A Robust Method for Multiple Outliers Detection in Multi-Parametric Models

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ABSTRACT: An analytical method consisting of concepts from the Danish robustified least squares, Pope's TAU statistic, and stepwise analysis is discussed. Simulated data were used to test the efficiency and reliability of the method. The collinearity condition in photogrammetry was the basic mathematical model. A flow diagram of the method and a summary of test results of test samples are included to provide more information on the method. The method can detect up to $(n - (n+u)/2$ outliers in small samples (where n is the sample size and u is the number of unknowns in the mathematical model). In large samples, more than $n - (n+u)/2$ outliers may be detectable. Geometric checkpoints were also used to reduce the effect of poor geometry of selected observations (especially in photogrammetry).

INTRODUCTION

MANY METHODS are now available for the detection of multiple outliers in multi-parametric estimation of photogrammetric and geodetic analytical models (e.g., Baarda (1968), Pope (1976), Hawkins (1981), and Cross (1985)). None of those, tested statistically, is suitable for samples having a large number of outliers. Five samples are given in Table 1 to show the largest normalized residual of each sample and the corresponding critical test values of Baarda's data snooping and Pope's TAU statistic. A large number of outliers may occur in observations (Chong, 1986); therefore, research was carried out to find a method which would be effective and reliable for analyzing observations with multiple outliers.

A brief description of each basic concept is presented, followed by a detailed description of the method. Results of a typical test sample are given to show the flow of data from observations to final solution.

ROBUSTIFIED LEAST SQUARES

Robustified least squares is a modification of the Danish method. The algorithm is shown below.

By making the weights, a function of ψ where $\psi = V_i/\sigma V_i$, then

$$P = P_i \, f(V_i, \sigma_{V_i}, \sigma_o, a) \tag{1}$$

i.e.,

$$P = \begin{cases} 1 \text{ for } |\psi| < a; \ a = \text{assigned constant} \\ 0 \text{ else} \end{cases}$$

where V_i = estimated residual of observation i,
P_i = *a priori* weight of observation i,
σ_{V_i} = estimated sigma of residual V_i,
σ_o = reference sigma, and
a = a predetermined numerical constant of Pope's TAU statistic.

Initially, a constant weight (P) is introduced in the traditional least-squares adjustment. New observation weights are computed from the ψ of the previous adjustment according to the requirement in Equation 1. A detailed description of the procedure is given in "methodology."

TAU STATISTIC

Pope (1976) discussed the TAU statistic in detail (the selection of TAU statistic was based on research findings in Chong (1986), p. 119). It is based on a distribution which he called the TAU distribution.

TAU is given as

$$\tau_\gamma = \frac{\gamma^{\frac{1}{2}} t_{\gamma-1}}{(\gamma - 1 + t^2_{\gamma-1})^{\frac{1}{2}}}$$

where t_γ = Student's-t distribution,
$\gamma = (n - u)$ or degrees of freedom,
n = sample size, and
u = number of unknown variables.

REJECTION TEST OF TAU

A critical value τ_c is selected in advance which is based on α, the probability of rejecting a true hypothesis (the probability of type I error). The probability of accepting a true hypothesis is the significance of the test, $1 - \alpha$, while the probabilities of

TABLE 1. EXAMPLES OF UNDETECTABLE MULTIPLE OUTLIERS BY SOME STATISTICAL METHODS

		Sample #1	Sample #2	Sample #3	Sample #4	Sample #5
	n	30	26	20	16	10
Test Sample	Number of Outliers	13	11	8	5	2
Information	Size of Outliers	67	50	50	50	50
Critical Test	Baarda's Data Snooping (ω_c)	1.96	1.96	1.96	1.96	1.96
Value at $\alpha = 0.05$	Pope's TAU (τ_c)	2.91	2.84	2.69	2.52	1.95
Largest Normalized Residuals (V/σ_V)*		1.88	1.88	1.61	1.86	1.43

*The critical test values of Baarda's Data Snooping (ω_c) and Pope's TAU (τ_c) were larger than the largest normalized residuals for all the five samples. The developed method was able to isolate all the outliers from the five samples successfully.

