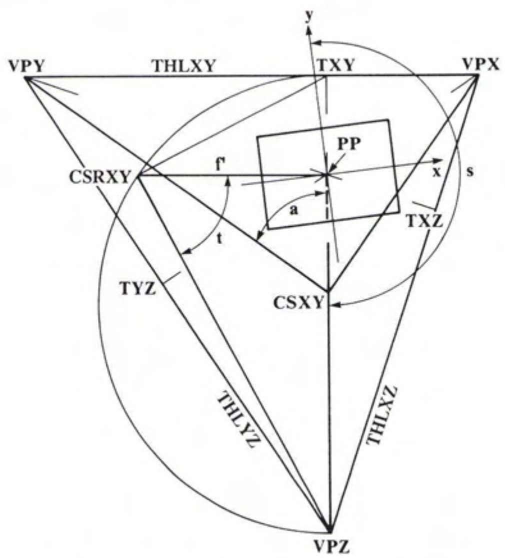

FIG. 4. Locating the camera attitude angles.

equivalent to projecting the box orthographically onto the horizontal reference plane.) Given this vertical translation of the box in object space, imagine the accompanying transformation of the visual rays from the **CS**. When the top becomes coincident with the horizon plane, the visual rays of the corners pass through the **THLXY**, and the points **A′**, **B′**, **C′**, **D′** (required for the orthographic construction) are established on **THLXY**. Likewise, to project the image vertical lines **FA**, **ED**, and **GB** from **VPZ** to the **THLXY** will produce the same **A′**, **B′**, **C′**, **D′** points. Then, revolving the image plane about **THLXY** into the **XY** horizon plane (with **CSXY** revolved back to **CS**) reveals that the visual rays from **CSXY** through **A′**, **B′**, **C′**, and **D′** intersect with the corresponding object-space corners of the raised box. To follow these steps, using object space, would produce a 1:1 orthographic projection. Because this is not practical, the concept of a model space is used.

A model space contains a scale model of the object space of concern, obtained by projecting object space points along the visual rays, while maintaining their relative orientation (see Figure 3). Model space is on the opposite side of the positive image plane from the **CS**. Model space preserves the angular dimensions of the visual rays from the **CS**, allowing the construction of practical orthographic projections. In Figure 2, the model space for the box is selected so the corner **A** of the box projects orthographically to the line **THLXY**. The other points of the model project orthographically to the **XY** horizon plane to produce the plan view **A′B″C″D″** as illustrated in Figure 2. Revolving the **XY** horizon plane about **THLXY** until it coincides with the positive image plane produces the construction in Figure 1. Each edge of the rectangle **ABCD** is parallel to either **CS-VPX** or **CS-VPY**: hence, the orthographic plan view of the rectangle is made by locating the box model-space corner **A″** at **A′** (on **THLXY**) and constructing the parallel lines through point **A′**. The orthographic corners **B″**, **C″**, and **D″** are constructed by the projecting lines from **CSXY** through the points on **THLXY**, as described in Step 5 and Step 6 above.

RETRIEVING OTHER PARAMETERS OF RESECTION

The procedures in Step 1 through Step 3 of the second section reconstruct the principal point of the image and the revolved camera station, **CSXY**. The effective focal length, **f′**, is readily determined from this construction. Also retrievable are the scaled camera-station coordinates in the object-space coordinate system generated at one corner of the solid (e.g., the corner at **A**). Finally, the attitude angles of the camera (azimuth, **a**, tilt, **t**, and swing, **s**) are retrievable relative to this coordinate system.

The scaled object-space **X** and **Y** coordinates of the camera station can be measured as in our previous paper (Williams and Brill, 1987) using the plan view points **A′**, **B″**, **C″**, **D″**, **and CSXY**. Similarly, once the plan views of the box are brought to a common scale, the **Z** coordinate of the camera station can be measured using the plan views of other faces of the box, which are constructed by the same projection procedure with **THLYZ** and **THLXZ** instead of **THLXY**.

To retrieve the relative focal length, **f′** (with units of the image coordinates), draw a line through **VPZ** and **PP** and extend this line to intersect **THLXY** at point **TXY** (see Figure 4). Then, draw a semicircle whose diameter is **VPZ-TXY**. Construct a line perpendicular to **VPZ-TXY** at the point **PP** and label as **CSRXY** the intersection of this line with the semicircle. The distance from **CSRXY** to **PP** is clearly the effective focal length, **f′**, for **CSRXY** is the image point obtained by revolving the camera station 90 degrees about the line **VPZ-TXY**. This can be checked by drawing an arc through **CSXY** with **TXY** as the radius point. The arc should pass through **CSRXY**.