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accepting the hypothesis when false, β, and rejecting a false hypothesis, $1 - \beta$, are the probability of type II error and the power of the test, respectively. According to Pope (1976), the rejection procedure is thought of as a test of the hypothesis that $V_i \sim n(0, \sigma_{Vi})$ for all i,

where $\sigma_{Vi} = L(\hat{\sigma}_{Vi}/\hat{\sigma}_o) = L(Q_{ii}/P_i)$ for some unspecified L, by use of the test

$$\text{accept } H_o \text{ if } \max(|V_i/\hat{\sigma}_{Vi}|) < \tau_c$$
$$\text{reject } H_o \text{ if } \max(|V_i/\hat{\sigma}_{Vi}|) \geq \tau_c$$

where $\mathbf{Q} = (\mathbf{I} - \mathbf{A}(\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P})$

and $\mathbf{A}$ is the design matrix (for further information, refer Pope to (1976).

τ_c is difficult to compute because it requires the distribution of max τ under the null hypothesis. Pope (1976) proposed a simplified derivation that evades many of the subtleties involved in a more thorough approach, i.e., $\alpha = P(\max \tau > \tau_c) = P(\text{one or more of } \tau_i > \tau_c) = 1 - P(\text{all } \tau_c \leq \tau_c)$. As such, $\alpha = 1 - (1 - a)^n$ ignoring the dependence of τ's.
where $a = P(\tau > \tau_c)$ or $a = 1 - (1 - \alpha)^{1/n} \cong \alpha/n$:
n = total number of observations

Computation of Critical Value

The critical values of TAU (τ_c) could be calculated by a subroutine TAURE, which was given by Pope (1976).

Significance Level

The significance level (α) was set at 0.001 (determined by simulation test). Other popular significance levels (e.g., 0.05 and 0.01) may be used. The number of iterations would probably increase with the increase in significance level. A lower significance level is not encouraged because the subroutine on TAU may not generate accurate critical values.

STEPWISE ANALYSIS

The stepwise analysis consists of two major steps. The first step (backward selection) involves the selection of a subsample of observations for outlier testing. The second step (forward selection) involves the selection of all tested inliers and one suspected outlier to test whether the suspected outlier is actually an outlier.

Backward Selection

From test results in Chong (1986), $(u + 2)$ is the most effective, reliable, efficient, and unbiased value for the initial subsample size. Therefore, $(u + 2)$ is adopted in the method.

Forward Selection

In this step, all outlying observations are added to the test inlier one at a time. The process continues until all outlying observations are tested statistically. It must be noted here that if the "smallest" (least $|V/\sigma_V|$) outlying observation is rejected, then normally all larger ones are candidates for rejection.

Test Model

The collinearity condition in photogrammetry was used to test the developed method. The condition may be expressed in the following form, which is known as the collinearity equation:

$$x_n = -f\left(\frac{m_{11}(X_N - X_L) + m_{12}(Y_N - Y_L) + m_{13}(Z_N - Z_L)}{m_{31}(X_N - X_L) + m_{32}(Y_N - Y_L) + m_{33}(Z_N - Z_L)}\right)$$

$$y_n = -f\left(\frac{m_{21}(X_N - X_L) + m_{22}(Y_N - Y_L) + m_{23}(Z_N - Z_L)}{m_{31}(X_N - X_L) + m_{32}(Y_N - Y_L) + m_{33}(Z_N - Z_L)}\right)$$

where
x_n, y_n = photo coordinates of image point N,
f = focal length of camera,
m_{ij} = elements of rotational matrix,
X_L, Y_L, Z_L = object space coordinates of exposure station of photo L, and
X_N, Y_N, Z_N = object space coordinates of point N.

The above notations are expressed as the following terms in the analysis, i.e.,

x_n, y_n = observations,

m_{ij}, X_L, Y_L, Z_L = unknown variables, and

f, X_N, Y_N, Z_N = known constants.

Test Data

Three hundred test samples of simulated data were used to test the efficiency and reliability of the method (refer to Chong (1986), pp. 75–87). Two hundred sixty of these test samples are mathematically simulated and the rest are obtained by laboratory simulation.

Mathematically simulated data are generated by computing the variables which would satisfy a photogrammetric model under certain conditions (e.g., to compute the photo coordinates (or observations) of point P, given the coordinates of exposure stations, values of rotational elements, ground coordinates of P, and focal length of cameras which would satisfy the collinearity condition). Simulated random errors and outliers (range from 1σ to 100σ) are added on to the computed photo coordinates (σ = standard deviation).

Laboratory simulated data are obtained by means of a control frame (three-dimensional model) and two Hasselblad MK-70 close-range cameras. A glass tank filled with water is placed between the cameras and the control frame. The "tank" is used to generate a local disturbance, i.e., objects which are obstructed by the "tank" are displaced from their actual positions on the exposures. Observed photo coordinates of displaced images become outliers.

METHODOLOGY OF THE ROBUST METHOD

The method was perfected after extensive testing. For further information relating to this method, one may refer to the dissertation by Chong (1986). Flow diagrams are presented in Figure 1 and Figure 2.

(1) Observations are initially analyzed by the method of least squares in the usual way.
(2) For each observation, the normalized residual $|V/\sigma_V|$ is computed.
(3) $(u + 2)$ observations having the least $|V/\sigma_V|$ values are selected and are assigned weights of 1. All other observations are assigned weights of 0.
(4) For analytical models where geometry is important (e.g., collinearity condition), initial approximate values of unknowns (e.g., Z coordinate of exposure station of collinearity equations), σ_{n-1} (estimated standard error of observations), or σ_o (design standard error of observations) may be used as the preselected test values for the geometry check. An example of the geometry check is given in Figure 2. If any of the checks fail, an outlying observation may be selected to replace the weak observation (one having the largest $|V/\sigma_V|$).
(5) All $|V/\sigma_V|$s are computed (1st iteration) with the assigned weights. All observations with weights of 1 are tested against τ_c at a sample size of $(u + 2)$. If one of these observations is rejected, it is replaced by the observation having the next lower $|V/\sigma_V|$ in the initial least-squares solution. If all replacements are rejected, then one may use the acceptable observations to compute the final solution. If all the initial $(u + 2)$ observations are acceptable, one may assign weights of 0 to any observation in the whole sample whose $|V/\sigma_V|$ is greater than or equal to $\tau_{c(u+2)}$ except one rejected observation having the least $|V/\sigma_V|$ (this observation is called "the selected outlying observation"). This observation together with other accepted observations are assigned weights of 1.
(6) Step 4 is repeated and all $|V/\sigma_V|$s are computed (the 2nd iteration) with the assigned weights. If the selected outlying observation is rejected, one may select from the previous iteration an out-

lying observation having the next lowest $|V/\sigma_V|$ (normally not necessary because once an outlying observation is rejected, others are also candidates for rejection). Again, if all replacements (one at a time) are rejected, then one can use the accepted observations to compute the final solution. If a selected outlying observation is accepted, then all the acceptable observations ($|V/\sigma_V| < \tau_c$) in that iteration are assigned weights of 1. Here again, one outlying observation having the lowest $|V/\sigma_V|$ is also assigned a weight of 1, and all other rejected observations are assigned weights of 0.

(7) All $|V/\sigma_V|$s are computed (3rd iteration and up) with the assigned weights and step 6 is repeated.

Test Results

Table 2 provides a summary of the test results. Each test sample carries $(n - (n+u)/2)$ outliers. Test samples are grouped according to the size of outliers (e.g., a sample in group $0\sigma - 25\sigma$ has outliers whose sizes range from 0σ to 25σ). The percentage of samples tested successfully was based on the number of test samples for each group. For further information, one may refer to Chong (1986).

CONCLUDING REMARKS

The developed robust test method has many unique qualities. Some of the qualities are as follows:

- It makes use of robustified least squares which is resistant to outliers;
- It makes use of all observations that are considered good observations with reference to a statistical distribution in the final solution;
- It uses the most advanced and very popular Pope's TAU statistic;
- It uses checkpoints to detect weak geometry in observations for geometry sensitive analytical models;

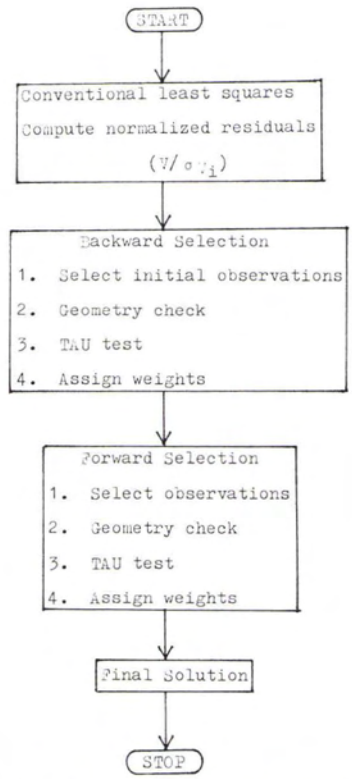

FIG. 1. Generalized flow diagram of robust test method.