The camera-attitude angles can also be determined as shown in Figure 4. The tilt angle, **t**, is defined as the angle between the image-plane normal (principal ray) and the object-space **Z** axis (nadir direction) in the principal plane. From the definition of **CSRXY** in Figure 4, it follows that the angle **PP-CSRXY-VPZ** is the tilt, **t**, as shown.

The azimuth, **a**, is defined as the angle between the object-space **X** axis and the line of intersection (**THLXY**) of the image plane with the object-space **XY** plane. The plan view **A′**, **B″**, **C″**, **D″** in Figure 1 reveals **a** to be the angle between **A′-B″** and **A′-VPX** (or between **THLXY** and **VPX-CSXY**). The latter definition is equivalent to the angle denoted **a** in Figure 4. Expressed in object-space terms, **a**, can be defined as the angle between the object-space **XZ** plane and the plane normal to the **XY** plane through the **THLXY**, as seen in **XY** plan view.

The swing, **s**, is the easiest angle to visualize, for it represents a transformation of coordinates within the image plane and not an orientation change of the plane itself. Swing is defined as the angle between the image-space **+y** axis and the line of intersection between the image plane and the principal plane (generated by the principal ray and the nadir direction). By convention (Slama, 1980), **s** is the angle of clockwise rotation about the principal ray from **+y** to the principal plane. Swing is readily found by drawing a ray in the image-space **+y** direction through **PP**, measuring the angle this line makes with the ray **PP-VPZ**, and denoting this angle by **s** as in Figure 4.

EXTENSION TO SOLIDS WITH TWO PARALLEL NONRECTANGULAR FACES

The methods of this paper can be applied, in slightly modified form, to structures (such as the Pentagon building in Washington, D.C., or the Flatiron building in New York City) whose walls are rectangular, but whose floors and ceilings do not form rectangles. Finding the camera parameters and plan views in these examples requires one known quantity to replace the rectangularity of the box top invoked earlier. In the example of the Pentagon, either one interior angle of the pentagon must be known, or a camera-attitude angle such as tilt, **t**, must be known.

We discuss the analyses proceeding from these two beginnings as separate cases (see Figures 5 and 6), and use for illustration a structure whose top and bottom are regular pentagons.

If a pentagon angle, **p**, is known (see **<215** in Figure 5), draw the vanishing points **VPZ**, **VP12**, and **VP15** by extending parallel lines until they meet. Here we are using single lines in the figure to represent sets of parallel lines. For example, line **1-2** represents a set of parallel lines from the tops and bottoms of windows, the parapet lines, and so on, which are omitted from the figures to simplify the illustrations; these lines converge at **VP12**. Arbitrarily assign object-space **Y** in the direction of the line **1-2** (so **VPY** = **VP12**). The first objective is to find **VPX**, the vanishing point for lines parallel to the plane of the pen-

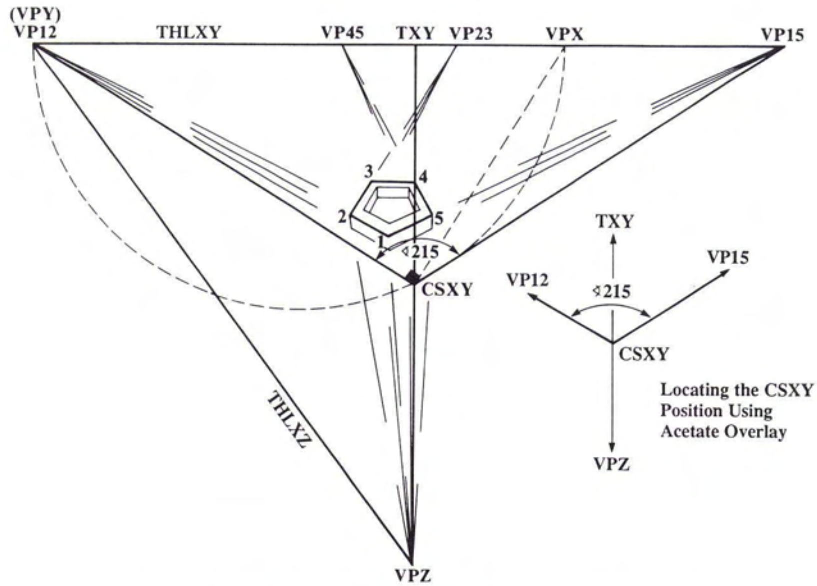

FIG. 5. Finding plan view of pentagon: pentagon angle known.