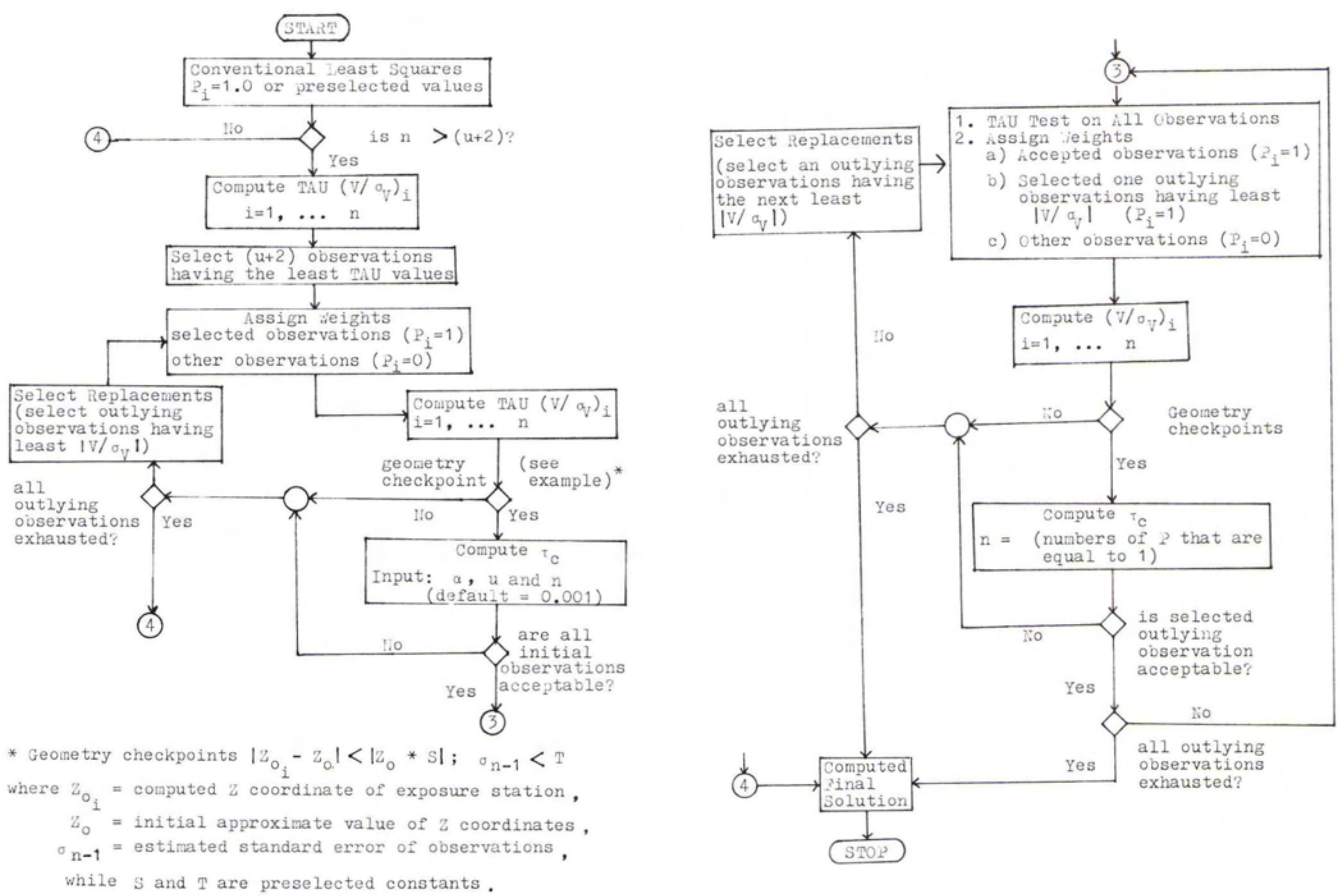

FIG. 2. A flow diagram for analyzing test data by the developed method.

TABLE 2. SUMMARY OF TEST RESULTS

Type of test data	Size of outliers (in σ)	Number of outliers in each sample	Number of sample	Number of successful case	Percent of successful test*
Mathematically	0 to 25	$(n - (n+u)/2$	65	55	84.6
simulated	25 to 50	for all the	65	65	100
	50 to 75		65	65	100
data	75 to 100	test samples	65	65	100
Laboratory	0 to 25		—	—	—
simulated	25 to 50	n = sample size	10	10	100
data	50 to 70	u = unknown	15	15	100
	75 to 100	variables	15	15	100

*The percentage of the samples that all outliers in each sample were successfully isolated.

- It can detect up to $(n - (n+u)/2)$ outliers in small samples. In large samples, more than $(n - (n+u)/2)$ outliers may be detectable.
- It can be implemented in any existing adjustment program which uses a conventional least-squares approach, by means of one or two subroutines.

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