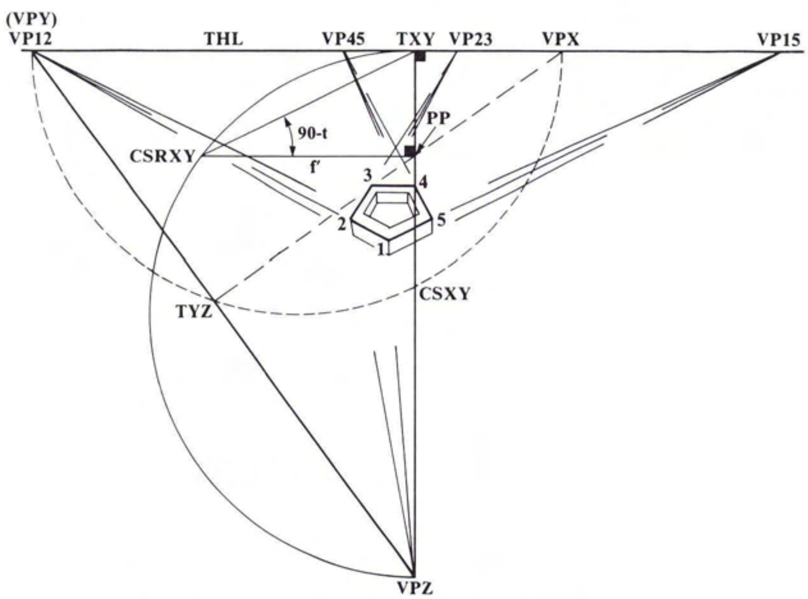

FIG. 6. Finding plan view of pentagon: tilt angle known.

tagon and perpendicular to edge **1-2**. **VPX** must lie on the line **VPY-VP15** because all sets of parallel lines in the same plane will converge to vanishing points on a common horizon line. The line **VPY-VP15** is hence **THLXY**, and the graphical accuracy with which it is drawn will be enhanced by drawing the other vanishing points, **VP45** and **VP23**, as indicated in Figure 5. Again, it should be noted that these vanishing-point locations are found using sets of parallel lines which are represented by a single line (for simplicity) in the illustrations of Figures 5 and 6. Now draw the perpendicular from **VPZ** to **THLXY**, and designate the intersection **TXY**. On a piece of clear plastic, draw two lines intersecting at angle **p**, and place the intersection point on the line **VPZ-TXY**. While keeping this intersection point on that line, adjust the position of the piece of plastic until the lines scribed on the plastic pass through **VPY** and **VP15**. The position of this intersection point on the line **VPZ-TXY** is **CSXY**, the camera station revolved 90 degrees about **THLXY**. A perpendicular to **VPY-CSXY** drawn through **CSXY** will then intersect with **THLXY** at **VPX** (see Figure 5). Once **VPX** is determined, the construction of plan views proceeds analogously with the presentation in the second section.

If the tilt angle is known (see Figure 6), construct **VPY, VP15, VPZ**, and **TXY** as before, and then draw the circle whose diameter is **VPZ-TXY**. Draw a ray from **TXY** making angle **t** with respect to the line **VPZ-TXY**; label as **CSRXY** the intersection of this line with the semicircle. Drawing the perpendicular to **VPZ-TXY** through the point **CSRXY** locates the principal point, **PP**, on **VPZ-TXY** as shown. Then, a line drawn through **PP** perpendicular to **THLYZ** intersects with **THLXY** at point **VPX**. The plan-view reconstruction then proceeds as before. This construction from the tilt angle can be understood in the same way as the other constructions, given that **CSRXY** represents the camera station revolved 90 degrees about **VPZ-TXY**, and **PP** is the intersection of the altitude of the triangle **VPX-VPY-VPZ**.

CONCLUSION

In the classical sense of photogrammetry, perspective is one of the first geometric properties of photography that is studied.

The emphasis of such studies is usually on aerial photography, where the aerial photographs may be vertical or low or high oblique views. Most courses of instruction quickly replace single photographs by multiple photograph studies because of the ease with which multiple photographs can be measured, computer programs run, and data analyzed. It is rare that much emphasis is put into discussion of single-photograph perspective required for close-range forensic photogrammetry.

In this paper, we have used specific aspects of single photograph perspective to present a fresh review of techniques commonly used in photogrammetric analysis of three-point perspective photography. Normally, these techniques are elaborated in books and journals on architectural drawing, and are used in transforming orthographic views into perspective views. We have taken these techniques (with a few modifications) and have applied them to transforming close-range perspective photography into orthographic views. The close-range perspective photograph, typical of forensic photography, is actually an auxiliary perspective of the area or object of concern. Given the auxiliary view, the photogrammetrist can use perspective techniques to construct the orthographic plan and elevation views (with all due concern for accuracy).

The use of single-image perspective techniques is not widely discussed in photogrammetric books or journals. We hope the present work will make the procedures easier for the photogrammetrists analyzing single images (particularly to forensic close-range applications), and also that it will stimulate further discussion of the application of perspective techniques in close-range photogrammetry.

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